CYCLES OF ALL POSSIBLE LENGTHS IN DIREGULAR BIPARTITE TOURNAMENTS[1]

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Abstract. In 1967 Alspach [1] proved that every arc of a diregular tournament is contained in cycles of all possible lengths. In this paper, we extend this result to bipartite tournaments by showing that every arc of a diregular bipartite tournament is contained in cycles of all possible even lengths, unless it is isomorphic to one of the graphs F_{4k}. Simultaneously, we also prove that an almost diregular bipartite tournament R is Hamiltonian iff $|V_1| = |V_2|$ and R is not isomorphic to one of the graphs F_{4k-2}, where (V_1, V_2) is a bipartition of R. Moreover, as a consequence of our first result it follows that every diregular bipartite tournament of order p contains at least $p/4$ distinct Hamiltonian cycles. The graphs $F_r = (V, A), (r \geq 2)$ is a family of bipartite tournaments with $V = \{v_1, v_2, \dots, v_r\}$ and $A = \{v_i v_j \mid j - i \equiv 1 \pmod 4\}$.

An oriented graph is a digraph without loops, multiple arcs, or cycles of length two. We shall also refer to oriented complete graphs, and oriented complete bipartite graphs, as tournaments and bipartite tournaments, respectively.

Let $R = (V, A)$ be an oriented graph with vertex set V and arc set A. For $v \in V$ and $S \subseteq V$, we define

$$N_s^-(v) = \{u \in S \mid uv \in A\},$$
$$N_s^+(v) = \{u \in S \mid vu \in A\}.$$

Also, $R - S$ is the subdigraph of R induced by $V(R) - S$. If R_1 is a subdigraph of R, we shall write $N_{R_1}^-(v)$, $N_{R_1}^+(v)$ and $R - R_1$ instead of $N_{v(R_1)}^-(v)$, $N_{v(R_1)}^+(v)$ and $R - V(R_1)$, respectively. Also, $d_R^-(v) = |N_R^-(v)|$ and $d_R^+(v) = |N_R^+(v)|$ denote the indegree and outdegree of v in R, respectively. For $u, v \in V, u \neq v$, we shall call u and v equivalent in R if $N_R^-(u) = N_R^-(v)$ and $N_R^+(u) = N_R^+(v)$, which is denoted by $u \overset{R}{\sim} v$. If S and T are disjoint subsets of V and $uv \in A$ for all $u \in S$ and $v \in T$ we write $S \Rightarrow T$ or $T \Leftarrow S$.

Furthermore, we shall refer to a directed path, and a directed cycle as a path, and a cycle, respectively. R is said to be k-diregular if $d_R^-(v) = d_R^+(v) = k$ for all $v \in V$; almost k-diregular if R is not k-diregular, $|d_R^-(v) - d_R^+(v)| \leq 1$ and $k - 1 \leq \max\{d_R^-(v), d_R^+(v)\} \leq k$ for all $v \in V$; and it is strong if there exists

[1] J.A. Bondy has informed us, after a recent visit to China, that a number of the results proved in this paper are also proved independently in a paper by Amar and Manoussakis, which will appear in the Journal of Combinatorial Theory (B). –The Editor

a $u - v$ path and a $v - u$ path in R for any $u, v \in V$. A component of R is a maximal strong subdigraph of R.

Let $C = x_1 y_1 x_2 y_2 \dots x_n y_n x_1$ be a cycle in a bipartite tournament. $N_c^-(v)$ or $N_c^+(v)$ is said to contain consecutive vertices if there exists an i such that x_i, x_{i+1} or $y_i, y_{i+1} \in N_c^-(v)$ or $N_c^+(v)$. Moreover, for integer $r \geq 2$, $F_r = (V, A)$ will denote a bipartite tournament with $V = \{v_1, v_2, \dots, v_r\}$ and $A = \{v_i v_j \mid j - i \equiv 1 \pmod 4\}$.

The oriented graph R is called pancyclic if it contain cycles of all possible lengths, and vertex (arc) pancyclic if every vertex (arc) of R is contained in cycles of all possible lengths. For an oriented bipartite graph, pancyclicity means even-pancyclicity. Moreover, it is easy to see that arc-pancyclicity implies vertex pancyclicity which implies pancyclicity. Other terms and symbols not explicitly defined in this paper may be found in [2].

In 1967, Alspach [1] proved that every diregular tournament is arc-pancyclic. We shall show in this paper an analog statement for bipartite tournaments. To this end we need the following results.

Theorem 1. *[3, Jackson]. Every strong bipartite tournament of minimum indegree h and minimum outdegree k contains a cycle of length at least $2(h + k)$.*

Theorem 2. *An almost diregular bipartite tournament R is Hamiltonian if and only if $|X| = |Y|$ for a bipartition (X, Y) of R and R is not isomorphic to one of the graphs F_{4r-2}, $r \geq 1$.*

Proof: The necessity is obvious. Let $R = (V, A)$ be an almost k-diregular bipartite tournament which satisfies the hypothesis of the Theorem 2, and let $V = X \cup Y$ be the bipartition of R. Then $|X| = |Y| = \frac{|V|}{2} = 2k - 1$. Moreover, we have

$$k - 1 \leq d_R^-(v),\ d_R^+(v) \leq k \quad \text{for all } v \in V.$$

We first prove that R is strong. Otherwise, R has at least two components, say $R_1, R_2, \dots, R_n (n \geq 2)$ with the bipartition $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ respectively. We may assume without loss of generality that $X_i \Rightarrow Y_j$ and $Y_i \Rightarrow X_j$ whenever $i < j$. If $V(R_1)$ or $V(R_n)$, $|V(R_1)| \leq 2k - 1$, say. Then it easily follows that

$$(k-1)|V(R_1)| \leq \sum_{v \in R_1} d_R^-(v) \leq \frac{|V(R_1)|^2}{4},$$

which implies

$$|V(R_1)| \geq 4(k-1).$$

If $k > 1$, then $|V(R_1)| > 2k - 1$. If $k = 1$, then $R \cong F_2$. This contradicts the initial assumption that $|V(R_1)| \leq 2k - 1$ and $R \ncong F_{4r-2}$, respectively. Thus, R is strong.

Now, by Theorem 1, it follows that R contains a cycle of length $4k-4$ or $4k-2$. Suppose that R is not Hamiltonian, that is, R contains no cycles of length $4k-2$. Let C be a cycle of length $4k-4$ in R, which is longest, say

$$C = x_1 y_1 x_2 y_2 \dots x_{2k-2} y_{2k-2} x_1.$$

Again, suppose u, v are two vertices in $R - C$ and, without loss of generality, $uv \in A$ and $v \in X$. Hence $u \in Y$. Put

$$S = \{y_i \in C \mid x_i \in N_c^-(u)\},$$
$$T = \{x_i \in C \mid y_i \in N_c^+(v)\}.$$

Then, since C is a longest cycle of R. We may conclude that

(i) $N_c^+(v) \cap S = \phi$ and

(ii) $N_c^-(u) \cap T = \phi$.

∎

Case 1. $N_c^-(u)$ *contains consecutive vertices.*

Without loss of generality, we may assume $x_1, x_2 \in N_c^-(u)$. Then $y_1, y_2 \in N_c^-(v)$ by (i). We first show that $N_R^-(y_1)$ and T are disjoint. Otherwise, $x_i \in N_R^-(y_1) \cap T$. Put

$$i_0 = \min\{i \mid y_i \in N_c^+(v)\}.$$

Then $vy_{i_0}, ux_{i_0}, y_{i_0-1}\, v \in A$, and hence

$$C = x_1 u x_{i_0} y_{i_0} \dots x_i y_1 x_2 \dots y_{i_0-1} v y_i \dots y_{4k-4} x_1.$$

is a Hamiltonian cycle in R, which contradicts the assumption. So that

$$N_R^-(y_1) \cap T = \phi.$$

Moreover, since $x_2, v \in N_R^+(y_1) \setminus T$, we get

$$d_R^+(y_1) \geq |T| + 2,$$

or, equivalently,

$$|T| \leq d_R^+(y_1) - 2.$$

However, $d_R^+(y_1) \leq k$, by the almost k-diregularity of R. Hence

$$|T| \leq k - 2.$$

But it is easy to see that $|T| = d_R^+(v)$, which contradicts the almost k-diregularity of R.

Case 2. *$N_c^-(u)$ contains no consecutive vertices.*

We may assume, without loss of generality, that $x_1 \in N_c^-(u)$. Notice that $d_R^-(u) = |N_R^-(u)| = |N_c^-(u)| \geq k-1$ and $|V(C)| = 4k-4$, we conclude in this case that

$$\text{(iii)}\ N_c^-(u) = \{x_1, x_3, \dots, x_{2k-3}\} \text{ and hence}$$
$$\text{(iv)}\ N_c^+(u) = \{x_2, x_4, \dots, x_{2k-2}\}.$$

By (i) and (iii), we get further

$$N_c^-(v) = \{y_1, y_3, \dots, y_{2k-3}\}.$$

Again, by $uv \in A$ and the almost k-diregularity of R, it follows that

$$N_c^+(v) = \{y_2, y_4, \dots, y_{2k-2}\}.$$

Consider the cycles $C_i^1 = x_{2i-1}ux_{2i}y_{2i}\dots x_{2i-1}$ obtained from C by replacing the edges $x_{2i-1}y_{2i-1}$, $y_{2i-1}x_{2i}$ with the edges $x_{2i-1}u$, ux_{2i}, for $i = 1, 2, \dots, k-1$, respectively. Since $y_{2i-1}\ v \in A$, so $y_{2i-1} \underset{\sim}{R} u$, $i = 1, 2, \dots, k-1$. Similarly for $C_i^2 = y_{2i-1}vy_{2i}\dots y_{2i-1}$, it follows that $x_{2i} \underset{\sim}{R} v$. We have got

$$\{y_1, y_3, \dots, y_{2k-3}, u\} \Rightarrow \{x_2, x_4, \dots, x_{2k-2}, v\}, \text{ so far.}$$

Again, by the almost k-diregularity of R, it follows that

$$\{x_1, x_3, \dots, x_{2k-3}\} \Rightarrow \{y_1, y_3, \dots, y_{2k-3}, u\} \text{ and}$$
$$\{x_2, x_4, \dots, x_{2k-2}, v\} \Rightarrow \{y_2, y_4, \dots, y_{2k-2}\}$$

and hence

$$\{y_2, y_4, \dots, y_{2k-2}\} \Rightarrow \{x_1, x_3, \dots, x_{2k-3}\}.$$

It follows that $R \cong F_{4k-2}$. This contradiction completes the proof. ∎

We now state and prove our main result.

Theorem 3. *Every diregular bipartite tournament is arc-even-pancyclic, unless it is isomorphic to one of the graphs F_{4r}, $r > 1$.*

Proof: Let $R = (V, A)$ be a k-diregular bipartite tournament and let uv be an arc of R. Again let $R' = R - \{u, v\}$. Then R' is clearly an almost k-diregular bipartite tournament. Suppose first that $R \not\cong F_{4k}$, which implies obviously $R' \not\cong F_{4k-2}$. Hence, by Theorem 2, R' contains a Hamiltonian cycle

$$C = v_1v_2\dots v_{4k-2}v_1, \text{ say.}$$

For integer m, $2 \leq m \leq 2k$, put

$$B_m = \{v_{4k-2m+i+1} \in C \mid v_i \in N_c^-(u)\},$$

where addition of the subscript of v is taken modulo $4k - 2$. Then we have $N_c^+(v) \cap B_m \neq \phi$, since otherwise, $N_R^-(v) \supseteq B_m \cup \{u\}$ leads to $d_R^-(v) = |N_R^-(v)| \geq k+1$ by $|B_m| = |N_c^-(u)| = |N_R^-(u)| = d_R^-(u) = k$, which contradicts the k-diregularity of R. Thus we may assume that $v_j \in N_c^+(v) \cap B_m$, that is, $v_i u$, $vv_j \in A$ and $j - i = 4k - 2m + 1$ (if $j \geq i$) or $j - i = -2m + 1$ (if $j \leq i$). Hence

$$C = v_i u v v_j v_{j+1} \dots v_i$$

is a cycle of length $2m$ ($2 \leq m \leq 2k$) containing uv.

On the other hand, it is easy to see that the graph F_{4k} is a k-diregular bipartite tournament which contains no cycles of lengths $4r + 2$ for $r = 1, 2, \dots, k-1$. Hence F_{4k} does not have the property described in the theorem.

The proof is complete. ∎

As the immediate consequences of Theorem 3 we have

Corollary 4. *[4, 5]. Every diregular bipartite tournament is vertex-even-pancyclic, hence even-pancyclic, unless it is isomorphic to one of the graphs F_{4r} ($r > 1$).*

Corollary 5. *Every diregular bipartite tournament of order p contains at least $\left[\frac{p^2}{8k}\right]$ distinct cycles of length $2k$ for all $2 \leq k \leq \frac{p}{2}$, unless it is isomorphic to the graph F_p.*

Proof: Let $R = (V, A)$ be a diregular bipartite tournament of order p. If $R \cong F_p$, then the conclusion is obviously untenable for $k \equiv 1 \pmod 2$. So, assume $R \not\cong F_p$. By Theorem 3, it follows that $A(R)$ is covered by cycles of length $2k$ in R. Since $|A(R)| = \frac{p^2}{4}$, R contains at least $\left[\frac{p^2}{8k}\right]$ distinct cycles of length $2k$, ($2 \leq k \leq \frac{p}{2}$). ∎

Corollary 6. *Every diregular bipartite tournament of order p contains at least $\frac{p}{4}$ distinct Hamiltonian cycles.*

Proof: The statement is easily verified for $R \cong F_p$. If $R \not\cong F_p$, the corollary follows immediately from Corollary 5. ∎

Corollary 7. *Every diregular bipartite tournament of order p contains at least $\frac{p^2}{4}$ distinct cycles of length $p - 2$, unless it is isomorphic to the graph F_p.*

Proof: Suppose that the graph R' obtained from a diregular bipartite tournament R of order p by deleting an arc together with two ends. Then R' is an almost diregular tournament of order $p - 2$. Since $R \not\cong F_p$ implies $R' \not\cong F_{p-2}$, by Theorem 2, R'

contains an Hamiltonian cycle which is also a cycle of length $p-2$ in R, Since R contains $\frac{p^2}{4}$ arcs, the corollary is obvious. ∎

In concluding, we would mention that Jackson [3] has conjectured that every diregular bipartite tournament is decomposable into Hamiltonian cycles. Clearly, this conjecture implies that every arc of a diregular bipartite tournament is contained in a Hamiltonian cycle. However, Theorem 3 indicates that the latter is true. This result supports to a certain extent the above conjecture. Moreover, we would also pose an interesing question: whether every result of diregular tournament may be extended to the multiple tournaments which is the oriented complete n-partite graphs, which give the corresponding version. In particular, we make the following conjectures.

Conjecture 1. *Every arc of a diregular 3-partite tournament R is contained in cycles of all lengths* $3, 6, 9, \ldots, |V(R)|$.

Conjecture 2. *If R is a k-diregular bipartite tournament and E is a set of at most $k-1$ arcs of R, then $R-E$ is Hamiltonian.*

Finally, we point out that Conjecture 1 is best possible in some sense in view of the oriented graphs with vertex-set $\{v_1, v_2, \ldots, v_{3k}\}$ $(k \geq 1)$ and arc set $\{v_i v_j \mid j-i \equiv 1 \pmod 3\}$. Whereas Jackson's conjecture mentioned above implies Conjecture 2.

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