Generalizations of Hadwiger's Conjecture

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Abstract. Conjectured generalizations of Hadwiger's Conjecture are discussed. Among other results, it is proved that if X is a set of 1, 2 or 3 vertices in a graph G that does not have K_6 as a subcontraction, then G has an induced subgraph that is 2-, 3- or 4-colourable, respectively, and contains X and at least a quarter, a third or a half, respectively, of the remaining vertices of G. These fractions are best possible.

1. Introduction

Let $\mathcal{G}_t$ denote the class of graphs that do not have the complete graph K_{t+1} as a subcontraction (minor). Hadwiger [2, 3] conjectured that every s-chromatic graph has K_s as a subcontraction, or equivalently, setting $t = s-1$, that every graph in $\mathcal{G}_t$ is properly vertex-t-colourable. For a discussion of this conjecture see, for example, [4]; the conjecture is known to be true for $t = 1, 2, 3$ or 4 (the last of these because of the four-colour theorem and Wagner's equivalence theorem), but it is not known whether it is true for any larger value of t.

In [5] I made the following conjecture, in which '+' denotes 'join'.

Conjecture A. *Let H be a connected graph with at least one edge, and let G be a graph that does not have $K_{t-1}+H$ as a subcontraction. Then G can be (improperly) vertex-t-coloured in such a way that no subgraph of G isomorphic to H has all its vertices the same colour.*

In [5] I proved this conjecture for $t = 2$ and $t = 3$, and it is trivially true for $t = 1$ if one allows the existence of the empty graph K_0 with the property that $K_0+H = H$ for every graph H. If $H = K_2$ then the conjecture is equivalent to Hadwiger's conjecture. So when $t = 4$ the conjecture is true for $H = K_2$, but it is not known whether it is true for any other graph H; and nothing is known about its truth for larger values of t.

Conjecture A contains Hadwiger's conjecture as a special case, but its other cases are apparently harder than Hadwiger's conjecture. In contrast, in the present paper I shall consider the following conjecture, which also contains Hadwiger's conjecture as a special case but is otherwise apparently simpler than Hadwiger's conjecture. From now on, all

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colourings are assumed to be proper vertex-colourings.

Conjecture B(r,s,t) ($0 \leqslant r \leqslant s \leqslant t$, $t > r$). *Let G be a graph in $\mathcal{G}_t$ and let X be a set of r or fewer vertices of G. Then there exists an s-colourable induced subgraph of G that contains X and at least*

$$\frac{s-r}{t-r}\,|V(G-X)| \tag{1}$$

vertices of $G-X$.

The number of vertices in (1) is the largest possible if, for example, the induced subgraph $\langle X\rangle \cong K_r$ and G is of the form $\langle X\rangle + kK_{t-r}$.

For fixed values of r and t, the conjecture is trivial if $s = r$ and becomes steadily harder as s increases, until the case $s = t$ which is equivalent to Hadwiger's conjecture itself. Note that all the conjectures would follow from Hadwiger's conjecture that G is t-colourable, and so they are all true if $t \leqslant 4$.

The purpose of this note is to prove Conjecture B(r,s,t) when (simultaneously) $0 \leqslant r \leqslant 3$ and $s = r+1 \leqslant t \leqslant r+4$. This includes (for $t > 4$) the cases B(1,2,5), B(2,3,5), B(3,4,5), B(2,3,6), B(3,4,6) and B(3,4,7), the first three of which are the results stated in the Abstract. The results B(1,2,5), B(2,3,6) and B(3,4,7) depend on the four-colour theorem, as do all the results B(r,s,4) with $r = 0$ or $s = 4$, and the result B(1,3,4). The very simple method used in the present paper only works when $t \geqslant 2s-r$, and so it would be particularly interesting to have a direct proof, not assuming the four-colour theorem, of B(1,3,4). This states that if v is a vertex in a graph G that does not have K_5 as a subcontraction, then G has a 3-colourable induced subgraph that contains v and at least two thirds of the vertices of $G-v$. Note that this result implies Albertson's theorem [1] that every planar graph contains an independent set comprising at least $\frac{2}{9}$ of its vertices. (Of course, the four-colour theorem implies that $\frac{2}{9}$ can be replaced here by $\frac{1}{4}$.)

2. Proofs

Lemma 1. *For fixed values of r, s and t such that $t \geqslant 2s-r$, let (G,X) be a counterexample to Conjecture B(r,s,t) such that G has as few vertices as possible. Let v be a vertex of $G-X$ and let u, w be neighbours of v.*
(a) *If u and w are both in X, then they are adjacent.*
(b) *If u is in X and w is in $G-X$, then u,w are adjacent.*

Proof. (a) Note that the condition $t \geqslant 2s-r$ is equivalent to

$$\frac{s-r}{t-r} \leqslant \frac{1}{2}. \tag{2}$$

Suppose that u and w are non-adjacent. Form (G',X') from (G,X) by contracting the edges uv and vw to form a new vertex x which belongs to X', and transferring an arbitrary vertex y from $G-X$ to X', so that $|X'| = |X|$ and $|V(G'-X')| = |V(G-X)|-2$. By hypothesis, (G',X') satisfies Conjecture B(r,s,t), and so G' has an s-colourable induced subgraph that contains X' and at least

$$\frac{s-r}{t-r}\,|V(G'-X')|$$

vertices of $G'-X'$. Remove x from the colour-class containing it and replace it by u and w to form an s-colourable induced subgraph of G that contains X and at least

$$\frac{s-r}{t-r}\,|V(G'-X')|+1 \geqslant \frac{s-r}{t-r}(|V(G'-X')|+2) = \frac{s-r}{t-r}\,|V(G-X)|$$

vertices of $G-X$, by (2). This contradicts the hypothesis that (G,X) is a counterexample to Conjecture B(r,s,t).

(b) This is identical except that contracting the edges uv and vw gives a set X' of the right size immediately, and so we do not need to transfer an additional vertex y to X'. □

Lemma 2. *If $t \geqslant 2s-r$, then Conjecture B(r,s,t) is true if and only if, for each q $(0 \leqslant q \leqslant r)$, every graph H in $\mathscr{G}_{t-q}$ has an $(s-q)$-colourable induced subgraph containing at least $|V(H)|(s-r)/(t-r)$ vertices of H.*

Proof. If some graph H in $\mathscr{G}_{t-q}$ does not have such a subgraph, then we obtain a counterexample G to Conjecture B(r,s,t) by taking $\langle X\rangle = K_r$ and forming G from $\langle X\rangle \cup H$ by adding edges joining all vertices of H to the same q vertices of X.

Suppose conversely that every graph in $\mathscr{G}_{t-q}$ does have such a subgraph, for each q $(0 \leqslant q \leqslant r)$. Let (G,X) be a minimal counterexample to Conjecture B(r,s,t), and let H be a component of $G-X$. It follows from Lemma 1 that each vertex x of X is adjacent to all or none of the vertices of H, and that two vertices of X that are adjacent to H are adjacent to each other. Thus, if q is the number of vertices of X that are adjacent to H, then $H \in \mathscr{G}_{t-q}$. By supposition, H has an $(s-q)$-colourable induced subgraph containing at least $|V(H)|(s-r)/(t-r)$ of its vertices. Colour the vertices of X with $|X| \leqslant r$ distinct colours, and, for each component H of $G-X$, colour the appropriate induced subgraph of H with $s-q$ colours that are not used on the q vertices of X that are adjacent to H. The coloured vertices induce an s-colourable subgraph of G that contradicts the supposition that (G,X) is a counterexample to Conjecture B(r,s,t). □

Lemma 3. *If $t \geqslant 2s-r$, then Conjecture $B(r,s,t)$ holds if, for each q $(0 \leqslant q \leqslant r)$, every graph H in $\mathcal{G}_{t-q}$ is $\lfloor f(q,r,s,t) \rfloor$-colourable, or has a $\lfloor \frac{1}{2} f(q,r,s,t) \rfloor$-colourable induced subgraph that contains at least half its vertices, where*

$$f(q,r,s,t) = \frac{(s-q)(t-r)}{s-r}.$$

Proof. Each of these conditions implies that H has an $(s-q)$-colourable induced subgraph that contains at least $|V(H)|(s-r)/(t-r)$ vertices of H. The result follows from Lemma 2. □

Theorem. *Conjecture $B(r,s,t)$ holds if $0 \leqslant r \leqslant 3$ and $s = r+1 \leqslant t \leqslant r+4$.*

Proof. Since $q \leqslant r = s-1 \leqslant t-1$ we have

$$f(q,r,s,t) = (r+1-q)(t-r) = (r-q)(t-r-1)+(t-q) \geqslant t-q,$$

and so B(r,s,t) holds by Lemma 3 if $t-q \leqslant 4$, since then every graph in $\mathcal{G}_{t-q}$ is $(t-q)$-colourable (by the four-colour theorem and Wagner's equivalence theorem). Now, $t \leqslant r+4$ implies that $t-q \leqslant 4$ unless $q \leqslant t-5 \leqslant r-1$, in which case we can write $q = r-1-p$ with $p = 0$, 1 or 2. But then

$$\tfrac{1}{2} f(q,r,s,t) = \frac{(2+p)(t-q-1-p)}{2} = t-q-1+\frac{p(t-q-3-p)}{2} \geqslant t-q-1$$

since $(t-q)-3-p \geqslant 5-3-2 = 0$. But I proved in [4] that, for each integer $m \geqslant 3$, every graph in $\mathcal{G}_m$ has an $(m-1)$-colourable induced subgraph that contains at least half its vertices, and so the result follows from Lemma 3. □

References

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