The Interior of an m-Convex Set

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For $m \geq 2$, a set $A \subset \mathbb{R}^2$ is said to be *m-convex* if, for any $x_1, \ldots, x_m \in A$, we have $[x_i, x_j] \subset A$ for some $i \neq j$, where as usual $[x, y]$ denotes the line segment $\{tx + (1-t)y: 0 \leq t \leq 1\}$. Thus a set is 2-convex iff it is convex, and for $n \geq m$ an m-convex set is n-convex. The notion of m-convexity was introduced by Valentine [5] in the case $m = 3$, and by Kay and Guay [2] for general m.

The structure of closed m-convex sets in $\mathbb{R}^2$ is well-understood. Indeed, Breen and Kay [1] showed that a closed m-convex set is a finite union of convex sets. However, not much is known about non-closed m-convex sets. (See Tattersall [3] for a general background).

Since the interior of a 2-convex set is 2-convex, it is natural to ask what happens for m-convex sets in general. It is easy to prove directly that the interior of a 3-convex set is 3-convex: one just mirrors the proof of the fact that the interior of a convex set is convex. However, there are 4-convex sets in $\mathbb{R}^2$ whose interiors are not 4-convex. For example, if $A = \mathbb{R}^2 - \{(0, y): y > 0\} - \{(1,0), (-1,0)\}$ then A is 4-convex, being the union of 3 convex sets, but int A is only 5-convex. Similar examples show that, for any $m \geq 4$, there are m-convex sets in $\mathbb{R}^2$ whose interiors are $(m+1)$-convex but not m-convex.

Tattersall [3, 4] raised the question as to whether there is a function $f: \mathbb{N} \to \mathbb{N}$ such that the interior of an m-convex set in $\mathbb{R}^2$ is always $f(m)$-convex. The aim of this note is to show that this is not the case. In fact, we show that there is a 6-convex set $A \subset \mathbb{R}^2$ such that int A is not m-convex for any $m \geq 2$, and is indeed not even ∞-convex, in the sense of [2], meaning that there is an infinite set $\{x_1, x_2, \ldots\} \subset$ int A such that no segment $[x_i, x_j]$, $i \neq j$ lies inside int A.

The proof of this result is probabilistic; to be precise, we use simple arguments about sets of measure zero to prove the existence of a desired set without explicitly constructing it.

The lemmas we need concern blocking sets. For disjoint $S, T \subset \mathbb{R}^2$, we say that *$S$ blocks T* if $[t_1, t_2] \cap S \neq \emptyset$ for all distinct $t_1, t_2 \in T$. We call S *m-blocking* if it blocks some m-set in $\mathbb{R}^2 - S$. Thus a set A is m-convex iff $\mathbb{R}^2 - A$ is not m-blocking.

We are particularly interested in minimal m-blocking sets.

Lemma 1. *Let $n > 2m$. Then*

$$\mu\{(s_1, \ldots, s_n) \in (R^2)^n : \{s_1, \ldots, s_n\} \text{ a minimal } m\text{-blocking set}\} = 0,$$

where μ denotes Lebesgue measure.

Proof. If S is a minimal m-blocking set then there is an m-set T such that S blocks T but no proper subset of S blocks T. In particular, for each $s \in S$ there are $x, y \in T$ and $\lambda \in (0,1)$ such that $s = \lambda x + (1-\lambda)y$.

Define a map $G: (R^2)^m \times (0,1)^n \to (R^2)^{[m]^{(2)} \times [n]}$ by setting

$$G(x, \lambda)_{ij,k} = \lambda_k x_i + (1-\lambda_k)x_j, \quad x \in (R^2)^m,\ \lambda \in (0,1)^n,$$

for each $i, j \in [m] = \{1, \ldots, m\}$, $i < j$ and $k \in [n]$. Here as usual $[m]^{(2)}$ denotes the set of 2-element subsets of $[m]$. For a function $f: [n] \to [m]^{(2)}$, define $P_f: (R^2)^{[m]^{(2)} \times [n]} \to (R^2)^n$ by

$$P_f(x) = (x_{f(1),1}, \ldots, x_{f(n),n}), \quad x \in (R^2)^{[m]^{(2)} \times [n]}.$$

Thus if $\{s_1, \ldots, s_n\}$ is a minimal m-blocking set then certainly $(s_1, \ldots, s_n)$ is in the image of $P_f \circ G$ for some f. However, each $P_f \circ G$ is a smooth map from an open subset of R^{2m+n} to R^{2n}, and $2m + n < 2n$, so the image of each $P_f \circ G$ has measure 0. As there are only finitely many f, we are done. ∎

In general, an m-blocking set can be quite small. However, this is not the case if the set is dispersed.

Lemma 2. *Let $\{s_1, \ldots, s_n\} \subset R^2$ be an m-blocking set such that no s_i is in the convex hull of $\{s_j : j \neq i\}$. Then $n \geq \binom{m}{2} - \frac{m-1}{2}$.*

Proof. Choose an m-set $\{x_1, \ldots, x_m\} \subset R^2$ which is blocked by $\{s_1, \ldots, s_n\}$. Thus for each $i \neq j$ there is a point $s_{ij} \in \{s_1, \ldots, s_n\} \cap [x_i, x_j]$. We note first that we cannot have $s_{ij} \in [x_k, x_l]$ if $\{k, l\} \cap \{i, j\} = \emptyset$, as then $s_{ij} \in \langle s_{ik}, s_{il}, s_{jk}, s_{jl} \rangle$, where $\langle \ldots \rangle$ denotes convex hull.

Now, since $\{s_1, \ldots, s_n\}$ does not contain 3 collinear points, $\{x_1, \ldots, x_m\}$ cannot contain 4 collinear points. Moreover, we claim that all collinear triples from $\{x_1, \ldots, x_m\}$ have the same middle point. Indeed, suppose that we have

$$x_i \in [x_j, x_k], \quad i \neq j, k$$

and

$$x_p \in [x_q, x_r], \quad p \neq q, r$$

with $i \neq p$. If $i \notin \{q, r\}$ and $p \notin \{j, k\}$ then

$$s_{ip} \in [x_i, x_p] \subset \langle s_{ij}, s_{ik}, s_{pq}, s_{pr} \rangle,$$

while if $i \in \{q, r\}$ or $p \in \{j, k\}$, say without loss of generality $i = q$, then

$$s_{ip} \in [x_i, x_p] \subset \langle s_{ij}, s_{ik}, s_{pr} \rangle.$$

Thus there are at most $\frac{m-1}{2}$ collinear triples in $\{x_1, \ldots, x_m\}$, and hence $n \geq \binom{m}{2} - \frac{m-1}{2}$. ∎

We now show the existence of some infinite sets which are not m-blocking.

Lemma 3. *Let $D_1, D_2, \ldots$ be open discs in $\mathbf{R}^2$ such that no D_i meets the convex hull of $\cup_{j \neq i} D_j$. Choose random points $Z_i \in D_i$ independently, with each Z_i having uniform distribution on D_i. Then $P(\{Z_1, Z_2, \ldots\}$ is m-blocking$) = 0$ for $m \geq 6$.*

Proof. If $\{Z_1, Z_2, \ldots\}$ is m-blocking then it contains a minimal m-blocking set: say $\{Z_{i_1}, \ldots, Z_{i_n}\}$, $n \leq \binom{m}{2}$. Since no Z_i is in the convex hull of $\{Z_j : j \neq i\}$, Lemma 2 gives $n \geq \binom{m}{2} - \frac{m-1}{2}$.

Suppose that $m \geq 6$, so that $\binom{m}{2} - \frac{m-1}{2} > 2m$. Then it follows from Lemma 1 that if $\{j_1, \ldots, j_r\}$ is any subset of $\mathbf{N}$ with $\binom{m}{2} - \frac{m-1}{2} \leq r \leq \binom{m}{2}$ then $P(\{Z_{j_1}, \ldots, Z_{j_r}\}$ a minimal m-blocking set$) = 0$. As there are only countably many such subsets of $\mathbf{N}$ we obtain $P(\{Z_1, Z_2, \ldots\}$ is m-blocking$) = 0$, as required. ∎

We are now ready to prove the main result of this note.

Theorem 4. *There is a 6-convex set $A \subset \mathbf{R}^2$ such that int A is not ∞-convex.*

Proof. It is easy to construct a sequence $(x_n)_{n \geq 1}$ in $\mathbf{C}$ such that $x_n \to 1$, $|x_n| > 1$ for all n, $0 < \arg x_n < \pi$ for all n, and $[x_j, x_k] \cap \mathbf{T} \neq \emptyset$ for all $j \neq k$, where $\mathbf{T} = \{z \in \mathbf{C} : |z| = 1\}$. Let $S = \mathbf{T} \cap \cup_{j \neq k} [x_j, x_k]$. Note that the set of accumulation points of S is $\mathbf{T} \cap \cup_j [x_j, 1]$, so that S itself consists of isolated points.

For $s \in S$, say $s = e^{it}$, choose an increasing sequence $t_1 < t_2 < \ldots$ such that $t_n \to t$ and $\{e^{i\theta} : t_1 < \theta < t\} \cap S = \emptyset$. For each n, choose an open disc $D_n^{(s)}$ in the segment enclosed by $[e^{it_n}, e^{it_{n+1}}]$ and $\{e^{i\theta} : t_n \leq \theta \leq t_{n+1}\}$.

Now, each member of the (countable) collection $\{D_n^{(s)} : n \in \mathbf{N},\ s \in S\}$ is separated by a chord of $\mathbf{T}$ from all the others, and therefore does not meet their convex hull. Thus by Lemma 3 there exist points $z_n^{(s)} \in D_n^{(s)}$, $n \in \mathbf{N}$, $s \in S$ such that the set $Z = \{z_n^{(s)} : n \in \mathbf{N},\ s \in S\}$ is not 6-blocking. However, $\overline{Z} = Z \cup \overline{S}$, so that $\overline{Z}$ blocks $\{x_n : n \in \mathbf{N}\}$. It follows that $A = \mathbf{R}^2 - Z$ is 6-convex while int A is not ∞-convex. ∎

As we remarked above, it is easy to show that the interior of a 3-convex set is 3-convex. This leaves the cases $m = 4$ and $m = 5$. One can show that if $S \subset \boldsymbol{R}^2$ is not 4-blocking and $\overline{S}$ blocks $\{x_1, \ldots, x_4\}$ then $x_1, \ldots, x_4$ are collinear. Based loosely on this, we conjecture that the interior of a 4-convex set in $\boldsymbol{R}^2$ is always 5-convex. In the case $m = 5$, we do not see any evidence to support a conjecture.

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