Edge Clique Graphs of Some Important Classes of Graphs[1]

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Abstract. In this paper we study the edge clique graph $K(G)$ of many classes of intersection graphs G—such as graphs of boxicity $\leq k$, chordal graphs and line graphs. We show that in each of these cases, the edge clique graph $K(G)$ belongs to the same class as G. Also, we show that if G is a W_4-free transitivity orientable graph, then $K(G)$ is a weakly θ-perfect graph.

Section 1: Introduction

In this paper we shall study the *edge clique graph of* G which is denoted by $K(G)$, and is derived from G in the following way: Let $V(K(G)) = E(G)$ and for every pair of edges e_i and e_j, let I_{ij} be the set of vertices of G upon which these two edges are incident, i.e., I_{ij} contains three or four vertices depending on whether or not e_i and e_j share a common vertex. Join e_i and e_j by an edge in $K(G)$ iff I_{ij} forms a clique (a complete subgraph) of G.

The edge clique graph was first introduced by Albertson and Collins [1984]. They have given results related to perfection of $K(G)$, as for example what properties of G will force $K(G)$ to be perfect.

Our interest in edge clique graphs was initiated in the study of the intersection number $i(G)$ of graph G. G is an *intersection graph* if we can assign a set $S(x)$ to each vertex x of G, such that $\{x,y\} \in E(G) \leftrightarrow S(x) \cap S(y) \neq \phi$. It is easy to see that every graph is an intersection graph. So we define the *intersection number* $i(G)$ of a graph to be the minimum cardinality of a set S such that G is the intersection graph of subsets of S. It can be shown that $i(G) = \theta_e(G) = \theta_v(K(G))$, where $\theta_e(G)$ and $\theta_v(G)$ are the minimum number of cliques required to cover $E(G)$ and $V(G)$ respectively. Since θ_v is a widely studied parameter, we investigate those classes of graphs for which $K(G)$ belongs to the same class as G so that $\theta_v(K(G))$ could be found by existing algorithms. This has been shown to be true for chordal graphs in Albertson and Collins [1984] and for chordal and strongly chordal graphs in Raychaudhuri [1988].

In section 2 we show that $K(G)$ preserves the structure of many intersection graphs namely chordal graphs and graphs of Boxicity $\leq k$. In section 3 we show

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that $K(G)$ is a line graph if G is one[2], and in section 4 we show that if G is a W_4-free transitively orientable graph, the $K(G)$ is a weakly θ-perfect graph [a graph for which $\theta_v(G) = \alpha(G)$, the maximum cardinality of a mutually nonadjacent set of vertices].

Section 2: Edge Clique Graph of Some Intersection Graphs

An important property of some intersection graphs is the Helly property which we describe next. A family $\{T_i\}_{i \in I}$ of subsets of a set T is said to satisfy the *Helly property* for J, if $J \subseteq I$, and $T_i \cap T_j \neq \phi$ for all $i, j \in J$ implies $\bigcap_{k \in J} T_k \neq \phi$.

Suppose G is the intersection graph of sets belonging to a family D, (for example, D may be the family of all real intervals). Is $S(x) \in D$ represents an intersection representation for G, the let $S(\{x, y\}) = S(x) \cap S(Y) \neq \phi$ be an assignment made to every edge $\{x, y\}$ of G. Then we have the following theorem.

Theorem 1. *Suppose G is the intersection graph of sets belonging to a family D. Suppose there is an intersection assignment $S(x)$ for G which satisfies the following conditions:*

(a) *$S(\{x, y\}) \in D$ for all $\{x, y\} \in E(G)$.*

(b) *S has the Helly property for all cliques of G, i.e., $\bigcap_{x \in K_i} S(x) \neq \phi$ for all cliques K_i of G.*

Then $K(G)$ is an intersection graph of sets belonging to family D.

Proof: Let $e = \{u, v\} \in E(G)$. Let e belong to the maximal cliques $K_1, K_2, \ldots,$ K_s. Then by (b) there is an intersection assignment S for which $S(K_i) = \bigcap_{x \in K_i} S(x) \neq \phi$ for all $i = 1, 2, \ldots, s$. Note that

$$S(K_i) \subseteq S(\{u, v\}), i = 1, \ldots, s \tag{1}$$

Associate with each edge $\{u, v\}$ of G, the set $S(\{u, v\})$. Note that $S(\{u, v\}) \in D$ by (a), and that

$$\begin{aligned} S(\{u, v\}) &\subseteq S(u) \\ S(\{u, v\}) &\subseteq S(v) \end{aligned} \tag{2}$$

We claim that $S(\{u, v\})$ is an intersection representation for the graph $K(G)$. To see why, suppose $\{e_i, e_j\} \in E(K(G))$. Then e_i, e_j belong to some maximal clique K_ℓ of G. Then $S(e_i)$ intersects $S(e_j)$ at $S(K_\ell)$ by (1). Next, suppose that $e_i = \{u, v\}$ and $e_j\{w, z\}$ and that $\{e_i, e_j\} \notin E(K(G))$. We claim that $S(e_i) \cap S(e_j) = \phi$, otherwise if $S(e_i) \cap S(e_j)$ contains a common point, say α, then by (2) $\alpha \in S(u), S(v), S(w)$ and $S(z)$ and therefore $\{u, v, w, z\}$ must be a clique of G, since S is an intersection assignment, which is a contradiction. ∎

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[2] We have recently become aware of an independent work [1988] by Chartrand, G., Kapoor, S.F., McKee, T.A., and Saba, F., in which similar results were obtained using different techniques.

Let us next recall an important characterization of *chordal graph* (a graph in which every cycle of length four or more has a chord) which says that G is a chordal graph iff G is the intersection graph of a family of subtrees of a tree. Also it is well known that the family of subtrees T_i of a tree T satisfy Helly property. (See Golumbic [1980]). Corollary 1.1 follows from the two above observations.

Corollary 1.1. *If G is a chordal graph, then $K(G)$ is a chordal graph.*

Boxicity of a graph G is the minimum k for which G is an intersection graph of k dimensional boxes in the Euclidean plane. ∎

Corollary 1.2. *If $k \geq 0$ and G is a graph of boxicity at most k then $K(G)$ has boxicity at most k.*

Proof: If $k = 0$, then Boxicity $(G) = 0$ means (by convention) G is a complete graph. Thus $K(G)$ is also a complete graph. So Boxicity $(K(G)) = 0$. Next suppose $k > 0$. Let $S(x)$ be an intersection assignment of G, where each $S(x)$ is a box of dimension $\leq k$. Every box of dimension k can be represented as the intersection of k intervals, so $S(x) = I_{x_1} \cap I_{x_2} \cdots \cap I_{x_k}$. We shall show that $S(x)$ satisfies conditions (a) and (b). Suppose $\{x, y\} \in E(G)$. Then $S(x) \cap S(y) \neq \phi$. Hence $\{I_{x_1} \cap I_{x_2} \cdots \cap I_{x_k}\} \cap \{I_{y_1} \cap I_{y_2} \cdots \cap I_{y_k}\} \neq \phi$. Thus, $I_{x_i} \cap I_{y_i} \neq \phi, i = 1, \ldots, k$. Then, $I_{x_i y_i} = I_{x_i} \cap I_{y_i}$ is an interval in the ith dimension and $S(\{x, y\}) = I_{x_1 y_1} \cap \cdots \cap I_{x_k y_k}$ is a k-dimensional box being being the intersection of k nonempty intervals in k different dimensions. Thus (a) is satisfied. Also (b) is satisfied since it is clear that if a family of boxes in k-space pairwise intersect then they have a nonempty intersection. ∎

By Corollary 1.1, the existing method to find $\theta_v(G)$ for a chordal graph G can be modified to find $i(G)$. Such a modification is given in Raychaudhuri [1988].

Section 3: Edge Clique Graph of Line Graphs

In this section we shall show that if G is a line graph then so is $K(G)$. A *line graph* of a graph G has as its vertex set the edge set of G and two edges of G are joined by an edge in the line graph if they share a common vertex in G. So a line graph is an intersection graph, where the set $S(x)$ corresponding to any vertex x is a two element set and $S(x) \neq S(y)$ is $x \neq y$. Such an assignment is a 2-r *set intersection assignment*, given by Steif [1982].

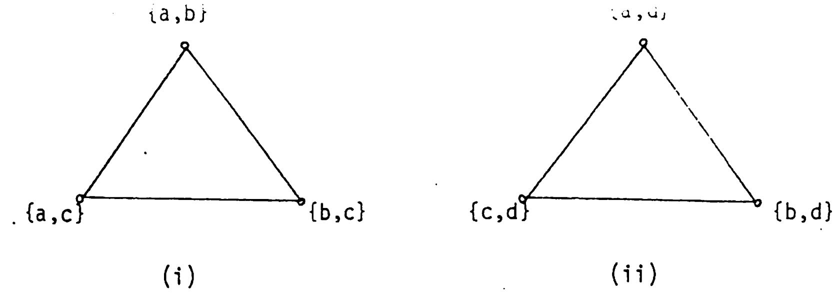
Next we prove some lemmas concerning line graphs, which are necessary to prove the main theorem of the section.

Lemma 1. *If K is a maximal clique in a line graph G, and $|K| \geq 4$, and if $S(x)$ is any 2-r set intersection assignment for G, then $\bigcap_{x \in K} S(x) \neq \phi$.*

Proof: If L is any clique of size 3 in a line graph G, then there are essentially only two 2-r set intersection assignments for the vertices in L as shown in Figure 1. One has $\bigcap_{x \in L} S(x) = \phi$ (Figure 1 (i)) and the other has $\bigcap_{x \in L} S(x) \neq \phi$

(Figure 1 (ii)). To see why Lemma 1 is true, consider any clique L of size 3 contained in K. If $\bigcap_{x\in L} S(x) \neq \phi = \{d\}$, as in Figure 1 (ii), then obviously $\bigcap_{x\in L} S(x) \neq \phi = \{d\}$. But if for all x in L, $S(x)$ is as shown in Figure 1 (i), then for all x in $K-L$, $S(x)$ must intersect with each of $\{a,b\}$, $\{b,c\}$ and $\{c,a\}$. Since $S(u) \neq S(v)$ whenever $u \neq v$, this vertex x cannot be given a 2-r set intersection assignment. Hence we must have $\bigcap_{x\in K} S(x) \neq \phi = \{d\}$. ∎

Figure 1. $S(x)$ for a clique of size 3 in the 2-r set intersection assignment of a line graph G



Lemma 2. *Any particular edge of a line graph G cannot be contained in more than one clique of size ≥ 4.*

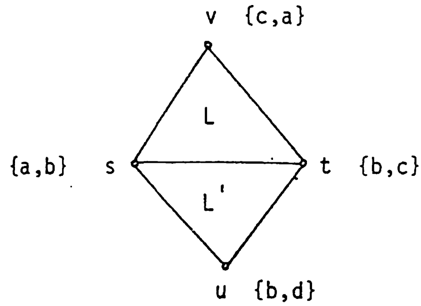
Proof: By contradiction, suppose that an edge $e = \{s,t\}$ of G is contained in two (or more) distinct cliques K and K', each of size ≥ 4. Then if $S(x)$ is a 2-r set representation of G and $S(s) = \{a,d\}$ and $S(t) = \{b,d\}$, then $\bigcap_{x\in K} S(x)$ and $\bigcap_{x\in K'} S(x) \neq \phi$, by Lemma 1, and the only element which may belong to these intersections is d. But since $K \neq K'$, this is a contradiction. ∎

Lemma 3. *Any particular edge of a line graph G cannot be contained in more than two cliques of size ≥ 3.*

Proof (by contradiction): Suppose $e = \{s,t\} \in E(G)$ belong to two distinct cliques L and L', where $|L|$ and $|L'|$ are ≥ 3. Since $L \neq L'$, there is a $v \in V(L)$ and there is a $u \in V(L')$ such that u and v are not adjacent in G. Then without loss of generality, the only possible 2-r set representation of vertices s,t,u,v of G are as shown in Figure 2. Note that $\bigcap_{x\in L'} S(x) \neq \phi = \{b\}$. If $\{s,t\} \in$ to a third maximal clique L'', L'' must contain a vertex w, which is not adjacent to some vertex in L'. So $b \notin S(w)$. But since w is adjacent to both s and t, for any 2-r set intersection assignment S of G, $S(w)$ must contain b which is a contradiction. Thus any edge of G can belong to at most two maximal cliques of size ≥ 3. ∎

From Lemmas 2 and 3 it follows that for any edge e of G, one and only one of the following cases may occur.

Figure 2

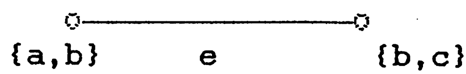


Case 1: e is maximal clique in G.
Case 2: e is in exactly one maximal clique, which is of size 3.
Case 3: e is in exactly one maximal clique which is of size ≥ 4.
Case 4: e is in exactly two maximal cliques, one of which is of size 3 and the other one is of size ≥ 4.
Case 5: e is in exactly two maximal cliques, each of size 3.

Theorem 2. *The edge clique graph $K(G)$ of a line graph G is a line graph.*

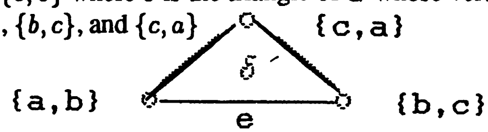
Proof: Suppose we are given a 2-r set intersection assignment $S(x)$ for the line graph G. With each edge e of G we shall associate a 2-r set S' in all the above five cases as shown and explained below:

Case 1: $S'(e) = \{b, e\}$

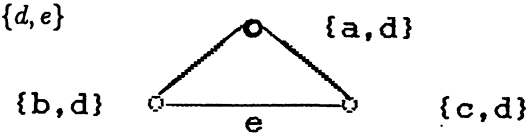


Case 2:

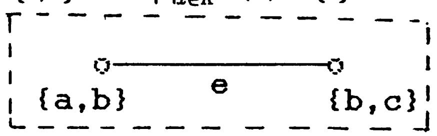
(a) $S'(e) = \{e, \delta\}$ where δ is the triangle of G whose vertices are represented by $\{a, b\}$, $\{b, c\}$, and $\{c, a\}$



(b) $S'(e) = \{d, e\}$

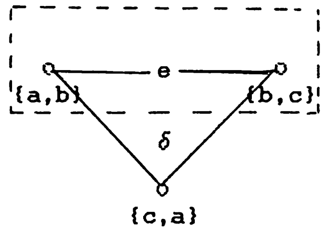


Case 3: $S'(e) = \{b, e\}$ where $\bigcap_{x \in K} S(x) = \{b\}$



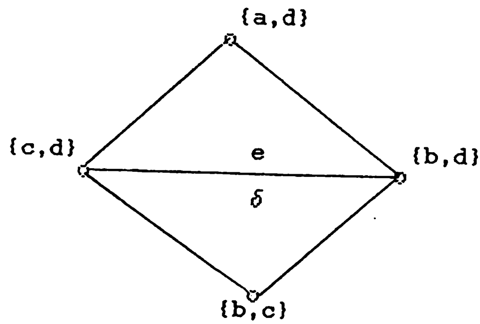
A maximal clique of K size ≥ 4

Case 4: $S'(e) = \{b, \delta\}$ where $\bigcap_{x \in K} S(x) = b$, and δ is the triangle of G represented by $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$



A maximal clique K of size ≥ 4

Case 5: $S'(e) = \{d, \delta\}$, where δ is the triangle represented by $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$.



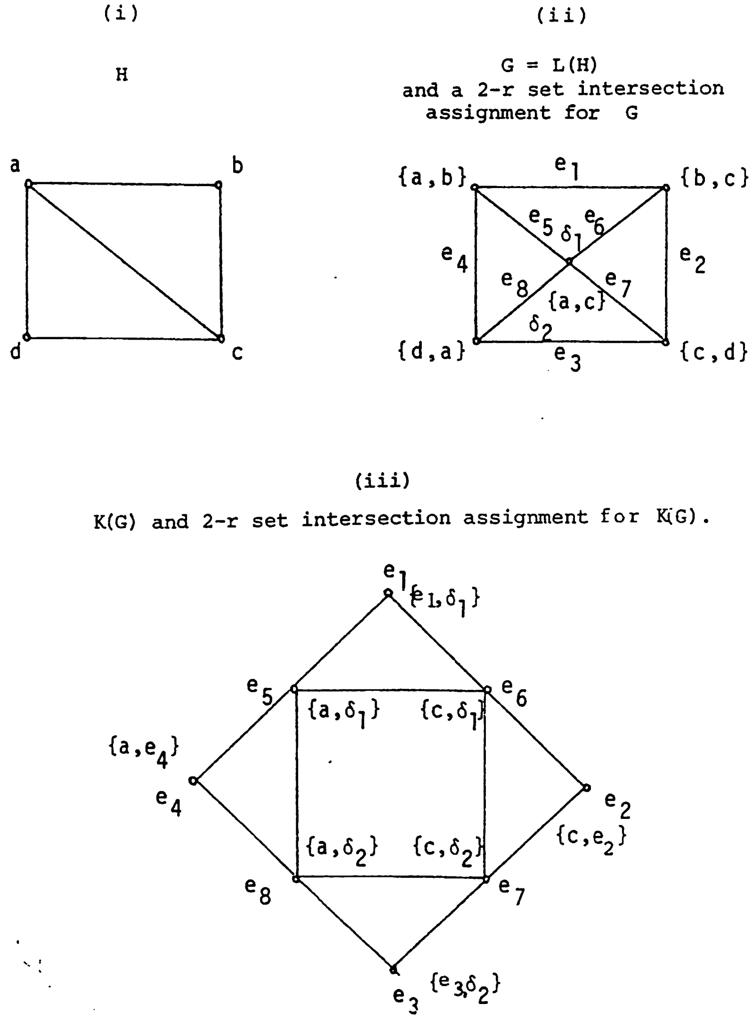
It is easy to see that $S'(e)$ is a 2-r set intersection assignment for $K(G)$. ∎

We have illustrated Theorem 2 with an example as shown in Figure 3. G is the line graph of H and $S(x)$ is a 2-r set intersection assignment for G shown in Figure 3 (ii). Then we give a 2-r set intersection assignment for $K(G)$ in Figure 3 (iii). Thus by Theorem 2, finding $i(G)$ for a line graph G is equivalent to finding $\theta_v(K(G))$, where $K(G) = L(H)$, the line graph of some graph H, whose construction is described in the Theorem. Unfortunately there is no known good algorithm to find θ_v for a line graph. But it is possible to find a lower bound for $i(G)$ in polynomial time, since $i(G) \geq \alpha(K(G)) = \alpha(L(H))$, which is the cardinality of a maximum cardinality matching in H, (See Lawler [1976] for discussion of such an algorithm).

Section 4: Intersection Number of W_4-free Transitively Orientable Graphs.

The important question that we ask in this section is: If G is a transitively orientable graph, which class does $K(G)$ belong to? We have been able to answer this question only partially. In particular we show that if G is a four wheel free transitively orientable graph, then $K(G)$ is weakly θ-perfect. Also we give an algorithm to find $i(G)$ for such graphs by solving a minimum flow problem in a network with lower capacities on its arcs.

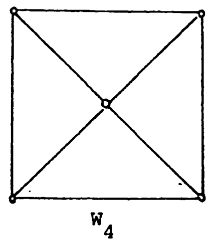
Figure 3



The graph W_4, i.e. the *wheel on four vertices*, is illustrated in Figure 4. Suppose G is transitively orientable graph. Let H be the Hasse diagram of some partial order associated with G. Then H has two obvious orientations, either from down to up or vice-versa. An *oriented Hasse diagram* $\hat{H}$ is a Hasse diagram with one of its obvious orientations. Then we have the following lemma.

Lemma 4. *Suppose G is a transitively orientable graph and G does not contain W_4 as an induced subgraph. Then no oriented Hasse diagram $\hat{H}$ of G can contain*

Figure 4. The wheel on four vertices

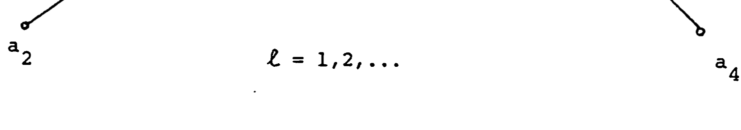


a generated subdigraph D isomorphic to the digraph D of Figure 5.

Proof: Suppose some oriented Hasse diagram $\hat{H}$ of G contains a generated subdigraph isomorphic to D of Figure 5. Then the graph generated by vertices a_1, a_2, a_3, a_4 and b_i, any $i \in \{1, 2, \ldots, \ell\}$, is a W_4. ∎

Given a W_4-free transitively orientable graph, G, draw an oriented Hasse diagram $\hat{H}$ of G. Let X and Y be respectively the maximal and minimal elements of H. If $|X|$ or $|Y| > 1$, add a source s or a sink t respectively and arcs (s, x)

Figure 5



for all x in X and arcs (y, t) for all y in Y. Otherwise let the unique elements of X and Y be the source and the sink respectively. Add a lower capacity of one on all arcs of $\hat{H}$. Let the resulting network be called the *associated network* and let it be denoted by N. We claim that N does not contain any generated subdigraph isomorphic to D of FIgure 5. To see why, suppose it did. Then some a_i or b_i of D must be s or t. Clearly s cannote be any b_i or a_3 or a_4. If $s = a_1$, then b_1 must be a maximal element of H, and a_2 is not, which is a contradiction since (a_2, b_1) is an arc of N. Similarly $s \neq a_2$. Similar reasoning will show that t cannot be any a_i or b_i. Thus N does not contain a subdigraph isomorphic to D. Then we have the following theorem.

Theorem 3. *If G is a W_4-free transitively orientable graph, then $i(G) = f^*$, the minimum flow in the associated network.*

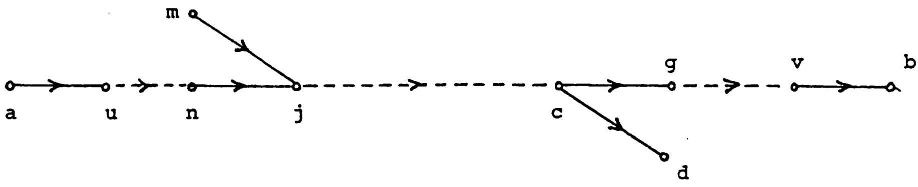
Proof: Take an edge clique covering of G of cardinality $i(G)$. Assume without loss of generality that the cliques in this edge covering are maximal. Since G is transitively orientable, every maximal clique of G corresponds to an $x-y$ directed path in $\hat{H}$ for some x in X and y in Y. Thus corresponding to the given edge clique covering, there are $i(G)$ paths in $\hat{H}$, $P_1, P_2, \cdots, P_{i(G)}$. Put a flow of one on each arc of these paths, and such flows on (s, x) and (y, t) which maintain conservation. Then this is a feasible flow in N of value $i(G)$, i.e., $i(G) \geq f^*$. To see why, note that every arc of $\hat{H}$ is in some P_m, $m = 1, 2, \cdots, i(G)$, since no arc of $\hat{H}$ is implied by transitivity by any other arc in $\hat{H}$.

Next, take a flow in N of value f^*. (Note that f^* is an integer since all capacities are integers.) Then f^* can be decomposed into f^* unit flows along $s-t$ paths. Thus correspondingly, we have f^* maximal cliques $K_1, K_2, \cdots, K_{f^*}$ which cover all arcs of $\hat{H}$. Suppose there is an edge $e = \{a, b\}$ of G which is not in any K_m, $m = 1, 2, \cdots, f^*$. Then $\{a, b\}$ must be an edge of G which is implied by transitivity by the arcs of $\hat{H}$. Thus without loss of generality there is a directed path P from a to b in $\hat{H}$ of length ≥ 2.

Let u be the first vertex following a on P and v be the last vertex preceding b on P. Then (a, u) and (v, b) does not belong to any common clique from $\mathcal{K} = \{K_1, K_2, \cdots, K_{f^*}\}$ otherwise $\{a, b\}$ belongs to a clique in $\mathcal{K}$. Thus if (a, u) is then K_i, then (v, b) is not in K_i, where $K_i \in \mathcal{K}$.

Among all maximal cliques in $\mathcal{K}$ which contain (a, u), let K_i be such that the last vertex $c \in K_i \cap P$ is furthest down on P. Then since $(v, b) \notin K_i$, c strictly precedes b on P. Thus c is not a minimal element of H. Let g and d be respectively the first vertices following c on P and K_i. Such a g and d always exist since c is not a minimal element. Also obviously $g \neq d$, and $\{g, d\} \notin E(H)$ since H does not contain a triangle. Since $\mathcal{K}$ is an edge clique covering, (c, g) belongs to some K_ℓ in $\mathcal{K}$. By our choice of K_i, $(a, u) \notin K_\ell$. Let j be the first vertex on $P \cap K_\ell$. The j strictly follows a on P. Thus j is not a maximal element of H. Let m and n be the first vertices preceding j on K_ℓ and P respectively. Since j is not a maximal element, such m and n always exist. Also, $m \neq n$, and $\{m, n\} \notin E(H)$. Then $\{m, n, j, c, g, d\} \cup$ all vertices on P between j and c generate a subdigraph isomorphic to D shown in Figure 6. Thus we have a contradiction. So we have f^* maximal cliques covering all edges of G. Thus $i(G) \leq f^*$. ∎

Figure 6



Next we quote a theorem by Dilworth [1950] which concerns the minimum number of paths in an acyclic directed graph which are sufficient to covers the arcs of the digraph.

Theorem 4 (Dilworth, 1950). *Let G be an acyclic directed graph and let A be a subset of its arcs. The minimum number of directed paths required to cover the arcs in A is equal to the maximum number of arcs in A no two of which are contained in a directed path in G.* ∎

Corollary 4.1. *If G is a W_4-free transitively orientable graph then its edge clique graph $K(G)$ is weakly θ-perfect.*

Proof: If G is a W_4-free transitively orientable graph, then $i(G)$ = minimum number of directed paths required to cover the arcs of the associated network = maximum number of arcs in the associated network no two of which are contained in a directed path in G. The first equality follows from Theorem 3 and the second from Theorem 4. So $i(G) = \beta(G)$ where $\beta(G)$ = maximum number of edges of G no two of which are contained in a common clique of G. So $\theta_v(K(G)) = i(G) = \beta(G) = \alpha(K(G))$. So $K(G)$ is weakly θ-perfect. ∎

References

Albertson, M.O., Collins, K.L., *Duality and perfection for edges in cliques*, J. Combinatorial Theory **36B** (1984), 298–309.

Chartrand, G., Kapoor, S.F., McKee, T.A., and Saba, F., *Edge Clique Graphs.* (to appear).

Dilworth, R.P., *A Decomposition Theorem for Partially Ordered Sets*, Annals of Math.% **51** (1950), 161–166.

Golumbic, M.C., "Algorithmic Graph Theory and Perfect Graphs", Academic Press, New York, 1980.

Lawler, Eugene, "Combinatorial Optimization, Networks & Matroid", Holt, Rhinehart & Winston, 1976.

Raychaudhuri, A., *Intersection number and edge clique graphs of chordal and strongly chordal graphs*, Congressus Numberantium **67** (1988), 197–204.

Steif, J.E., "Frame dimension, Generalized Competition Graphs, and Forbidden Sublist Characterizations", Henry Rutgers Thesis, Rutgers University, 1982.