Anticlusters and Intersecting Families of Sets and t-valued Functions

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Abstract

It is shown that if $[n] = X_1 \cup X_2 \cup ... \cup X_l$ is a partition of $[n]$ and if S_t is a family of t-valued functions intersecting on at least one element of k (circularly) consecutive blocks, then $|S_t| \le t^{n-k}$. If given $a_1 < a_2 < ... < a_k \le l$, S_t' is a family of t-valued functions intersecting on at least one element of $X_{a_1+m}, X_{a_2+m}, ..., X_{a_k+m}$ for some m with $1 - a_1 \le m \le n - a_k$, then $|S_t'| \le t^{n-k}$. Both these results were conjectured by Faudree, Schelp and Sós [FSS]. The main idea of our proofs is that of anticlusters introduced by Griggs and Walker[GW] which we discuss in some detail. We also discuss several related intersection theorems about sets, 2-valued functions and t-valued functions.

1. Introduction and Preliminaries

Let $[n]$ denote the set $\{1, 2, 3, ..., n\}$ and let $2^{[n]}$ denote the collection of all subsets of $[n]$. The collection of all subsets of $[n]$ of cardinality k is denoted by $\binom{[n]}{k}$. The study of intersecting families of subsets of $[n]$ is by now a very well established area of combinatorics which was started in 1961 by Erdös, Ko and Rado when they proved the following theorem which is stated below in its simplest form .

Theorem 1 . If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family and $n \ge 2k$, then $|\mathcal{F}| \le \binom{n-1}{k-1}$.

If there is no restriction on the size of the members of $\mathcal{F}$ then one can show the following.

Theorem 2 . If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting family then

$$|\mathcal{F}| \leq 2^{n-1}.$$

Proof. For any $A \subseteq [n]$, A and $[n] - A$ both cannot be in $\mathcal{F}$ and the theorem follows. □

Since the original paper of Erdös, Ko, and Rado [EKR] various intersection theorems have been obtained where one asks for the largest size of the family under certain conditions , e.g. family satisfies certain closure relations, solutions of various relations such as $F_1 \subseteq F_2$ are excluded, any two sets in the family can have intersection only of certain cardinalities (or only of certain *type*) etc.

One such question was raised by Paul Erdös, who asked whether $\mathcal{F} \subseteq 2^{[n]}$, which has the property that the intersection of any two members of the family contains two consecutive integers, has size at most 2^{n-2}? The following elegant argument due to Graham answered this question in the affirmative.

Theorem 3 . If $\mathcal{F} \subseteq 2^{[n]}$ and given $F_1, F_2 \in \mathcal{F}$, $F_1 \cap F_2$ contains two consecutive integers, then

$$|\mathcal{F}| \leq 2^{n-2}.$$

Proof. Let E and O denote the set of evens and odds in $[n]$ and consider

$$\mathcal{F}_E = \{E \cap F \mid F \in \mathcal{F}\} , \mathcal{F}_O = \{O \cap F \mid F \in \mathcal{F}\}.$$

If $\mathcal{F}$ satisfies the given condition then both $\mathcal{F}_E$ and $\mathcal{F}_O$ are intersecting families and hence

$$|\mathcal{F}| \leq |\mathcal{F}_E|\,|\mathcal{F}_O| \leq 2^{|E|-1} . 2^{|O|-1} = 2^{n-2}. \qquad \square$$

The above theorem is clearly the best possible since all sets containing, say $\{1,2\}$, have the desired property and there are 2^{n-2} of them.

Also there is an extensive literature on intersection theorems on sets where the possible sizes of intersections are restricted [DEF][DF] [FW][RW].

Given $\mathcal{B} \subseteq 2^{[n]}$, a family $\mathcal{F} \subseteq 2^{[n]}$ is said to be an *intersecting family over* $\mathcal{B}$ if for every $F_1, F_2 \in \mathcal{F}$ there exists some $B \in \mathcal{B}$ such that $B \subseteq F_1 \cap F_2$. Griggs and Walker [GW] and Chung et. al. [CGFS] proposed the following most general question : Given $\mathcal{B}$, what is the largest size of an intersecting family over $\mathcal{B}$? Let us denote this number by $v(\mathcal{B})$. Throughout this paper we will restrict ourselves to the case of all members of $\mathcal{B}$ having the same size. If $\mathcal{B}$ consists of a single set $B \subseteq [n]$ then clearly $v(\mathcal{B}) = 2^{n-|B|}$. On the other hand, suppose that $\mathcal{B} = \binom{[n]}{k}$, then an intersecting family $\mathcal{F}$ over $\mathcal{B}$ has the property that for any $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \geq k$. The largest possible cardinality of such a family was determined by Katona [K1] in 1964.

Theorem 4.

$$v\left(\binom{[n]}{k}\right) = \begin{cases} \sum_{j=p}^{n} \binom{n}{j} & \text{if n+k = 2p} \\ \sum_{j=p}^{n} \binom{n}{j} + \binom{n-1}{p-1} & \text{if n+k = 2p-1 .} \end{cases}$$

A family consisting of all possible supersets of a set is called a *kernel system*. In this paper we restrict ourselves to families $\mathcal{B}$ for which $v(\mathcal{B})$ equals the size of the kernel system containing a set of the smallest size. Theorem 2 and Theorem 3 provide examples of such families over singletons and the family of all sets $\{i, i+1\}$ $(1 \leq i \leq n-1)$, respectively.

2. Results

Since it is easy to determine v in extreme cases, the first question of interest is to consider some intermediate families $\mathcal{B} \subseteq \binom{[n]}{k}$ containing more than one set but not all. Suppose that $X \in \binom{[n]}{k}$ with $X = \{x_1 < x_2 < ... < x_k\}$. Let $\mathcal{B}_n(X)$ denote the collection of all cyclic translates of $X \in [n]$, that is, the sets $X + i$ where addition is carried out modulo n. The set of *ordinary* translates of X, that is, the sets $X + i$ for $1 - x_1 \leq i \leq n - x_k$, is denoted by $\mathcal{B}_n^*(X)$. Then we have immediately that

$$2^{n-k} \leq v(\mathcal{B}_n^*(X)) \leq v(\mathcal{B}_n(X)) .$$

R. Graham has conjectured that equality holds in the above equation and for a proof of this he offers USD 100.

Conjecture 1. For all $X \in \binom{[n]}{k}$, $v(\mathcal{B}_n(X)) = 2^{n-k}$.

Instead of families of sets one can ask the same question for families of 2-valued functions and, more generally, for families of t-valued functions where t is some fixed integer. Two functions f and g *intersect* if $f(i) = g(i)$ for some i with $1 \leq i \leq n$, and a family of functions is said to be *intersecting* if any two functions in it intersect.

Let $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$ be a partition of $[n]$ into disjoint l blocks and let $k(\leq l)$ be a fixed integer. In [FSS], Faudree, Schelp and Sós proved the following :

Theorem 5. Given $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$ and $0 < k \leq l$, let $\mathcal{S}_2$ be an intersecting family of 2-valued functions such that given $f, g \in \mathcal{S}_2$, there are k consecutive sets $X_j, X_{j+1}, \ldots, X_{j+k-1}$ (where the indices are taken modulo l) and elements $x_i \in X_i$ such that $f(x_i) = g(x_i)$ for $j \leq i \leq j+k-1$, then

$$|\mathcal{S}_2| \leq 2^{n-k} .$$

In this paper we give a simple proof of this theorem using anticlusters and prove the t-valued analogue of Theorem 5 which was conjectured by Faudree, Schelp and Sós at the end of [FSS].

Theorem 6. If in the statement of Theorem 5, the given family is of t-valued functions denoted by $\mathcal{S}_t$, then

$$|\mathcal{S}_t| \leq t^{n-k} .$$

In an attempt to generalize Theorem 5, Faudree, Schelp and Sós conjectured the following.

Conjecture 2. Given $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$ and positive integers $a_1 < a_2 < \ldots < a_k \leq l$, let $\mathcal{S}'$ be an intersecting family of 2-valued functions such that given $f, g \in \mathcal{S}'$ there exists a nonnegative integer m and elements $x_i \in X_{a_i+m}$ (the indices $a_i + m$ taken modulo l) such that $f(x_i) = g(x_i)$ for each i $(1 \leq i \leq k)$, then

$$|S'| \leq 2^{n-k} .$$

Note that Theorem 5 is a special case of this conjecture when $a_i = i$ for all i. We prove the ordinary translate version of this conjecture not only for 2-valued functions, but more generally for intersecting families of t-valued functions.

Theorem 7. Given $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$ and positive integers $a_1 < a_2 < \ldots < a_k \leq l$, let S'_t be an intersecting family of t-valued functions such that given $f, g \in S'_t$ there exists an integer m, $1 - a_1 \leq m \leq n - a_k$, and elements $x_i \in X_{a_i+m}$ such that $f(x_i) = g(x_i)$ for each i $(1 \leq i \leq k)$, then

$$|S'_t| \leq t^{n-k} .$$

We also prove :

Theorem 8 . Given $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$ and $15 \leq k \leq l$, let $\mathcal{Z}$ be an intersecting family of t-valued functions with $k \geq t+1$, such that given $f, g \in \mathcal{Z}$ there are k sets $X_{j_1}, X_{j_2}, \ldots, X_{j_k}$ and elements $x_i \in X_{j_i}$ such that $f(x_i) = g(x_i)$ for each i, $1 \leq i \leq k$, then

$$|\mathcal{Z}| \leq t^{n-k} .$$

3. Anticlusters

One possible approach to Conjecture 1 led Griggs and Walker to introduce the notion of anticlusters. If we look at the proof of Theorem 2 once again, we notice that $2^{[n]}$ has been partitioned into 2^{n-1} pairs such that any intersecting family can have at most one set from each pair since sets of a given pair do not intersect. Proof of Theorem 3 also does nothing but partition $2^{[n]}$ into 2^{n-2} blocks each of size 4 such that no two sets in a block have consecutive integers in common. Thus, any family interescting in consecutive integers can have at most 2^{n-2} members – one from each block. In fact the same argument proves that if S is a family of 2-valued functions on $[n]$ such that any two functions of S coincide on two consecutive integers, then $|S| \leq 2^{n-2}$.

Definition. A collection $\mathcal{A} \subseteq 2^{[n]}$ is said to be an *anticluster* for $\mathcal{B} \subseteq 2^{[n]}$ if for every $A_1, A_2 \in \mathcal{A}$ such that $A_1 \neq A_2$, the set $A_1 \cap A_2$ includes no set B in $\mathcal{B}$.

Then for any intersecting family $\mathcal{F}$ over $\mathcal{B}$ and any anticluster $\mathcal{A}$ for $\mathcal{B}$, $|\mathcal{F} \cap \mathcal{A}| \leq 1$. It follows that for any $\mathcal{B}$, $v(\mathcal{B})$ is at most the minimum number of anticlusters needed to partition $2^{[n]}$. A partition $2^{[n]} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_m$ is called an *anticluster decomposition over* $\mathcal{B}$ if each $\mathcal{A}_i$, $1 \leq i \leq m$, is an anticluster for $\mathcal{B}$. A decomposition with the smallest possible m is called a *minimal anticluster decomposition.* In [GW], Griggs and Walker conjectured the following.

Conjecture 3 . For all $X \in \binom{[n]}{k}$ there exists a partition of $2^{[n]}$ into 2^{n-k} anticlusters over $\mathcal{B}_n(X)$.

It is clear that Conjecture 3 implies Conjecture 1. That the above conjecture is not true for general $\mathcal{B}$ was shown by Griggs and Walker. They showed that for $n = 6$ and $\mathcal{B} = \binom{[6]}{2}$ it is necessary and sufficient to take 24 anticlusters, but $v(\mathcal{B})$ is only 22. In proving all the results mentioned in the previous section, however, we will in fact be able to provide an suitable anticluster decomposition. For this we will need a reformulation of Conjecture 3 in terms of matrices which is very convenient to use. For completeness, next we discuss this reformulation closely following Griggs and Walker [GW].

A subset $B \subseteq [n]$ can be identified with its characteristic vector $(a_1, a_2, ..., a_n)$ where $a_i = 1$ if $i \in B$ and $a_i = 0$ otherwise , viewed as an element of Z_2^n. Of course, $\mathcal{A} \subseteq Z_2^n$ forms an anticluster over $\lfloor$ if for no two elements of $\mathcal{A}$, both have 1's in every component indexed by some $B \in \mathcal{B}$. Thus one can seek to partition Z_2^n into 2^{n-k} affine "subspaces" of size 2^k, each parallel to a particular k-dimensional subspace, such that each corresponds to an anticluster for $\mathcal{B}(X)$. Thus, we want a k-dimensional subspace T such that the projection of T into the k-dimensional subspace generated by the standard basis vectors indexed by elements of B is one to one and onto. In other words, we seek a $k \times n$ matrix M such that every $k \times k$ submatrix of M consisting of columns indexed by B is nonsingular. We call this matrix M an *anticluster matrix* over $\mathcal{B}$. Thus Conjecture 3 is implied by the following conjecture.

Conjecture 4. For all k and n, $0 \leq k \leq n$, and all k-subsets $X \subseteq [n]$, there exists a k by n matrix M such that for all i, the k columns of M indexed by the cyclic translate $X + i \pmod{n}$ are linearly independent over $GF(2)$.

We have been viewing Z_2^n as containing the subsets of $[n]$ and the group operation corresponds to the symmetric difference of sets. Given $F_1, F_2 \subseteq [n]$ define the operation $\bigtriangledown$ by

$$F_1 \bigtriangledown F_2 = \overline{F_1 \triangle F_2} = (F_1 \cap F_2) \cup (\overline{F_1} \cap \overline{F_2})$$

where $\overline{A} = [n] - A$. For a given family $\mathcal{B}$ of subsets of $[n]$, $\mathcal{F}$ is called a $\bigtriangledown$-family over $\mathcal{B}$ if for every $F_1, F_2 \in \mathcal{F}$, $F_1 \bigtriangledown F_2$ contains some $B \in \mathcal{B}$. Let $\overline{v}(\mathcal{B})$ denote the cardinality of the largest $\bigtriangledown$-family $\mathcal{F}$ over $\mathcal{B}$. A collection $\mathcal{A}$ is said to be an $\bigtriangledown$ – *anticluster* for $\mathcal{B}$ if for every $A_1, A_2 \in \mathcal{A}$ such that $A_1 \neq A_2$, the set $A_1 \bigtriangledown A_2$ includes no set $B \in \mathcal{B}$.

One can see that if the matrix M of Conjecture 4 exists then it would in fact give a $\bigtriangledown$-anticluster decomposition. Also the existence of such a matrix over Z_t would imply an $\bigtriangledown$-anticluster decomposition over Z_t or, in other words, an anticluster decomposition for intersecting t-valued functions. Note that it is not necessary that all characteristic vectors form a vector space. In fact for the above argument it suffices that they form a module over Z_t.

4. Anticluster Matrix and Proofs of Theorem 6 and Theorem 8

The statement of Theorem 6 was conjectured by Faudree, Schelp and Sós at the end of [FSS]. Somewhat surprisingly their proof for the case $t = 2$ can be straightforwardly extended for any t. Though we will shortly see a simple proof using anticlusters, first we discuss this extension. We need an analogue of Lemma 6 of [FSS] and for completeness we outline its proof here.

Lemma 1 . For $l = n \leq 2k$, if $\mathcal{L}$ is a family of t-valued functions from $[n]$ to $\{1,2,...,t\}$ such that any two members of $\mathcal{L}$ intersect on k circularly consecutive integers, then

$$|\mathcal{L}| \leq t^{n-k} .$$

Proof. Let $Y \subseteq [n]$ be the set on which all functions of $\mathcal{L}$ agree. If $|Y| \geq k$ then the lemma is obvious. So assume $|Y| < k$. Suppose there are integers i and j, $n-k \leq |i-j| \leq k$ such that both are not in Y. Then there exist two functions say f and g such that $f(j) \neq g(j)$ and we may assume $f(i) = 0$. Since f and g agree on k consecutive integers and there are less than k integers between i and j, we must also have $g(i) = 0$. Because $i \notin Y$, there is an $h \in \mathcal{L}$ such that, say, $h(i) = 1$. Since f and g have distinct values on j, h must disagree with one of them, say f, and we obtain that h and f do not agree on k consecutive integers. It follows that if i is not in Y, then $2k-n+1$ consecutive integers, namely $i+n-k, i+n-k+1, ..., i+k$, must be in Y. Therefore a single element in $[n] - Y$ gives rise to $2k - n + 1$ elements in Y and each additional element in $[n] - Y$ will give rise to at least one more element in Y. Hence

$$|Y| \geq 2k - n + |[n] - Y| > 2k - n + n - k \geq k ,$$

which is a contradiction and the lemma follows. □.

Proof of Theorem 6. For $l = pk + r$, $0 \leq r < k$, partition the index set $\{1, 2, \ldots l\}$ into $k + r$ subsets $Y_1, Y_2, \ldots, Y_{k+r}$ by letting $Y_i = \{i, k+i, \ldots, (p-1)k + i\}$ for $1 \leq i \leq k$ and $Y_{i+k} = \{pk + i\}$ for $1 \leq i \leq r$. Note that any two distinct integers in the same block differ by at least k, so any k (circularly) consecutive integers will be in k consecutive terms of this partition modulo $k + r$. Let $W_1, W_2, \ldots, W_{k+r}$ be the partition of $[n]$ defined by $W_i = \cup_{j \in Y_i} X_j$ for $1 \leq i \leq k+r$. Given any two functions $f, g \in S_t$ there exist elements $w_j, w_{j+1}, ..., w_{j+k-1}$ (indices taken modulo $k+r$) with $w_i \in W_i$ such that $f(w_i) = g(w_i)$ for $j \leq i \leq j + k - 1$.

For a given ordered partition of $W_i = A_i^1 \cup A_i^2 \cup \ldots \cup A_i^t$ we will only consider the functions which take value $j \in [t]$ on every element of A_i^1, $j + 1$ on every element of A_i^2 and in general $j + m - 1$ (modulo t) on A_i^m. Such a function will be called a *cyclic* function on

W_i with respect to $A_i^1, A_i^2, ..., A_i^t$. Given an ordered partition there are t cyclic functions since they are completely determined by their value on A^1. Now we consider the set of functions on $[n]$ which are cyclic on each W_i with respect to the given ordered partitions. More precisely, let $S((A_1^1, A_1^2, ..., A_1^t), (A_2^1, ..., A_2^t), ...(A_{k+r}^1, ..., A_{k+r}^t))$ (in short, S) denote the set of those functions which are cyclic on each W_i with respect to $A_i^1, ..., A_i^t$. There are t^{k+r} such functions and the set of all t- valued functions can be partitioned into t^{n-k-r} such sets.

Now, any function f on S can be identified with a t-valued function f' on $\{1, 2, ..., k+r\}$ where f' is defined by $f'(i) = f(A_i^1)$ for each i. This identification also identifies any map in $S_t \cap S$ to a t-valued function on $\{1, 2, ..., k+r\}$ such that any two such functions intersect in k consecutive integers (modulo $k+r$). Therefore, by Lemma 1, we have $|S_t \cap S| \leq t^r$. This inequality is valid for each of the t^{n-k-r} classes and hence we obtain

$$|S_t| \leq t^{n-k-r} . t^r = t^{n-k} . \quad \square$$

Proof of Theorem 5 . Note that Theorem 5 is equivalent to the existence of a $k \times n$ matrix M of zeros and ones such that given a partition of the column indices $[n] = X_1 \cup X_2 \cup ... \cup X_l$ all the $k \times k$ submatrices consisting of one column each from k (circularly) consecutive blocks have determinant 1 (mod 2). Simply take a $k \times l$ matrix M' constructed using Pascal's Triangle (mod 2) as follows : without loss of generality assume first k columns of M' to form an identity matrix and let last $n-k$ columns be given by $m_{i,i+1} = 1$, for all i, $m_{k,j} = 1$, for all $j \geq k+1$, $m_{i,j} = m_{i+1,j} + m_{i,j-i}$ (mod 2), for $1 \leq i \leq k-1$, $k+2 \leq j \leq n$. It can be checked that any k consecutive columns of M' form a matrix with determinant $\equiv 1$ (mod 2). For details regarding construction of this matrix and why it works, see [GW](pp.94-96).

Now M can be constructed by repeating column 1 of M' $|X_1|$ times, column 2 $|X_2|$ times and in general column j $|X_j|$ times for $1 \leq j \leq l$. This matrix has the desired property and the theorem follows. $\square$

Notice that the proof of Theorem 6 partitions the set of all functions in S_t into t^{n-k-r} classes such that from each class we can have at most t^r functions. Lemma 1 can be used to further decompose each of those subclasses into t^r anticlusters. An explicit anticluster decomposition can also be given using the matrix approach.

Proof 2 of Theorem 6. Proceed exactly as in the proof of Theorem 5 to construct a matrix M with entries in Z_t *using Pascal's triangle (mod t)* such that given a partition of the column indices $[n] = X_1 \cup X_2 \cup \ldots \cup X_l$, all $k \times k$ submatrices consisting of one column each from k (circularly) consecutive blocks are nonsingular. □

Proof of Theorem 8. As in the proof of Theorem 6 let $S((A_1^1, A_1^2, \ldots, A_1^t), (A_2^1, \ldots, A_2^t), \ldots (A_l^1, \ldots, A_l^t))$ (in short S) denote the set of all functions which are cyclic on each X_i with respect to the ordered partition $A_i^1, A_i^2, \ldots, A_i^t$. There are t^l such functions and the set of all t-valued functions can be partitioned into t^{n-l} sets of this type. Any function of $S \cap \mathcal{Z}$ can be naturally identified with a collection of t-valued functions on $[l]$ such that any two functions agree on at least k points. Under the hypothesis, a theorem of Frankl and Furedi[FF] stated below (Theorem 9) gives that $|S \cap \mathcal{Z}| \leq t^{l-k}$. Hence we obtain

$$|\mathcal{Z}| \leq t^{n-l} \cdot t^{l-k} = t^{n-k} . \quad \square$$

Theorem 9 [FF]. The maximum number of integers sequences $(a_1, a_2, \ldots, a_n)$ such that $1 \leq a_i \leq t$ for $1 \leq i \leq n$, and any two sequences agree in at least $k (\geq 15)$ positions is t^{n-k} if and only if $t \geq k+1$.

The above theorem for $t \leq k$ is not true in general. The special case $t = 2$ is of particular interest since for this the problem reduces to the following : What is the maximum number of subsets of $[n]$ such that the symmetric difference of any two has cardinality at most $n - k$? This problem was posed by Erdös and solved by Kleitman [K2].

Theorem 10 [K2]. Let $\mathcal{Z}$ be a family of 2-valued functions such that any two intersect on at least k elements then

$$|\mathcal{Z}| \leq \begin{cases} \sum_{i=0}^{r} \binom{n}{r} & \text{for n+k=2r} \\ \sum_{i=0}^{r} \binom{n}{i} + \binom{n-1}{r} & \text{for n+k=2r+1.} \end{cases}$$

5. Families Over Ordinary Translates and the Proof of Theorem 7

Though it is not yet known whether one can prove Conjecture 1 using anticlusters, Griggs and Walker have used this to show that a kernel system is optimal over ordinary translates of a set. Their approach is to select the columns of M (the matrix in Conjecture 3) one at a time. In this greedy approach one selects any column vector for column j that is, for every i such that $j \in (X+i) \pmod{n}$, independent of the subspace generated by the column indexed by the set $(X+i) (\mathrm{mod}\ n) \cap [j-1]$. This means that we choose any k-vector as column j, given columns $1, 2, ..., j-1$, that gives rise to a suitable $k \times j$ matrix thus far (see [GW] pp. 96-97). Griggs and Walker thus obtained

Theorem 11 [GW]. For all k and n with $0 \leq k \leq n$, and any k-subset X of $[n]$, $v(\mathcal{B}_n^*(X)) = 2^{n-k}$.

We use this to prove the following conjecture of Faudree, Schelp and Sós.

Theorem 12. Given $[n] = X_1 \cup X_2 \cup ... \cup X_l$ and positive integers $a_1 < a_2 < ... < a_k \leq l$ let S' be an intersecting family of 2-valued functions such that given $f, g \in S'$ there exists an integer m, $1 - a_1 \leq m \leq n - a_k$, and elements $x_i \in X_{a_i+m}$ such that $f(x_i) = g(x_i)$ for each i $(1 \leq i \leq k)$, then

$$|S'| \leq 2^{n-k} .$$

Proof of Theorem 12. First construct a $k \times l$ matrix M' with $X = \{a_1, a_2, ..., a_k\}$ using the greedy approach of Theorem 11. It has the property that the matrix consisting of columns indexed by any translate of X is nonsingular. Now we construct a $k \times n$ matrix

M by first putting $|X_1|$ copies of column 1 of M', next $|X_2|$ copies of column 2 of M' and so on until $|X_l|$ copies of the last column of M' in the end. The new matrix M so constructed has the property that any $k \times k$ submatrix formed by k columns where the indices are taken one each from $X_{a_1+m}, X_{a_2+m}, \ldots, X_{a_k+m}$, for some m with $1 - a_1 \leq m \leq n - a_k$, is nonsingular. The theorem follows from an argument used in the previous proof. □

Proof of Theorem 7. We use the same trick of partitioning all t-valued functions into sets of cyclic functions. Let $S((A_1^1, A_1^2, \ldots, A_1^t), (A_2^1, \ldots, A_2^t), \ldots (A_l^1, \ldots, A_l^t))$ (in short S) denote the set of all functions which are cyclic on each X_i with respect to the ordered partition $A_i^1, A_i^2, \ldots, A_i^t$. There are t^l such functions and the set of all t-valued functions can be partitioned into t^{n-l} sets of this type. Any function of $S \cap S_t'$ can be naturally identified with a collection of t-valued functions on $[l]$ such that any two functions agree on a (ordinary) translate of $X = \{a_1, a_2, \ldots, a_k\}$. The theorem follows from the following Lemma. □

Lemma 2. Given $0 < k \leq l$ and $0 < a_1 < a_2 < \ldots < a_k \leq l$ let $\mathcal{T}$ be a family of t-valued functions on $[l]$ such that for every $f, g \in \mathcal{T}$ there exists an integer m with $1 - a_1 \leq m \leq l - a_k$ such that $f(a_i + m) = g(a_i + m)$ for each i $(1 \leq i \leq k)$, then

$$|\mathcal{T}| \leq t^{l-k}.$$

Proof. We imitate the proof of Theorem 12 over Z_t. The only crucial step is to construct a $k \times l$ anticluster matrix M' over Z_t using the greedy approach of section 5 of [GW](see pp. 96-97). Note that if t is a prime power then the lemma can be proved exactly as Theorem 11. So we assume that t is not a prime power.

Also assume that $k > 0$ because the lemma is trivial for $k = 0$. Each element j of $[l]$ appears at most once as the rth largest element of a translate of $X = \{a_1, a_2, \ldots, a_k\}$, for each r with $1 \leq r \leq k$. To select column j given the first $j-1$ columns of M', we must only be sure that for each such r, column j is not an element of the "subspace" generated by the columns indexed by the $r-1$ elements less than j in the translate of X. There are at most $t^{r-1} - 1$ distinct nonzero vectors which are linear combinations of these $r-1$ vectors. In addition, column j must not be a vector with *order* less than t or, what we may call by abuse of language, a *zero divisor*. Clearly, no component of a zero divisor can be relatively prime to t and therefore the total number of zero divisors cannot be more than $(t-\phi(t))^k$, where ϕ is the Euler-phi function. So the number of columns which will work as column j is at least

$$t^k - \left((t-\phi(t))^k + \sum_{r=1}^{k}(t^{r-1}-1) \right).$$

For $t \geq 4$ this expression can be seen to be greater than 0 and the lemma follows. □

Of course, one can directly construct a suitable matrix M for Theorem 7 from a matrix for Lemma 2.

6. Discussion and Open problems

The essential point of all the results proved in previous sections is that if a kernel system is optimal over $\mathcal{B}$ consisting of translates (cyclic or ordinary) of some fixed set, then the same bound continues to hold over $\mathcal{B}'$ which includes all sets obtained by replacing each element of the underlying set by a set and taking one element from each block indexed by a member of $\mathcal{B}$.

If the partition in Theorem 8 has all the blocks of equal size then one obtains a subfamily $\mathcal{B}$ of $\binom{[n]}{k}$ of cardinality $\frac{n^k}{l^{k-1}}$ $(= O(\binom{n}{k}))$ which is much larger (for suitable l and k) than, say the family of all cyclic translates of a particular set, and still has $v(\mathcal{B})$ equal to the size of a kernel system. One may ask : What is the size of the largest family $\mathcal{B} \subseteq \binom{[n]}{k}$ such that $v(\mathcal{B}) = 2^{n-k}$? Same question for families of t-valued functions.

The most general question, of course, is to classify all families for which a kernel system is optimal.

Acknowledgements

This paper has been adapted from Chapter 5 of our doctoral thesis written under the supervision of Professor Daniel J. Kleitman, who is always very helpful and full of encouragement.

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