A Lemma In Studying Chromaticity

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Abstract. In this note we introduce a lemma which is useful in studying the chromaticity of graphs. As examples, we give a short proof for a conclusion in [3].

Let G be a graph, $g(G)$ be its girth, $P(G)$ be its chromatic polynomial. We have

Lemma. *Let G and H be chromatically equivalent graphs, i.e. $P(G) = P(H)$. Then $g(G) = g(H)$ and the number of cycles in G and H with length $g(G)$ are equal.*

Proof: The conclusion is only a consequence of the following

Theorem [1]. *Let G be a graph with order p, size q, girth g and having k cycles of length g, b_i be the coefficient of λ^{p-i} in $P(G)$, $i = 0, \ldots, p$. Then*

1. $(-1)^i b_i = \binom{q}{i}, \qquad i = 0, 1, \ldots, g-2$
2. $(-1)^{g-1} b_{g-1} = \binom{q}{g-1} - k$

This Lemma is very useful in studying the chromaticity of graphs and is called the Mini-cycle Lemma. As examples, we use it to give a short proof for a conclusion in [3]. In this note $K_4(i, j, k, l, m, n)$ denotes the K_4 homeomorph shown in Figure 1. Let $y = 1 - \lambda$. By the formula of [4], we have

$$\begin{aligned} P(K_4(i,j,k,l,m,n)) = (-1)^q \frac{y}{(1-y)^2}(&y^{q-1} - y^{i+j+l} - y^{l+m+n} \\ &- y^{n+k+i} - y^{j+k+m} - y^{i+m} - y^{j+n} - y^{l+k} \\ &+ (y+1)(y^i + y^j + y^k + y^l + y^m + y^n) - (y^2+3y+2)) \end{aligned}$$

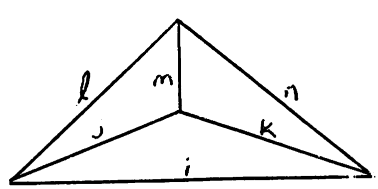

Fig. 1

Now we give a short proof for the following

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Conclusion [3]. *Suppose that $G = K_4(i,j,k,l,m,n)$ and $H = K_4(i',j',k',l',m',n')$ are chromatically equivalent homeomorphs such that two multisets (i,j,k,l,m,n) and (i',j',k',l',m',n') are the same. Then H is isomorphic to G.*

Proof: We solve the equation $P(G) = P(H)$ to confirm our claim. We consider three cases, *i.e.* both G and H have mini-cycle (i,j,k), G has mini-cycle (i,j,k) but H does not and G has mini-cycle (i,j,m,n).

Case 1. $H = K_4(i,j,k,l',m',n')$ where (l',m',n') is a permutation of (l,m,n). If H is not isomorphic to G, it is needed to consider two cases of the permutation: one element remains the same or (l',m',n') is a cycle of (l,m,n).

Case 11. $H = K_4(i,j,k,m,n,l)$. Then $P(G) = P(H)$ implies

$$\begin{aligned}&y^{i+j+l} + y^{j+k+m} + y^{i+k+n} + y^{i+m} + y^{j+n} + y^{k+l}\\ =&y^{i+j+m} + y^{j+k+n} + y^{i+k+l} + y^{i+n} + y^{j+l} + y^{k+m}\end{aligned}$$

Assume $i > j > k$ and $m > l$. Then either $i + j + m = i + k + n$ or $j + n$. In both cases we have $n > m$. Hence $l + k < i + j + m, j + k + n, i + k + l, i + n, j + l$ and $k + m$. There is no solution.

Case 12. $H = K_4(i,j,k,l,n,m)$.

$$y^{i+k+n} + y^{j+k+m} + y^{i+m} + y^{j+n} = y^{i+k+m} + y^{j+k+n} + y^{i+n} + y^{j+m}$$

Assume $n > m$. Then $i+k+n > i+k+m$ and $i+n$. So $i+k+n = j+k+n$ or $j+m$.

Case 121. $i+k+n = j+k+n, i = j$. H is isomorphic to G.

Case 122. $i+k+n = j+m$. Then $j+k+n = i+m$. There is no solution.

Case 2. G has mini-cycle (i,j,k) but H does not. First, since (i,j,k) is a mini-cycle of G, we have $l + m \geq i + k$, $m + n \geq i + j$ and $l + n \geq j + k$. So $l+m+n \geq i+j+k$. If the equality holds, equalities $l+m = i+k, m+n = i+j$ and $l + n = j + k$ must also hold. Hence $k = l, i = m, j = n$, and G has 4 cycles with length $i + j + k$. It is easy to see that H is isomorphic to G. Now $l + m + n > i + j + k$. One superedge of mini-cycle of H must be with length i, j or k. Suppose its length is i. We prove that H has a supertriangle with length $i + j + k$.

If the conclusion failed, the mini-cycle of H is a supersquare, which can be assumed to be (i,j,m,n) or (i,m,j,n), *i.e.* $m+n = k$. Since G has two cycles with length $i+j+k$, H has two supersquares with length $i+j+k$. If $H = K_4(i,j,l,k,m,n)$, then $i+j+m+n = i+l+k+m = i+j+k$, $k = m+n$, $j = l+m$ and $l+m+n < i+j+k$. A contradiction. If $H = K_4(i,m,l,k,j,n)$, since $i+l+j+k > i+j+k$ and $m+l+k+n > k+m+n > i+j+k$, H has only one supersquare whose length conld be equal to $i + j + k$. Also a contradiction. Therefore H has a supertriangle with length $i + j + k$.

This supertriangle must be (i, l, m) or (i, m, n). Without loss of generality, we suppose (i, l, m) is a supertriangle of H. Then $l + m = j + k$, $j > i$ and we can assume $m > k \geq j > l$. H is one of the following 4 graphs: $K_4(i, l, m, j, k, n)$, $K_4(i, l, m, k, j, n)$, $K_4(i, l, m, n, k, j)$ or $K_4(i, l, m, n, j, k)$.

Case 21.

$$y^{i+m} + y^{j+n} + y^{l+k} + y^{j+k+m} + y^{i+k+n}$$
$$= y^{i+k} + y^{l+n} + y^{m+j} + y^{l+m+k} + y^{i+m+n}$$

So $m + j = j + n$ or $i + k + n$.

Case 211. $m = n$ implies both G and H have mini-cycle (i, l, m). That is case 1.

Case 212. $m + j = i + k + n$. We have $j + k + m = i + m + n$, $j + k = i + n$, $k < n$, $l + k = i + k$, $l = i$ and $i + n = l + m$. So $m = n$. That is the case 211.

Case 22.

$$y^{i+m} + y^{j+n} + y^{l+k} + y^{j+k+m} + y^{i+k+n} + y^{i+j+l}$$
$$= y^{i+j} + y^{k+m} + y^{l+n} + y^{i+m+n} + y^{i+l+k} + y^{j+l+m}$$

$j + k + m = l + n$ or $i + m + n$.

Case 221. $j + k = i + n = l + m$, $k < n$, $l + k = i + j$ and $j + l + m = i + k + n$ imply $j = k$ and $i = l$, *i.e.* H is isomorphic to G.

Case 222. $j + k + m = l + n$ implies $n = 2m$; $l + k = i + j$ implies $i + m = i + l + k$ and $m = l + k$. There is no solution.

Case 23.

$$y^{i+m} + y^{j+n} + y^{l+k} + y^{j+k+m} + y^{i+k+n} + y^{i+j+l}$$
$$= y^{i+k} + y^{j+l} + y^{m+n} + y^{i+m+j} + y^{i+l+n} + y^{k+l+m}$$

$j + k + m = i + l + n$ or $m + n$.

Case 231. $j + k + m = i + l + n$. Then $2m = i + n$, $i + k + n = m + n$ and $i + k = l + k$ $(n > m > i)$. There is no solution.

Case 232. $j + k + m = m + n$ implies $n = j + k$. There is no solution.

Case 24.

$$y^{i+m} + y^{j+n} + y^{j+k+m} + y^{i+k+n} + y^{i+j+l}$$
$$= y^{i+j} + y^{m+n} + y^{i+m+k} + y^{i+l+n} + y^{j+l+m}$$

$j + l + m = i + k + n$ or $j + n$. In both cases $i < n$ and y^{i+j} can not be cancelled.

Case 3. Now suppose that the mini-cycle of G is a supersquare (i, j, m, n). By the above proof, we can assume that the mini-cycle of H is also a supersquare.

Assume $i + j \leq m + n$ and $i + n \leq j + m$. Then $k > m + n$ and $l > j + n$. It is easy to see H must have a supersquare (i, j, m, n) or (i, m, j, n). Hence $H = K_4(i, j, l, k, m, n)$ or $K_4(i, m, k, l, j, n)$ or $K_4(i, m, l, k, j, n)$.

Case 31.

$$y^{i+j+l} + y^{i+k+n} + y^{l+m+n} + y^{j+k+m} = y^{i+j+k} + y^{i+l+n} + y^{k+m+n} + y^{j+l+m}$$

No matter whether $i + j + l = i + j + k, i + l + n$ or $j + l + m$, H is isomorphic to G. If $i + j + l = k + m + n, i + k + n = j + l + m, l + m + n = i + j + k$ and $j + k + m = j + l + n$, H is also isomorphic to G.

Case 32.

$$y^{i+m} + y^{j+n} + y^{i+j+l} + y^{l+m+n} = y^{i+j} + y^{m+n} + y^{i+m+l} + y^{l+j+n}$$

$i + j = i + m$ or $j + n$. Both cases imply that H is isomorphic to G.

Case 33.

$$y^{i+m} + y^{j+n} + y^{i+j+l} + y^{i+k+n} + y^{l+m+n} + y^{j+k+m}$$
$$= y^{i+j} + y^{m+n} + y^{i+m+k} + y^{i+l+n} + y^{m+j+l} + y^{k+j+n}$$

$i + j = i + m$ or $j + n$. In both cases we have the conclusion that H is isomorphic to G. The proof has been completed.

References

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