Abelian 3-DCI groups of even order

Xin-Gui Fang
Department of Mathematics
Yantai University
Yantai, Shandong, China

Let G be a finite group. A subset S of G is called a *Cayley subset* if $S \neq \Phi$ and $1 \notin S$. Given G and a Cayley subset S of G, we define the Cayley digraph $X = X(G,S)$ of G with respect to S by

$$V(X) = G,$$
$$E(X) = \{(a,b) \mid a,b \in G,\ ba^{-1} \in S\}.$$

Given a Cayley subset S of G and an $\alpha \in \operatorname{Aut} G$, clearly α induces a graph isomorphism from $X(G,S^\alpha)$ onto $X(G,S)$. Conversly, a Cayley subset S of G is called a "Cayley Isomorphism" subset if for any graph isomorphism $X(G,S) \cong X(G,S')$ of Cayley digraphs there exists an $\alpha \in \operatorname{Aut} G$ such that $S^\alpha = S'$.

Let m be a positive integer. We call G an *m-DCI group* if every Cayley subset S with $|S| \leq m$ is a "Cayley Isomorphism" subset, and we call G a *DCI-group* if it is m-DCI for all $m \leq |G|$.

A. Ádám [1] conjectured that the cyclic group Zn of order n is a DCI-group. Elspas and Turner [2] disproved this conjecture by showing that Z_8 is not 3-DCI. Recently, the author [3] determined all finite abelian 2-DCI groups, and Min-Yao Xu and the author [4] obtained necessary and sufficient conditions under which an abelian group of odd order is 3-DCI. Our main results are stated below.

Theorem A. ([3]).

a.) *A finite abelian group G is 1-DCI if and only if every Sylow subgroup of G is homocyclic;*

b.) *A finite abelian group G is 2-DCI if and only if G is 1-DCI and the Sylow 2-subgroup of G is cyclic or elementary abelian.*

Theorem B. ([4]). *An abelian group of odd order is 3-DCI if and only if every Sylow subgroup of G is homocyclic and the Sylow 3-subgroup is cyclic or elementary abelian.*

The purpose of this paper is to prove the following.

Theorem. *Let G be an abelian group of even order, $G = H \times T$, where H is the Sylow 2-subgroup of G. Then G is 3-DCI if and only if T is 3-DCI and H is cyclic of order 4 or elementary abelian.*

A finite group G called *homogenous* if for any isomorphic subgroups H and K of G and any group isomorphism $\sigma : H \to K$, σ can be extended to an automorphism of G.

ARS COMBINATORIA 32(1991), pp. 263-267

Lemma. ([3]). *A finite abelian group is homogenous if and only if every Sylow subgroup of G is homocyclic, that is G is 1-DCI.*

Proof of the theorem: Suppose first that G is an even order abelian 3-DCI group. Then G must be 2-DCI according to the definition of m-DCI. Hence every Sylow subgroup of G is homocyclic and the Sylow 2-subgroup H is cyclic or elementary. Thus T is 3-DCI by Theorem B. Since Z_8 is not 3-DCI, H must be cyclic of order 4 or elementary abelian.

Conversely suppose that $G = H \times T$ with H cyclic of order 4 or elementary abelian and T an odd order abelian 3-DCI group. Then G is 2-DCI by Theorem A . Thus it is sufficient to prove that an arbitrary Cayley 3-subset $S = \{a, b, c\}$ of G is a "Cayley isomorphism" subset. Suppose that $S' = \{a', b', c'\}$ is a Cayley 3-subset such that $X(G,S) \cong X(G,S')$. We shall show that there exists an $\alpha \in \mathrm{Aut}\, G$ such that $S^\alpha = S'$.

For convenience, we use the following notation. For a Cayley digraph $X = X(G,S)$, an element $x \in G$ and a positive integer i, we write

$X_i(x) = \{y \in G|$ there is a directed walk of length i from x to y in $X(G,S)\}$,

$X_{-i}(x) = \{y \in G|$ there is a directed walk of length i from y to x in $X(G,S)\}$.

Clearly, $X_1(x) = xS = \{xa, xb, xc\}$, $X_{-1}(x) = \{a^{-1}x, b^{-1}x, c^{-1}x\}$ and $X_2(x) = xS^2 = \{xa^2, xb^2, xc^2, xab, xac, xbc\}$. Assume that σ is a graph ismorphism from $X(G,S)$ onto $X(G,S')$. Since Cayley digraphs are vertex-transitive, without loss of generality, we may assume that $1^\sigma = 1$. Hence $S^\alpha = S'$ and we may assume that

$$a^\sigma = a', \quad b^\sigma = b', \quad c^\sigma = c'.$$

Because a, b and c are distinct so are ab, ac and bc. Thus we get $3 \le |X_2(1)| \le 6$. When $|X_2(1)| = 3$ and 6, the same argument as in [4] will give the desired result. So we need only treat $|X_2(1)| = 4$ and 5. We shall consider the two cases separately.

Case 1: $X_2(1) = 4$. Without loss of generality we may assume $X_2(1) = \{ab, ac, bc, a^2\}$, where $a^2 = b^2 = c^2$ or $a^2 = b^2$, $c^2 = ab$. Observe that $a^2 = b^2 = c^2$ implies H elementary abelian, and that $a^2 = b^2$, $c^2 = ab$ implies that $H = Z_4$. Then conclude that S and S' must satisfy the same types of equations since $X_2(1) = 4$ for S'. It is trivial to show that there exists an $\alpha \in \mathrm{Aut}\, G$ such that $S^\alpha = S'$ when $a^2 = b^2 = c^2$. If $a^2 = b^2$ and $c^2 = ab$, may suppose $H = \langle x \rangle$ and $|x| = 4$. In this case, S and S' are one of (a), (b), (c) and (a'), (b'), (c') respectively.

$(a)\ \{u, ux, ux^2\}$, $(b)\ \{u, ux, ux^3\}$, $(c)\ \{ux, ux^2, ux^3\}$,

$(a')\ \{u', u'y, u'y^2\}$, $(b')\ \{u', u'y, u'y^3\}$, $(c')\ \{u'y, u'y^2, u'y^3\}$, where $u, u' \in T$ and $\langle x \rangle = \langle y \rangle$.

Since $|\langle S \rangle| = |\langle S' \rangle|$, we get $|u| = |u'|$. By 1-DCI of G and $\mathrm{Aut}\, G \cong (\mathrm{Aut}\, H) \times (\mathrm{Aut}\, T)$, there exists an $\alpha \in \mathrm{Aut}\, G$ such that $u'^{\alpha} = u$, $y^{\alpha} = x$. So may assume that S and S' are one of (a), (b), and (c) respectively. Now we need only to show that σ is not a graph isomorphism from $X(G,S)$ onto $X(G,S')$ if $S \neq S'$. We shall show this. We give full details of the argument for the case

$$S = \{u, ux, ux^2\}. \quad S' = \{u, ux, ux^3\} \tag{1}$$

First, let $T_i = \{u^i, u^i x, u^i x^2, u^i x^3\}$, i is a non-negative integer and $|u| = m$ is odd. From $1^{\sigma} = 1$, $S^{\sigma} = S'$, it follows that $X_{-1}(S) = X_{-1}(S')$, that is $T_0^{\sigma} = T_0$. Then by induction on i, we obtain $T_i^{\sigma} = T_i$, $i = 0, 1, 2 \cdots$. Now we shall show that

$$(ux^3)^k \longrightarrow (ux^2)^k, \tag{*}$$

for any positive integer k.

Again we do this by induction on k. When $k = 1$, from $T_1^{\sigma} = T_1$ and $S^{\sigma} = S'$, we get $(ux^3)^{\sigma} = ux^2$. Assume (*) is true for k and consider $k+1$. Suppose $k \equiv r \pmod 4$, $r \in \{0, 1, 2, 3\}$. If $r = 0$, then $(ux^3)^k = u^k$ and $(ux^2)^k = u^k$. The inductive hypothesis $(ux^3)^k \mapsto^{\sigma} (ux^2)^k$ will give $X_1(u^k)^{\sigma} = X_1'(u^k)$. Since $T_{k+1}^{\sigma} = T_{k+1}$, we get $(u^{k+1}x^3)^{\sigma} = u^{k+1}x^2$, that is $(ux^3)^{k+1} \overset{\sigma}{\mapsto} (ux^2)^{k+1}$. We can repeat what we did in $r = 0$ to show that (*) holds for $r \equiv 1, 2, 3 \pmod 4$.

On other hand, $|ux^3| = 4m$ and $|ux^2| = 2m$. From (*), we get $(ux^3)^{2m} \mapsto (ux^2)^{2m} = 1$. This contradicts $1^{\sigma} = 1$. Thus (1) does not happen. Similarly, we can show (2), (3) do not happen, when

$$S = \{u, ux, ux^2\}, \quad S' = \{ux, ux^2, ux^3\}, \tag{2}$$

$$S = \{u, ux, ux^3\}, \quad S' = \{ux, ux^2, ux^3\}. \tag{3}$$

So the theorem holds for $|X_2(1)| = 4$.

Case 2: $|X_2(1)| = 5$. Without loss of the generality may assume $X_2(1) = \{ab, ac, bc, b^2, c^2\}$, and $a^2 = b^2$ or $a^2 = bc$. We shall treat the two cases separately.

(A): $X_2(1) = \{ab, ac, bc, a^2 = b^2, c^2\}$.

In this case, we have the following facts:

1. $X_2'(1) = \{a'b', a'c', b'c', a'^2 = b'^2, c'^2\}$ in $X(G, S')$.

Because $X_1(a) \cap X_1(b) = \{a^2 = b^2, ab\}$ and $a^{\sigma} = a'$, $b^{\sigma} = b'$. It follows that $|X_1(a') \cap X_1(b')| = 2$ in $X(G, S')$. Hence $a'^2 = b'^2$ or $a'^2 = b'c'$ or $b'^2 = a'c'$. Obviously, $X_1(a') \cap X_1(b') \cap X_1(c') \neq \Phi$ in $X(G, S')$ if $a'^2 = b'c'$ or $b'^2 = a'c'$. But $X_1(a) \cap X_1(b) \cap X_1(c) = \Phi$ in $X(G, S)$ and $(X_1(a) \cap X_1(b) \cap X_1(c))^{\sigma} = X_1(a') \cap X_1(b') \cap X_1(c')$. Thus we have $a'^2 = b'^2$.

2. The graph isomorphism σ satisfies equations (**) below

$$(ac^k)^\sigma = a'c'^k, \quad (bc^k)^\sigma = b'c'^k, \quad (c^{k+1})^\sigma = c'^{k+1}, \qquad (**)$$

for $k = 0, 1, 2, \cdots, |c| - 1$.

Again by induction on k. (**) holds when $k = 0$ since $S^\sigma = S'$ and $a^\sigma = a'$, $b^\sigma = b'$, $c^\sigma = c'$. Assume (**) holds for $k-1$ and consider k. We obtain $X_1(ac^{k-1})^\sigma = X_1(a'c'^{k-1})$ and $X_1(bc^{k-1})^\sigma = X_1(b'c'^{k-1})$ by the inductive hypothesis . Hence $X_1(ac^{k-1})^\sigma \cap X_1(bc^{k-1})^\sigma = X_1(a'c'^{k-1}) \cap X_1(b'c'^{k-1})$. Because $X_1(ac^{k-1}) - X_1(bc^{k-1}) = \{ac^k\}$ and $X_1(a'c'^{k-1}) - X_1(b'c'^{k-1}) = \{a'c'^k\}$, it follows that $(ac^k)^\sigma = a'c'^k$. For the same reason, $(bc^k)^\sigma = b'c'^k$. Finally, from the inductive hypothesis $(c^k)^\sigma = c'^k$, we get $X_1(c^k)^\sigma = X_1(c'^k)$, that is $\{ac^k, bc^k, c^{k+1}\}^\sigma = \{a'c'^k, b'c'^k, c'^{k+1}\}$. Thus $(c^{k+1})^\sigma = c'^{k+1}$.

3. ([3]). Let G be an abelian 2-DCI group. $\{a, b\}$ and $\{a', b'\}$ are two Cayley 2-subset. σ is a group isomorphism from $X(G, \{a, b\})$ onto $X(G, \{a', b'\})$ and $1^\sigma = 1$, $a^\sigma = a'$, $b^\sigma = b'$. Then σ satisfies

$$(a^i b^j)^\sigma = a'^i b'^j$$

where i, j are non-negative integers.

Now we shall complete the proof of (A). First of all, we show that

$$(xc)^\sigma = x^\sigma c^\sigma = x^\sigma c', \qquad (***)$$

for arbitrary $x \in \langle S \rangle$.

Because $\langle S \rangle = \langle a, b, c \rangle$ and $a^2 = b^2$, we may assume that $x = ab^j c^k$ or $b^{j+1} c^k$, where j, k are non-negative integers. In fact, we need only prove

$$\{ab^j c^k, b^{j+1} c^k\}^\sigma = \{a'b'^j c'^k, b'^{j+1} c'^k\}. \qquad (***')$$

we do this by inductive induction on j. (***') holds by 2. when $k = 0$. Assume $j > 0$. By the inductive hypothesis, we obtain

$$[X_1(ab^j c^k) \cap X_1(b^{j+1} c^k)]^\sigma = X_1(a'b'^j c'^k) \cap X_1(b'^{j+1} c'^k),$$

that is

$$\{ab^{j+1} c^k, b^{j+2} c^k\}^\sigma = \{a'b'^{j+1} c'^k, b'^{j+2} c'^k\}.$$

Hence (***') holds.

Then, (***) implies that the restriction of σ to $\langle S \rangle$ is a graph isomorphism from $X(\langle S \rangle, \{a, b\})$ onto $X(\langle S' \rangle, \{a', b'\})$. Furthermore, σ is a graph isomorphism from $X(G, \{a, b\})$ onto $X(G, \{a'b'\})$. Now 3. tells us that σ satisfies

$(a^i b^j)^\sigma = a'^i b'^j$ for arbitrary non-negative integers i, j. Combining this with (***), find that

$$(a^i b^j c^k)^\sigma = a'^i b'^j c'^k, \qquad (****)$$

for all i, j, k non-negative integers.

Finally, by (****), it follows that the restriction of σ to $\langle S\rangle$ is a group isomorphism from $\langle S\rangle$ onto $\langle S'\rangle$. From lemma, $\sigma|_{\langle S\rangle}$ can be extended to an $\alpha \in \mathrm{Aut}\, G$, such $\alpha|_{\langle S\rangle} = \sigma|_{\langle S\rangle}$. Thus we have shown $S^\alpha = S'$.

(B): $X_2(1) = \{ab, ac, bc = a^2, b^2, c^2\}$.

The same statement as in the proof of [4] for $|X_2(1)| = 5$ will give the proof of (B).

This compleles the proof of the sufficiency of the theorem.

Acknowledgements.

The author wishes to express his sincere appreciation to Professor C. E. Praeger and Professor Ming-yao Xu for their helpful directions.

References

1. A. Ádám, *Research problem 2–10*, J. combin. Theory **2** (1967), p. 393.
2. B. Elspas and J. Turner, *Graphs with circulant adjacency Matrices*, J. Combin. Theory **9** (1970), 297–307.
3. Xin-Gui Fang, *A characterization of abelian 2-DCI groups*, J. Math (PRC), Vol. **8** (1988), 315–317.
4. Xin-Gui Fang and Ming-Yao Xu, *Abelian 3-DCI groups of odd order*, Ars Combinatoria **28** (1989), 247–251.