ON CONVEX HULLS OF GRAPHS

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Abstract. The convex hull of graph G, a notion born in the theory of random graphs, is the convex hull of the set in xy-plane obtained by representing each subgraph H of G by the point whose coordinates are the number of vertices and edges of H.

In the paper the maximum number of corners of the convex hull of an n-vertex graph, bipartite graph, and $K(r)$-free graph is found. The same question is posed for strictly balanced graphs.

1. INTRODUCTION

The following result from the theory of random graphs gave rise to the notion of the convex hull of a graph. Let $K(n,p)$ be a random graph obtained from a complete graph $K(n)$ by deleting each edge independently with probability $1-p$. Further let $P(n,p,G)$ be the probability that $K(n,p)$ contains no subgraph isomorphic to G. Throughout the paper $|G|$ and $e(G)$ stand for the number of vertices and edges of G. Setting, $p = p(n)$, $n \to \infty$, we call subgraph H of G leading if $e(H) > 0$ and for all $F \subseteq G$, $e(F) > 0$,

$$n^{|H|}p^{e(H)} = 0(n^{|F|}p^{e(F)}).$$

The main result in |2| says that if H is a leading subgraph of G then there are constants $c_1, c_2 > 0$ such that

$$-c_1 n^{|H|}p^{e(H)} < \log P(n,p,G) < -c_2 n^{|H|}p^{e(H)}.$$

A complete characterization of the subgraphs of G which become leading for some range of $p(n)$ can be derived by simple geometric means.Let $\Omega_G = \{(|H|, e(H)) : H \subseteq G, |H| > 1\}$ and let C_G be the convex hull of Ω_G in the Cartesian xy-plane. We are only interested in the upper boundary of C_G which is called here "the roof" and denoted by R_G. The shape of the roof is determined by the points $T_s = (s, e_s)$, where $e_s = \max\{e(H) : H \subseteq G, |H| = s, s = 2, \ldots, |G|\}$. Not every T_s lies on the roof and we set $I_G = \{s : T_s \in R_G\}$. It is easily verified that a subgraph H of G is leading for some range of $p(n)$ if and only if it corresponds to a point of R_G, i.e. $e(H) = e_s$ and $s = |H| \in I_G$.

Moreover, the appropriate range of $p(n)$ can be read out from the slopes to the left and to the right of T_s.

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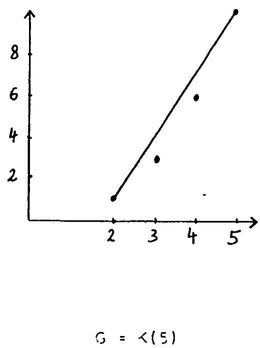
In this paper we investigate properties of R_G. Clearly, R_G consists of straight line segments whose endpoints are $T_s, s \in I_G$. Note, first, that $|I_G| = 2$ for complete graphs and $|I_G| = |G| - 1$ for trees (see Figures 1 and 2 below).



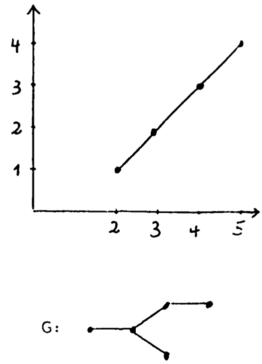


Figure 1 Figure 2

There is no gap between the two extremes as for all $2 \leq t \leq n-1$ one can draw graph G with $|G| = n$ and $|I_G| = t$ (take $K(n+2-t)$ with pendant path of length $t-2$).

It is more interesting to ask about the number of corners R_G. For $s \in I_G$, let $a_s^-(a_s^+)$ be the slope of the segment of R_G whose right (left) endpoint is $T_s (a_2^- = \infty, a_{|G|}^+ = 0$, for convenience). We set

$$J_G = \{s \in I_G : a_s^- > a_s^+\}$$

and search for $\gamma_n(\mathcal{F}) = \max\{|J_G| : |G| = n, G \in \mathcal{F}\}$, where $\mathcal{F}$ is a specified family of graphs.

In Section 2 this problem is solved for graphs, bipartite graphs and, asymptotically, for $K(r)$-free graphs.

Another class of graphs we deal with are strictly balanced graphs. Graph G is strictly balanced if for all $H \subseteq G, d(H) < d(G)$, where $d(H) = e(H)/|H|$. With the exception of disjoint unions of $K(2)$, all graphs satisfying $|J_G| = 2$ are strictly balanced, but the inverse is not true as Figure 3 shows. What is the maximum number of corners a strictly balanced graph may have? Unable to anwer this question, in Section 3 we give crude bounds on $\gamma_n(S)$ where S is the family of strictly balanced graphs.

Graph G contains isolated vertices iff $a_{|G|}^- = 0$. Therefore, everywhere in the paper we restrict ourselves to graphs without isolated vertices. Hence, always $J_G \supseteq \{2, |G|\}$.

The smallest integer not smaller than x is designated by $\lceil x \rceil$.

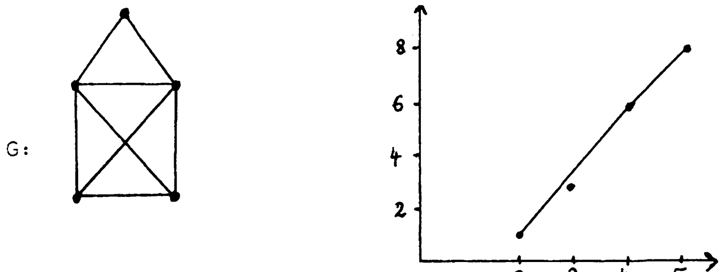


Figure 3

2. MAXIMUM NUMBER OF CORNERS

In this Section we find the exact value of $\gamma_n(\mathcal{F})$ for $\mathcal{F} = \mathcal{G}$-the family of all graphs and for $\mathcal{F} = \mathcal{B}$-the family of all bipartite graph. The latter happens to coincide with $\gamma_n(\mathcal{F}_3)$ where $\mathcal{F}_r$ is the family of $K(r)$-free graphs. Finally, we calculate the limit of $\gamma_n(\mathcal{F}_r)/n$ for $r \geq 3$.

THEOREM 1. *For* $n = 5m - i, m \geq 2, i = 0, \ldots, 4,$

$$\gamma_n(\mathcal{G}) = 2m + 2 - \lceil i/2 \rceil.$$

Consequently, $\gamma_n(\mathcal{G})/n \to 2/5$ *as* $n \to \infty$.

THEOREM 2. *For* $n = 7m - i, m \geq 2, i = 0, \ldots, 6,$

$$\gamma_n(\mathcal{B}) = \gamma_n(\mathcal{F}_3) = 2m + 2 - \lceil i/4 \rceil$$

.

THEOREM 3.

$$\lim_{n \to \infty} \gamma_n(\mathcal{F}_r)/n = \frac{2r-4}{5r-8}, r \geq 3$$

.

We call graph G $K(2)$*-balanced if for all* $H \subseteq G, e(H) > 1, d'(H) \leq d'(G)$ *holds, where*

$$d'(H) = \frac{e(H) - 1}{|H| - 2}.$$

Trees, cycles, complete graphs, and r-partite complete graphs are $K(2)$- balanced and, obviously, G is $K(2)$-balanced iff $|J_G| = 2$.

In the proofs the following contruction will be crucial. ($V(G)$ is the vertex-set of G and $G[V]$ stands for the subgraph of G induced by $V, V \subset V(G)$).

Construction

Let G_o be an arbitrary $K(2)$-balanced graph, $d_o = d'(G_o)$ and $m = \lceil d_o \rceil - 1$. Notice that $|G_o| \geq 2d_o - 1$ and $m \geq d_o$. Let $V = \{v_m, u_{m-1}, w_{m-1}, v_{m-1}, u_{m-2} w_{m-2}, \ldots, v_1, u_o, w_o\}$ be disjoint from $V(G_o)$. We construct graph G so that $V(G) = V(G_o) \cup V, G[V(G_o)] = G_o, u_i$ is joined to w_i and each of v_i, u_i, w_i is joined to an arbitrary set of i vertices of $G_o, i = 0, \ldots, m$. For $G_o = K(4)$ the graph G if presented in Figure 4.

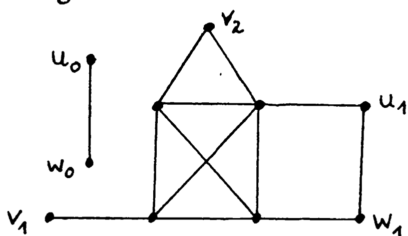


Figure 4

LEMMA.

For every graph G constructed as above

$$J_G = \{2\} \cup \{|G_o| + j : j \not\equiv 2(\bmod 3), j = 1, \ldots, m\}.$$

Proof: Consider the function $f(H) = d_o(|H| - 2) - e(H) + 1$. Obviously $d'(H) < d_o$ iff $f(H) > 0$. Let $H \subseteq G, H \neq G_o, H_o = H \cap G_o$ and $x = |H| - |H_o|$. Then $f(H) = f(H_o) + d_o x - e(H) + e(H_o) > 0$, since $f(H_o) \geq 0, e(H) - e(H_o) \leq xm < xd_o$ and at least one inequality is strict. Thus $J_G \cap \{2, \ldots, |G_o|\} = \{2, |G_o|\}$. Let $H_x = G[V(G_o) \cup V_x]$, where V_x is the set of first x elements of V. To complete the proof we will show that $e(H_x) = e_s, s = |H_x|$. Let $H \subseteq G, |H| = s, y = s - |H \cap G_o| > x$. Denote by k_x the number of edges joining V_x to G_o or contained in V_x. Then

$$f(H) \geq f(H_o) + d_o y - k_y > d_o y - k_y > d_o x - k_x = f(H_x).$$

Hence $e(H_x) > e(H)$. ∎

Proof of Theorem 1:

The lower bound is immediate by the above construction with $G_o = K(2m)$. Then $|G| = 5m$ and $|J_G| = 2m + 2$. Deleting $v_1, \{u_o, w_o\}, \{u_o, w_o, v_1\}$, or $\{u_o, w_o, v_1, w_1\}$, respectively, we achieve the required size of $|J_G|$ also in the cases $i = 1, 2, 3, 4$.

To prove the upper bound asume that $J_G = \{n_1, \ldots, n_t\}, n_1 = 2, n_t = |G| = n$. The sequence $a_{n_i}^-, i = 2, \ldots, t$ is positive, strictly decreasing and

$$a = a_{n_2}^- = \frac{e_{n_2} - 1}{n_2 - 2} \le \frac{1}{2}(n_2 + 1). \tag{1}$$

The proof is based on the simple idea that small difference $n_i - n_{i-1}$ accelerate the decay of slopes, whereas large values of $n_i - n_{i-1}$ increase the number of "non-corner" points T_s. In detail, set

$$r_s = |\{i : n_i - n_{i-1} = s, i = 3, \dots, t\}|, s = 1, 2, \dots$$

If $n_i - n_{i-1} = 1$ then $a_{n_i}^-$ is an integer and so

$$r_1 < a \tag{2}$$

For a similar reason, $r_1 + \frac{1}{2}(r_2 - r_1) < a$, or equivalently

$$r_1 + r_2 < 2a \tag{3}$$

Observe that

$$t = 2 + \sum_{s \ge 1} r_s = n - n_2 + 2 - \sum_{s \ge 2} (s-1) r_s. \tag{4}$$

Therefore, by (1) and (3)

$$n = n_2 + \sum_{s \ge 1} s r_s \ge r_1 + r_2 + \sum_{s \ge 1} s r_s \ge 3t - 6 - r_1$$

and

$$t \le \frac{1}{3}(n + r_1) + 2. \tag{5}$$

On the other hand, by (1), (2), and (4).

$$t \le n - n_2 + 4 - t \le n - r_1 + 4 - t,$$

so

$$t \le \frac{1}{2}(n - r_1) + 2 \tag{6}$$

The inequalities (5) and (6) imply that

$$t \le \frac{2}{5} n + 2 = 2m + 2 - \frac{2i}{5}$$

and the theorem follows. ∎

Proof of Theorem 2 and 3:

Let G be a $K(r)$-free graph. By Turan's theorem

$$a \leq \frac{(r-2)n_2^2 - 2(r-1)}{2(r-1)(n_2-2)}$$

and by similar arguments

$$\gamma_n(\mathcal{F}_3) \leq \frac{2}{7}n + 2\frac{2}{7} = 2m + 2 + \frac{2-2i}{7}$$

and

$$\gamma_n(\mathcal{F}_r) \leq \frac{2r-4}{5r-8}n + c, c > 0, r \geq 4.$$

For the lower bound we use our construction with G_o being the Turan graph with $r-1$ parts of size $2m/(r-2)$ each ($2m$ is assumed to be divisible by $r-2$). Then $d_o = m + \in, \frac{1}{2} \leq \in < 1$, and

$$|J_G|/|G| \sim \frac{2r-4}{5r-8} \text{ as } m \to \infty.$$

Moreover, G may be $(r-1)$ chromatic, so $K(r)$-free. In the case $r = 3$ we start with $G_o = K(2m, 2m)$ - a complete bipartite graph and then $d_o = m + 1/2, |G| = 7m, |J_G| = 2m + 2$. Deleting $v - 1$ we prove our result for $i = 1$. Switching $K(2m, 2m)$ to $K(2m, 2m-1)$, $K(2m-1, 2m-1)$ and $K(2m-1, 2m-2)$ we still have $d_o > m - \frac{1}{2}$ and this time deleting u_m, we cover the cases $i = 2, 3, 4$. For $i = 5, 6$ we additionally remove v_1 and $\{u_o, w_o\}$, respectively. ∎

3. STRICTLY BALANCED GRAPHS

Let us recall that a graph G is strictly balanced if $d(H) < d(G)$ for an $H \subseteq G$, where $d(H) = e(H)/|H|$. Strictly balanced graphs play an important role in the theory of random graphs, as they are the only graphs for which,

$$P(n,p,G) \sim exp\{-\mu_n(G)\}$$

holds on the threshold, i.e. when $np^{d(G)} \to c > 0$, where $\mu(G)$ is the expectation of the number $X_n(G)$ of subgraphs of $K(n,p)$ isomorphic to G. It follows from the more general result that, on the threshold, $X_n(G)$ has Poisson limit distribution iff G is strictly balanced (|1|).

Let S be the family of strictly balanced graphs. In particular, S includes all k-trees and connected regular graphs. Below we find a lower and upper bound for $\gamma_n(S)$. Unfortunately they are far apart, and it remains an open problem to determine the correct order of magnitude of $\gamma_n(S)$.

THEOREM 4.
For n sufficiently large,

$$(2n)^{1/3}+1<\gamma_n(S),2n^{2/3}+1.$$

Proof:

Upper bound
Let G be strictly balanced and $J_G=\{n_1,\ldots,n_t=n\}$. We abbreviate $a^-_{n_i}=a_i$ and $e_{n_i}=e_i$, for convenience. We have

$$a_t<a_2=\frac{e_2-1}{n_2-2}.$$

On the other hand, for $i=2,\ldots,t$,

$$a_i>\frac{e(G)-e_{i-1}}{n-n_{i-1}},$$

which implies

$$a_i>\frac{e_{i-1}}{n_{i-1}}=d_{i-1}$$

(here we use the fact that G is strictly balanced). Last inequality is equivalent to $d_i>d_i-1$. Thus the lower and upper bound for a_i move toward each other. But we only utilize the fact that $a_t>d_2$. Hence

$$a_2-a_t<\frac{e_2-1}{n_2-2}-\frac{e_2}{n_2}\leq 1.$$

Suppose $t\geq 2n^{2/3}+1$ and let

$$x=|\{i:n_i-n_{i-1}\geq n^{1/3},i=2,\ldots,t\}|$$

If $x\geq n^{2/3}$ then

$$n-2=(n_2-n_1)+\cdots+(n_t-n_{t-1})\geq xn^{1/3}\geq n,$$

a contradiction. If $x<n^{2/3}$ then

$$|\{i:n_i-n_{i-1}\leq n^{1/3},i=2,\ldots,t\}|=t-1-x>n^{2/3}.$$

By pigeon-hole principle there is s, $1\leq s<n^{1/3}$, such that

$$|\{i:n_i-n_{i-1}=s\}|>\lceil n^{1/3}\rceil.$$

Therefore

$$a_2-a_t\geq(\lceil n^{1/3}\rceil-1)\frac{1}{s}\geq 1,$$

again a contradiction.

Lower bound

Let G be a connected graph obtained from vertex-disjoint cycles $C_o, \ldots, C_t$, $|C_o| = |C_1| = \binom{t}{2} + 1, |C_i| = \binom{t}{2} + i, i = 1, \ldots, t \geq 3$
by connecting them with t disjoint edges $\epsilon_1, \ldots, \epsilon_t$ so that ϵ_i joins C_{i-1} to C_i. It can be checked that $|G| = \frac{1}{2}(t+1)t^2 + 1$ and

$$J_G = \{2, |C_o|, |C_o| + |C_1|, |C_o| + |C_1| + |C_2|, \ldots, |G|\}.$$

Hence $|J_G| = t + 2$ and the theorem follows. ∎

References

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