SOME CYCLIC PROPERTIES IN BIPARTITE DIGRAPHS WITH A GIVEN NUMBER OF ARCS

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Abstract. We deal with conditions on the number of arcs sufficient for bipartite digraphs to have cycles and paths with specified properties.

1. Introduction.

Many sufficient conditions for the existence of paths and cycles of various lengths in digraphs are known (see, for example, [3], [5]). The object of this work is to point out some corresponding results for bipartite digraphs with conditions on the number of arcs. For conditions involving out-degrees and in-degrees, the reader is encouraged to consult [2].

Throughout this paper, $D = (X, Y, E)$ denotes a bipartite digraph of order n with bipartition (X, Y), where $|X| = a \leq b = |Y|$ and $n = a + b$. Then, $V(D)(= X \cup Y)$ denotes the set of vertices and, $E(D)$ denotes the set of arcs of D. If x and y are vertices of D, then we say that x dominates y if the arc (x, y) is present. For $A, B \subseteq V(D)$, we define $E(A \to B) = \{(x,y) \mid x \in A,\ y \in B,\ (x,y) \in E(D)\}$ and $E(A,B) = E(A \to B) \cup E(B \to A)$. The out-degree, in-degree and degree of a vertex x are defined as $|E(x \to D)|$, $|E(D \to x|$ and $|E(x,D)|$, respectively, and are denoted $d^+(x)$, $d^-(x)$ and $d(x)$, respectively. The independence number α of D is the cardinality of a maximum independent set A. A cycle is called dominating, if for every vertex x of D we have $E(x, C) \neq \emptyset$. Let x and y be two vertices of the same bipartition set of D. The digraph $D_{xy,s}$ is obtained from D by replacing the vertices x and y by a new vertex s and by adding all the arcs in such a way that $E(D_{xy,s} \to s) = E(D \to x)$ and $E(s \to D_{xy,s}) = E(y \to D)$.

The following results are used in this paper.

[1] Research partially supported by PRC Math-Info.

Theorem a [7]. *Let D be a digraph of order n and independence number at least $\alpha, n \geq 2\alpha$. If $|E(D)| > n(n-1) - (n-\alpha) - \alpha(\alpha-1)$, then D is hamiltonian.*

Theorem b [7]. *Let D be a digraph of order n and independence number at least $\alpha, n \geq 2\alpha + 1$. If $|E(D)| > n(n-1) - (n-\alpha) - \alpha(\alpha-1) + 1$, then D is Hamilton connected.*

Lemma c [6]. *Let D be a dipartite which contains a cycle C of length $2r, 2r < n$. Let x be a vertex not contained in C. If $|E(x,C)| > r$, then D contains cycles of every even length m, $2 \leq m \leq 2r$, through x.*

2. Cycles and paths in bipartite digraphs with a given number of arcs.

We begin with the following proposition on cycles of prescribed lengths in bipartite digraphs with a given number of arcs.

Proposition 2.1. *Let $D = (X, Y, E)$ be a bipartite digraph of order n and k an integer, $a \geq k$. If $|E(D)| \geq ab + (k-1)b + 1$, then D has cycles of every even length $m, 2 \leq m \leq 2k$.*

Proof: We use induction on n. For $n = 3, 4$ the statement is easily verified. For $k = 1$ we have $|E(D)| \geq ab + 1$ and therefore D has a cycle of length 2. Also, the case $a = b = k$ was proved in [4]. Moreover, assume that D has exactly $ab + (k-1)b + 1$ arcs, for otherwise we may consider a spanning subgraph of D with exactly $ab + (k-1)b + 1$ arcs instead of D. Now if for every vertex y in Y we have $d(y) \geq a + k$, then $|E(D)| = \sum_{i=1}^{b} d(y_i) = b(a+k) = ab + bk$, which is a contradiction. If, on the other hand, for some vertex y of Y we have $d(y) \leq a + k - 1$, then $D - y$ has $a(b-1) + (k-1)(b-1) + 1$ or more arcs. Hence, $D - y$ has cycles of every even length $m, 2 \leq m \leq 2k$, by induction, as required. ∎

The above conditions are the best possible. Indeed, let B, A and C be independent sets of cardinalities b, $k-1$ and $a-k+1$, respectively. Now consider the complete bipartite digraph with bipartition (A, B) and then add all the arcs from C to B. The resulting bipartite digraph has exactly $2ab - (a-k+1)b$ arcs, however it has no cycle of length $2k$.

For strong bipartite digraphs the following may hold.

Conjecture 2.2. *Let $D = (X, Y, E)$ be a strong bipartite digraph and k an integer $a \geq k \geq 2$. If $|E(D)| \geq ab + (k-2)b + a - k + 3$, then D has a cycle of length at least $2k$.*

This conjecture, if true, would be the best possible. To see that, let us take independent sets B, A and C of cardinalities b, $k-2$ and $a-k+2$, respectively. Now consider the complete bipartite digraph with bipartition (A, B) and, then add all the arcs from C to B and all the arcs from exactly one vertex of B to all vertices of C. The resulting bipartite digraph has $ab + (k-2)b + a - k + 2$ arcs; however, it has no cycle of length $2k$.

Theorem 2.3. *Let $D = (X, Y, E)$ be a bipartite digraph with $2ab - a + 1$ or more arcs and let y be any vertex of D. Then*

(i) *every subset of $V(D)$ of cardinality a is contained in a cycle of length $2a$;*

(ii) *D contains cycles of every even length m, $2 \leq m \leq 2a$, through y;*

(iii) *if $b > a$, then for any two vertices y_1 and y_2 of Y, there is a path from y_1 to y_2 of every even length m, $2 \leq m \leq 2a$; and*

(iv) *if x_1 and x_2 are distinct vertices of X, then there is a path from x_1 to x_2 of every even length m, $2 \leq m \leq 2a - 2$.*

Proof of Part (i): Let S be any subset of $V(D)$ of cardinality a. Consider a subset A of Y such that $Y \cap S \subseteq A$ and $|A| = a = |X|$. The bipartite digraph D' with bipartition (X, A) satisfies $|E(D')| \geq |E(D)| - 2(b - a)a = 2a^2 - a + 1$ and therefore is hamiltonian by Proposition 2.1. It follows that all the vertices of A (and consequently all the vertices of S) are on a cycle of length $2a$, as required.

Proof of Part (ii): It is by induction. The proof is trivial for small values of n.

Assume first that $b > a$ and that y is a vertex of Y. Then the digraph $D - y$ satisfies $|E(D - y)| \geq |E(D)| - 2a = 2a(b - 1) - a + 1$ and therefore has a cycle of length $2a$, by Proposition 2.1. On the other hand, we have $d(y) \geq |E(D)| - 2(b - 1)a = a + 1$ and then Lemma c permit us to complete the proof.

Assume next that either $a = b$ or y is a vertex of X. Clearly there exists a cycle of length $2a$ through y in D by (i). If $d(y) = 2a$, then $D - y$ has a cycle of length $2a - 2$ by Proposition 1 and, by Lemma c, we complete the argument. If, on the other hand, $d(y) \leq 2b - 1$, then take two distinct vertices x and z of D such that y dominates z and is dominated by x. Let now D' denote the bipartite digraph obtained from $D - \{x, y, z\}$ by adding a new vertex s and the arcs $\{(w, s) | (w, x) \in E(D)\} \cup \{(s, w) | (z, w) \in E(D)\}$. Then $|E(D')| \geq |E(D)| - 2a - 2b + 3 = 2(a - 1)(b - 1) - (a - 1) + 1$. It follows that there are cycles of every even length m, $2 \leq m \leq 2(a - 1)$, through s in D' by induction, so the conclusion follows for y.

Proof of Part (iii): Let y_1, y_2 be two vertices of Y. Then the digraph $D_{y_2 y_1, s}$, defined in the introduction, satisfies $|E(D_{y_2 y_1, s}| \geq |E(D)| - 2a = 2a(b - 1) - a + 1$ and therefore there are cycles of all even lengths m, $2 \leq m \leq 2a$ through s in $D_{y_2 y_1, s}$ by (ii). Now it is extremely easy to see that the conclusion of (iii) is also verified.

Proof of Part (iv): It is very similar to that of (iii).

The proof of the theorem is complete. ∎

The bound in the above theorem is best possible. To see that take the complete bipartite digraph $K^{\star}_{a,b}$ and, then delete all the incoming arcs to a vertex y in the larger bipartition set. The resulting bipartite digraph has $2ab - a$ arcs, however it has no path with terminus y.

From Theorem 2.3 we obtain the two corollaries below. Note that Corollary 2.4 generalises a result proved in [4].

Corollary 2.4. *Let $D = (X, Y, E)$ be a bipartite digraph such that $|X| = |Y| = \frac{n}{2}$. If $|E(D)| \geq \frac{n^2-n}{2} + 1$, then any vertex of D is contained in a cycle of each even length m, $2 \leq m \leq n$.*

Corollary 2.5. *Let D be a digraph of order n and independence number at least α. If $|E(D)| \geq n(n-1) - (n-\alpha) - \alpha(\alpha-1) + 1$, then any set of $n-\alpha$ vertices is contained in a cycle of length at least* $\min(n, 2(n-\alpha))$ *in D.*

Proof: If $n \geq 2\alpha$, then D is hamiltonian by Theorem a. Assume $n \leq 2\alpha - 1$. Let S be an independent set of cardinality α in D. Now consider a spanning subgraph D' of D with arc-set $E(D') = E(D) - \{(x,y) \mid x, y \in V(D) - S, (x,y) \in E(D)\}$. Then D' is bipartite with bipartition $(S, D-S)$ and satisfies $|E(D')| \geq |E(D)| - (n-\alpha)(n-\alpha-1) \geq 2\alpha(n-\alpha) - (n-\alpha+1$. Consequently, it follows from Theorem 2.3 that each set with $n-\alpha$ vertices is contained in a cycle of length $2(n-\alpha)$ in D', so the conclusion easily follows for D. ∎

Theorem 2.6. *Let $D = (X, Y, E)$ be a bipartite digraph with $2ab - a + 2$ or more arcs and let x, y be two vertices of D. Then:*

(i) *any set of $a-1$ vertices is contained in a path of length at least $2a-2$ from x to y; and*
(ii) *for every vertex x of X and every vertex y of Y, there are paths from x to y and from y to x of every odd length m, $3 \leq m \leq 2a-1$.*

Proof of (i): Let x and y be two vertices of D and also let S be any set of $a-1$ vertices in $D - \{x, y\}$. We distinguish between three cases (a), (b) and (c).

(a) Both the vertices x and y are in X.

In this case we have $|E(D_{yx,s})| \geq |E(D)| - 2b \geq 2(a-1)b - (a-1) + 1$. It follows from Theorem 2.3 that S is contained in a cycle of length $2a-2$ in $D_{yx,s}$. This cycle necessarily contains the vertex s and therefore S is contained in a path from x to y of length $2a-2$ in D.

(b) Both the vertices x and y are in Y.

The proof is very similar to that of (a).

(c) The vertex x is in X and the vertex y is in Y.

Consider the digraph $D - y$. It satisfies $|E(D-y)| \geq 2ab - a + 2 - 2a = 2a(b-1) - a + 2$. Consider now a vertex z which dominates y. Then, if $a = b$ it follows from (iii) of Theorem 2.3 that there is a hamiltonian path from x to z. If, on the other hand, $b > a$ it follows from case (a) above that there is a path from x to z which contains all the vertices of S. Finally in both cases, using the arc (z, y), we can find a path from x to y which contains S.

Proof of (ii): We shall prove that there are paths from x to y of every odd length m, $3 \leq m \leq 2a-1$. Let z be a vertex in $(Y-y)$ which is dominated by x. Now let D' denote the digraph obtained from $D - \{x,y,z\}$ by adding a new vertex s and the arcs $\{(w,s)\,|\,(w,y) \in E(D)\} \cup \{(s,w)\,|\,(z,w) \in E(D)\}$. Then $|E(D')| \geq |E(D)| - 2a - 2b + 2 \geq 2(a-1)(b-1) - (a-1) + 1$. It follows from Theorem 2.3 that there are cycles of every even length m, $2 \leq m \leq 2(a-1)$, through s and, then it is easy to see that the conclusion of this case is verified. This completes the proof of Theorem 2.6. ∎

In the above theorem, $2ab - a + 2$ can not be replaced by $2ab - a + 1$. To see that, take two independent sets A and B of cardinalities a and $b-1$, respectively, and then consider the complete bipartite digraph D with bipartition (A,B). Next, add a new vertex y and all the arcs from y to A and an arc from exactly one vertex x of A to y. Clearly, D has $2ab - a + 1$ arcs, however it has no path from x to y of length more than one.

From Theorem 2.6 we obtain the following corollary for digraphs.

Corollary 2.7. *Let D be a digraph of order n and independence number at least α. Let x and y be two vertices of D. If $|E(D)| \geq n(n-1) - (n-\alpha) - \alpha(\alpha-1) + 2$, then any set of $n-\alpha-1$ vertices is contained in a path of length at least $\min(n-1, 2(n-\alpha-1))$ from x to y in D.*

Proof: If $n \geq 2\alpha + 1$, then D is Hamilton-connected by Theorem b. Assume $n \leq 2\alpha$. Let S be an independent set of cardinality α. Consider a spanning subgraph D' of D with arc-set $E(D') = E(D) - \{(x,y)\,|\,x,y \in V(D) - S, (x,y) \in E(D)\}$. Clearly, D' is bipartite with bipartition $(S, D-S)$ and moreover $|E(D')| \geq |E(D)| - (n-\alpha)(n-\alpha-1) \geq 2\alpha(n-\alpha) - (n-\alpha) + 2$. It follows that D' satisfies the conclusion of Theorem 2.6 and consequently D does also. ∎

We shall conclude this section with the following result on dominating cycles.

Theorem 2.8. *Let $D = (X,Y,E)$ be a bipartite digraph with $2ab - 2a + 1$ or more arcs. Then D has a dominating cycle of length at least $2a-2$ unless $a = b = 2$, $V(D) = \{x_1, x_2, y_1, y_2\}$ and $E(D) = \{(x_1,y_1),(y_1,x_1),(y_1,x_2),(x_2,y_2),(y_2,x_2)\}$.*

Proof: (The current proof was suggested by the referee and is shorter and easier than the original.) For $a = 1$, we always have a cycle of length 2. For $a = 2$ and $b = 2$, the theorem is easily verified. In what follows, assume $a \geq 2$ and $b \geq 3$. Let C be a longest cycle. The length of C is either $2a-2$ or $2a$. If it is $2a$, then any $y \in V(D) - V(C)$, is in Y. If $E(y,C) = \emptyset$, then y is isolated in D and $|E(D)| \leq 2ab - 2a$, a contradiction. ∎

Assume C has length $2a-2$. If C is dominating, then there is a y in $V(D) - V(C)$ such that $E(y,C) = \emptyset$. Since $|E(D)| \geq 2ab - 2a + 1$ and $E(y,C) = \emptyset$,

$D-y$ is a complete bipartite digraph with possibly one arc missing. Furthermore, $E(x,y) \neq \emptyset$ for all x not on C and in a different bipartition set from y. Let x be such a vertex. Since at most one arc is missing in $D-y$ and the length of C is at least four, a new cycle C' can be formed by replacing a segment $z_1 z_2 z_3$ of C with $z_1 x z_3$. It is easy to see that C' is dominating.

Theorem 2.8 is the best possible because of the bipartite digraph consisting of the disjoint union of the complete bipartite digraph with bipartition (A, B), where $|A| = a$ and $|B| = b-1$, and an isolated vertex.

3. Cycles and paths with conditions on the number of arcs involving out-degrees and in-degrees.

Theorem 3.1. *Let $D = (X, Y, E)$ be a bipartite digraph such that for every vertex x we have $d^+(x) \geq r$, $d^-(x) \geq r$, where $b \geq a \geq r$. Then:*

(i) *if $|E(D)| \geq f(a,b,r) = 2ab - (r+1)(a-r) + 1$, $a-1 \geq r \geq 0$, then D has a cycle of length $2a$; and*

(ii) *if $|E(D)| \geq g(a,b,r) = 2ab - r(a-r) + 1$, $a-1 \geq r \geq 1$, then for any two vertices x and y which are not in the same bipartition set of D there is a path of length $2a-1$ from x to y.*

Proof: The proof is by induction on n. ∎

Notation: By $\Phi(a,b,r)$ (resp. $\Omega(a,b,r)$) we denote the family of digraphs with out-degrees and in-degrees at least r which satisfy the conditions of (i) (resp. of (ii)).

It is easy to see that the conclusion of the Theorem is true for any digraph with no more than 5 vertices. In addition, any digraph either in $\Phi(a,b,0)$ or in $\Phi(a,b,a-1)$ is hamiltonian by Theorem 2.3 and any digraph either in $\Omega(a,b,1)$ or in $\Omega(a,b,a-1)$ verifies the conclusion of (ii) by Theorem 2.6. The proof is based on the following two claims.

Claim 1. *If every digraph in $\Phi(a-1,b-1,r-1)$ has a cycle of length $2a-2$, then every digraph in $\Omega(a,b,r)$ satisfies the conclusion of (ii).*

Proof: Assume that D is in $\Omega(a,b,r)$. Let y be a vertex of Y and x, z be distinct vertices of X such that z is dominated by y. Now let D' denote the digraph obtained from $D - \{x,y,z\}$ by adding a new vertex s in X and the arcs $\{(s,p) | p \in V(D'), (z,p) \in E(D)\} \cup \{(p,s) | p \in V(D'), (p,x) \in E(D)\}$. Then D satisfies $|E(D')| \geq |E(D)| - 2a - 2b + 2 \geq f(a-1, b-1, r-1)$ and consequently D' has a cycle of length $2a-2$ through s by induction. It follows that there is a path from y to x in D of length $2a-1$, as required. ∎

Claim 2. *If every D in $\Omega(a,b,r)$ satisfies the conclusion of (ii), then every D in $\Phi(a,b,r)$ has a cycle of length $2a$.*

Proof: We distinguish between two cases.

First case: For every vertex z of D we have $d^+(z) \geq r+1$ and $d^-(z) \geq r+1$.

Assume first $b > a$. Let y be a vertex of Y. Then $|E(D-y)| \geq f(a,b,r) - 2a = f(a,b-1,r)$, and hence $D-y$ has a cycle of length $2a$, by induction.

Assume now $a = b$. We may assume $r < a-1$ or else $D = K^*_{a,a}$ which certainly has a cycle of length $2a$. Since $|E(D)| \geq f(a,a,r)$, there exist $x \in X$ and $y \in Y$ such that both (x,y) and (y,x) belong to $E(D)$. Let $D' = D - \{x,y\}$. Now $|E(D)| \geq f(a-1,a-1,r)$ so that D' has a $(2a-2)$-cycle C' by induction. Let (x_i,y_i), $i = 1,2,\ldots,a-1$, by arcs of C' from $X-x$ to $Y-y$. If for some i, both (x_i,y) and (x,y_i) are arcs of D, then replacing (x_i,y_i) in C' with a path (x_i,y,x,y_i) yields a cycle of length $2a$ in D as required. Otherwise, for each i, $i = 1,2,\ldots,a-1$, add to $E(D)$ whichever of (x_i,y) or (x,y_i) is missing to form a new digraph D''. Then $|E(D'')| \geq |E(D)| + a - 1 \geq g(a,b,r)$ and by hypothesis, D'' has a path of length $2a-1$ from y to x. The path can use only arcs of D so that adding the arc (x,y) yields a cycle of length $2a$ in D.

Second case: There is a vertex z such that either $d^+(z) = r$ or $d^+(z) = r$ holds.

Without loss of generality assume that $d^-(z) = r$ holds. Suppose first that z is in X. Let D' denote the digraph obtained from D by adding the arcs $\{(p,z)\,|\,p \in Y$ and $(p,z) \notin E(D)\}$. Then $|E(D')| \geq |E(D)| + b - r \geq g(a,b,r)$. Consider now a vertex y which dominates z. It follows by hypothesis that ther is a path from the vertex z to y of length $2a-1$ in D' and it uses only arcs of D. Therefore by using this path and the arc (y,z) we can find a cycle of length $2a$ in D.

Suppose now that the vertex z is in Y. Add the arcs $\{(p,z)\,|\,p \in X$ and $(p,z) \notin E(D)\}$ and let D' denote the resulting digraph. Then as above we can easily complete the proof. ∎

In the above Theorem we do not know if the bound is the best possible. Nevertheless, we shall see later, particularly in Theorem 3.5 and Theorem 3.6, that under the conditions of Theorem 3.1, we can get more information for D which is, in a sense, best possible. In fact, perhaps the following holds.

Conjecture 3.2. *Let $D = (X,Y,E)$ be a bipartite digraph such that for every vertex x we have $d^+(x) \geq r$ and $d^-(x) \geq r$, where $b \geq a \geq r \geq 1$. Then D has a cycle of length $2a$, in both cases (i) and (ii) below:*

(i) $b \geq r(a-r+1)$ *and* $|E(D)| \geq f_1(a,b,r) = 2ab - r(a+b-2r) + 1$; *and*

(ii) $b \leq r(a-r+1)$ *and* $|E(D)| \geq f_2(a,b,r) = 2ab - (r+1)(b-r) + 1$.

Remark: It was proved by D. Amar and the second author in [2] that a bipartite digraph D with out-degrees and in-degrees not less than $\frac{a+1}{2}$ has a cycle of length $2a$ unless it is isomorphic to some specified digraphs. It follows from this result that Conjecture 3.2 is true for $a \leq 2r-1$.

The conditions given in (i) (resp. in (ii)) in the above Conjecture would be the best possible, for $a \geq 2r$ (resp. for $a \geq 2r+1$). To see that, let us define the following digraphs.

For Part (i): Let Y_1, Y_2, X_1 and X_2 be independent sets of order $b-r, r, a-r, r$, respectively. Consider the disjoint union of Y_1, Y_2, X_1, X_2 and add all the arcs in such a way that every vertex of X_1 dominates every vertex of $Y_1 \cup Y_2$ and is dominated by every vertex of Y_1, while every vertex of X_2 is dominated by every vertex of $Y_1 \cup Y_2$ and dominates every vertex of Y_2. The resulting bipartite digraph has out-degrees and in-degrees at least r and $f_1(a,b,r)-1$ arcs. However, it has not a cycle of length $2a$ since it is not strong.

For Part (ii): Consider the extremal digraph of (i) with $|Y_1| = b-r$, $|Y_2| = r$, $|X_1| = a-r-1$ and $|X_2| = r+1$ and add all the arcs from Y_2 to X_1. Clearly the resulting digraph has $f_2(a,b,r)-1$ arcs, but it has no cycle of length $2a$.

The following corollary is an easy consequence of Theorem 3.1.

Corollary 3.3. *Let $D=(X,Y,E)$ be a bipartite digraph with out-degrees and in-degrees at least r such that $|X|=|Y|=\frac{n}{2}$. If $|E(D)| \geq \frac{n^2}{2}-(r+1)(\frac{n}{2}-r)+1$, then D is hamiltonian.*

In view of Theorem 3.5, we shall establish the following Lemma.

Lemma 3.4. *Let $x_1, y_1, \ldots, x_m, y_m, x_1$ be a cycle of length $2m < n$ in a bipartite digraph D and let x be a vertex in $D-C$. If r is an integer, $r \geq 0$, and if $|E(x,C)| \geq m+r$, then $|\Omega(x)| \geq r$, where $\Omega(x) = \{x_i | (y_{i-1}, x) \in E(D)$ and $(x, y_i) \in E(D)\}$ (the indices are taken modulo m).*

Proof: Let $\delta_i(x) = |E(y_{i-1} \to x)| + |E(x \to y_i)|$. It follows from the definition of $\delta_i(x)$ that

$$\delta_i(x) = \begin{cases} 2 & \text{if } x_i \in \Omega(x) \\ \leq 1 & \text{otherwise.} \end{cases}$$

Then if $\overline{\Omega}(x)$ is the complement of $\Omega(x)$ we have

$$\sum_{i=1}^{m} \delta_i(x) \leq 2|\Omega(x)| + \overline{\Omega}(x)| = 2|\Omega(x)| + (m - |\Omega(x)|).$$

On the other hand,

$$\sum_{i=1}^{m} \delta_i(x) = \sum_{i=1}^{m} |E(y_{i-1}, x)| + \sum_{i=1}^{m} |E(x, y_i)| = |E(x,C)| = m + r.$$

It follows that $|\Omega(x)| \geq r$, which is the desired result. ∎

Theorem 3.5. *Let $D = (X, Y, E)$ be a bipartite digraph with out-degrees and in-degrees at least r, where $b \geq a \geq r+1$. If $|E(D)| \geq 2ab - (r+1)(a-r)+1$, then every $r+1$ vertices of D are on a common cycle of length $2a$.*

Proof: Let $S = \{s_1, s_2, \ldots, s_{r+1}\}$, $S \subseteq V(D)$, be a set of cardinality $r+1$. We shall prove by induction on r and contradiction that all vertices of S are on a common cycle of length $2a$.

For $r = 0$ we have Theorem 2.3. Assume, in what follows, the property true for r, but not true for $r+1$. Moreover, we may assume $S \cap X = \emptyset$, that is, $S \subseteq Y$, since D has a cycle of length $2a$ by Theorem 3.1. We see that for some vertex S_{r+1} of S, we have $d(S_{r+1}) \geq a+r+1$, since $\sum_{i=1}^{r+1} d(S_i) \geq |E(D)| - 2a(b-r-1) \geq (r+1)(a+r)+1$. Next consider a cycle C of length $2a$ which contains all the vertices of S, except S_{r+1}. Then $|\Omega(S_{r+1})| \geq r+1$ by Lemma 3.4; in other words, we may replace a vertex c of $C - (S - S_{r+1})$ by S_{r+1}, hence we get a contradiction of our assumption. ∎

The conditions given in the above theorem are, in a sense, the best possible. To see it consider the extremal digraph of (i) of Conjecture 3.2 with $|Y_1| = r+1$, $|Y_2| = b-r-1$, $|X_1| = r$ and $|X_2| = a-r$ on which we add all the arcs from Y_2 to X_1. In this digraph there is no cycle which contains all the vertices of Y_1.

Both Lemma 3.6 and Lemma 3.7 are used in the proof of Theorem 3.8.

Lemma 3.6. *Let $y_1, x_1, \ldots, x_a, y_{a+1}$, $y_i \in Y$ and $x_i \in X$, be a path P of length $2a < n-1$ in a bipartite digraph D and let y be a vertex in $Y - P$. If $|E(y,P)| \geq a+r+1$ and $a \geq r+1$, then $|\Delta(y)| \geq r$, where $\Delta(y) = \{y_i \mid (x_{i-1}, y) \in E(D)$ and $(y, x_i) \in E(D)$, $2 \leq i \leq a\}$.*

Proof: It is very similar to the proof of Lemma 3.4 ∎

Lemma 3.7. *Let $D = (X, Y, E)$ be a bipartite digraph such that $b > a$ and for every vertex x, $d^+(x) \geq r$, $d^-(x) \geq r$. If $|E(D)| \geq 2ab - r(a-r)+1$, then for any two vertices x and y of Y, there is a path from y to x of length $2a$.*

Proof: Let x and y be two vertices of Y. We have $E(D_{xy,s}) \geq 2a(b-1) - r(a-r+1)+r+1$ and therefore there is a cycle of length $2a$ through s in $D_{xy,s}$, by Theorem 3.5. The conclusion easily follows. ∎

Theorem 3.8. *Let $D = (X, Y, E)$ be a bipartite digraph with out-degrees and in-degrees at least r, where $b \geq a \geq r+1$. Let x and y be two vertices of D. If $|E(D)| \geq 2ab - r(a-r)+1$, than any set of r vertices is contained in a path of length at least $2a-2$ from x to y.*

Proof: Let $S = \{s_1, s_2, \ldots, s_r\}$, $S \subseteq V(D)$, be a set of cardinality r. If x and y are any two vertices of $D - S$, then we have to prove that all the vertices of S are on a common path from x to y. We distinguish between three cases (a), (b) and (c).

(a) Both the vertices x and y are in X.

For any vertex z in $D_{xy,s}$ we have $d^+(z) \geq r-1$ and $d^-(z) \geq r-1$. In addition, $|E(D_{xy,s})| \geq |E(D)| - 2b \geq 2(a-1)b - r(a-1-r+1)+1$, hence any r vertices are on a common cycle of length $2(a-1)$ through s, in $D_{xy,s}$, by Theorem 3.7. The conclusion follows immediately for D.

(b) The vertices x and y are not in the same bipartition set of D.

Assume that x is a vertex of X and y is a vertex of Y (the case $x \in Y$ and $y \in X$ is similar). Let z be a vertex which dominates y and, then let D' denote the digraph obtained from $D - \{x, y, z\}$ by adding a vertex s and the arcs $\{(s, w) | w \in D'$ and $(x, w) \in E(D)\} \cup \{(w, s) | w \in D'$ and $(w, z) \in E(D)\}$. Then, as in case (a) we can easily complete the proof.

(c) Both the vertices x and y are in Y.

First, it follows from Lemma 3.7 that there is a path from x to y of length $2a$. Thus we may assume $S \cap X = \emptyset$. Next, for some vertex S_r of S, we have $d(S_r) \geq a + r$ since $\sum_{i=1}^{r} d(S_i) \geq |E(D)| - 2a(b-r) \geq r(a+r)+1$. Moreover, contracting the vertices x and y as in case (a), we can see that the set $S - S_r$ is contained in a path P of length $2a$ from x to y. It follows from Lemma 3.6 that $\Delta(S_r) \geq r$. Then we can replace a vertex z of $P - (S - S_r)$ by S_r, which is the desired result. ∎

The conditions given in Theorem 3.8 are best possible. To see that, consider the extremal digraph of Theorem 3.5 with $|Y_1| = |X_1| = r$, $|Y_2| = b - r$ and $|X_2| = a - r$. In this digraph there is no path with both extremities in X_1, which contains all the vertices of Y_1.

Acknowledgement.
We wish to thank the referee of this paper for all his remarkable suggestions.

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