The λ-Designs with $e_1 = 4$

Xiang-dong Hou[1]
Department of Mathematics
University of Wyoming
Laramie, Wyoming 82071
U.S.A.

Abstract. A λ-design is an $n \times n$ (0,1)-matrix A satisfying $A^tA = \lambda J + \text{diag}[k_1 - \lambda, \ldots, k_n - \lambda]$, where A^t is the transpose of A, J is the $n \times n$ matrix of ones, $k_j > \lambda > 0$ $(1 \le j \le n)$, and not all k_j's are equal. Ryser [4] and Woodall [6] showed that such an A has precisely two row sums r_1 and r_2 $(r_1 > r_2)$ with $r_1 + r_2 = n + 1$. Let e_1 be the number of the rows of A with sum r_1. It is shown that if $e_1 = 4$, then $\lambda = 3$.

1. Introduction

A λ-design is a family of subsets $S_1, S_2, \ldots, S_n$ of $\{1, 2, \ldots, n\}$ such that $|S_i| = k_i > \lambda > 0$ $(1 \le i \le n)$, $|S_i \cap S_j| = \lambda$, $(1 \le i \ne j \le n)$, and not all k_i's are equal. In terms of the point-block incidence matrix, it can be viewed as an $n \times n$ (0,1)-matrix A such that

$$A^tA = \lambda J + \text{ diag } [k_1 - \lambda, \ldots, k_n - \lambda] \tag{1.1}$$

where A^t is the transpose of A, J is the $n \times n$ matrix of ones. The fundamental structure of λ-designs, established by Ryser [4] and Woodall [6], is that A has precisely two row sums: e_1 rows with sum r_1; e_2 rows with sum r_2, where $r_1 > r_2$ and $r_1 + r_2 = n + 1$. More properties of λ-designs are discussed in [**1, 3, 4, 6**].

To complement a (0,1)-matrix with respect to a fixed column is to subtract the fixed column from all the other columns and identify -1's with 1's. Complementing the incidence matrix of a symmetric (v, k, λ')-design (not of the form $(4\lambda - 1, 2\lambda, \lambda)$) with respect to a fixed column gives a λ-design with $\lambda = k - \lambda'$. (If the symmetric design is of the form $(4\lambda - 1, 2\lambda, \lambda)$, the result is again a symmetric $(4\lambda - 1, 2\lambda, \lambda)$-design. cf. Theorem 2 of [3].) All the known examples of λ-designs are obtained in this way. Such λ-designs are called type-1 λ-designs according to [1]. The "λ-design conjecture" says that all λ-designs are of type-1. The conjecture has been verified for $1 \le \lambda \le 9$ ([2]) and for all prime values of λ ([5]). It is easily seen that $\lambda \le e_1$ if the conjecture is true. On the other hand, the proof of "$\lambda \le e_1$" would be a considerable step towards the proof of the "λ-design conjecture". It was proved that $e_1 = 1$ implies $\lambda = 1$, that $e_1 = 2$ is impossible ([1]), and that $e_1 = 3$ implies $\lambda = 2$ ([7]). Here we prove that if $e_1 = 4$, then $\lambda = 3$. Hence all the λ-designs with $e_1 \le 4$ are of type-1.

---

[1] Present Address: Department of Mathematics, University of Illinois, Chicago, Illinois 60680, U.S.A.

By a suitable permutation, we can always assume that a λ-design A is in the form

$$A = \begin{bmatrix} \overbrace{A_0}^{f_0} & \overbrace{A_1}^{f_1} & \cdots & \overbrace{A_{e_1}}^{f_{e_1}} \\ B_0 & B_1 & \cdots & B_{e_1} \end{bmatrix} \}e_1 \tag{1.2}$$

where $[A_0 A_1 \ldots A_{e_1}]$ has row sum r_1, $[B_0 B_1 \ldots B_{e_1}]$ has row sum r_2, A_i has column sum i $(0 \leq i \leq e_1)$. (B_i has constant column sum by (1.3) below.) Let k'_j, k^*_j be the column sums of the j^{th} columns of $[A_0 A_1 \ldots A_{e_1}]$ and $[B_0 B_1 \ldots B_{e_1}]$ respectively. Then $k'_j + k^*_j = k_j$. Let $\rho = \frac{r_1 - 1}{r_2 - 1} > 1$. The following facts ((1.3)–(1.10)) are from [1], [4] and [6]:

$$k^*_j = \lambda - \rho(k'_j - \lambda) \tag{1.3}$$

$$\sum_{j=1}^{n} \frac{1}{k_j - \lambda} = \frac{\lambda(1+\rho)^2 - \rho}{\lambda\rho} \tag{1.4}$$

$$e_1 = \frac{\lambda(1+\rho)^2 - (\rho + n)}{\rho^2 - 1} \tag{1.5}$$

$$r_1 = \frac{n\rho + 1}{\rho + 1}, \quad r_2 = \frac{n + \rho}{\rho + 1} \tag{1.6}$$

$$(\det A)^2 = \frac{\lambda(1+\rho)^2}{\rho} \prod_{j=1}^{n} (k_j - \lambda) \tag{1.7}$$

$$\rho \leq \lambda, \text{ if } e_1 > 1 \tag{1.8}$$

$$A \begin{bmatrix} \frac{1}{k_1 - \lambda} & & \\ & \ddots & \\ & & \frac{1}{k_n - \lambda} \end{bmatrix} A^t = I + \begin{bmatrix} \rho J_{e_1} & J \\ J & \frac{1}{\rho} J_{e_2} \end{bmatrix}. \tag{1.9}$$

In (1.9), J_{e_1}, J_{e_2} are the square matrices of ones of orders e_1 and e_2, the remaining two J's are the matrices of ones of suitable sizes.

If A has two column sums, namely $A = \begin{bmatrix} A_{i_1} & A_{i_2} \\ B_{i_1} & B_{i_2} \end{bmatrix}$, then

$$A_{i_1}, A_{i_2}, B_{i_1}, B_{i_2} \text{ all have constant row sums.} \tag{1.10}$$

We define the balanced inner product (BIP) of two vectors $(a_i, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ to be $\sum_{j=1}^{n} \frac{a_j b_j}{k_j - \lambda}$. For $1 \leq i, j \leq n$, let BIP(i, j) be the BIP of the i^{th} and the j^{th} rows of A, BIP$(i, \underline{j})$ the BIP of the i^{th} row of A and the complement of the j^{th} row of A, BIP($\underline{i}, \underline{j}$) the BIP of the complements of the i^{th} and the j^{th} rows of A. All these BIP's can be found using (1.9) and (1.4) (cf. [3]). We list the ones to be applied in the following section:

$$\text{BIP}(i, i) = 1 + \rho\,, \qquad 1 \le i \le e_1 \tag{1.11}$$

$$\text{BIP}(i, j) = \rho, \qquad 1 \le i, j \le e_1, \quad i \ne j \tag{1.12}$$

$$\text{BIP}(i, \underline{j}) = \frac{1}{\rho}, \qquad e_1 < i \le n, \quad 1 \le j \le e_1 \tag{1.13}$$

$$\text{BIP}(\underline{i}, \underline{j}) = \frac{\lambda - \rho}{\lambda\rho}, \qquad 1 \le i, j \le e_1, \quad i \ne j \tag{1.14}$$

$$\text{BIP}(\underline{i}, \underline{j}) = \rho - \frac{1}{\lambda}, \qquad e_1 < i, j \le n, \quad i \ne j \tag{1.15}$$

2. $\lambda = 3$ when $e_1 = 4$

The aim of this section is to prove that $\lambda = 3$ when $e_1 = 4$ (Theorem 2.4). We assume $e_1 = 4$ from now on. It follows from (1.5) and (1.6) that

$$n = (\lambda - 4)\rho^2 + (2\lambda - 1)\rho + (\lambda + 4) \tag{2.1}$$

$$r_1 = (\lambda - 4)\rho^2 + (\lambda + 3)\rho + 1 \tag{2.2}$$

$$r_2 = (\lambda - 4)\rho + (\lambda + 4) \tag{2.3}$$

Recall that in (1.2), A_i has column sum i $(0 \le i \le 4)$. Let ℓ_i^* and ℓ_i be the column sums of B_i and $\begin{bmatrix} A_i \\ B_i \end{bmatrix}$ respectively $(0 \le i \le 4)$. Using (1.3), we have the following table:

i	0	1	2	3	4
ℓ_i^*	$\lambda + \lambda\rho$	$\lambda + \lambda\rho - \rho$	$\lambda + \lambda\rho - 2\rho$	$\lambda + \lambda\rho - 3\rho$	$\lambda + \lambda\rho - 4\rho$
ℓ_i	$\lambda + \lambda\rho$	$\lambda + \lambda\rho - \rho + 1$	$\lambda + \lambda\rho - 2\rho + 2$	$\lambda + \lambda\rho - 3\rho + 3$	$\lambda + \lambda\rho - 4\rho + 4$
No. of columns of A_i	f_0	f_1	f_2	f_3	f_4

Table 2.1. Values of ℓ_i^* and ℓ_i

Let Z be the set of integers.

Lemma 2.1. $\rho \in Z$.

Proof: First suppose $\exists\ 0 \le i \le 3$ such that $f_i \ne 0$ and $f_{i+1} \ne 0$. Since $f_i \ne 0$ implies $\ell_i^* \in Z$, we have $\ell_i^* - \ell_{i+1}^* = \rho \in Z$. Now it is sufficient to consider the following cases: (i) $f_0 = f_2 = f_3 = 0$; (ii) $f_0 = f_2 = f_4 = 0$; (iii) $f_1 = f_2 = f_4 = 0$; (iv) $f_1 = f_3 = 0$.

(i) $f_0 = f_2 = f_3 = 0$. $A = \begin{bmatrix} A_1 & A_4 \\ B_1 & B_4 \end{bmatrix}$. By (1.10)

$$A_1 = \begin{bmatrix} \overbrace{1\dots1}^{\frac{1}{4}f_1} & \overbrace{}^{\frac{1}{4}f_1} & \overbrace{}^{\frac{1}{4}f_1} & \overbrace{}^{\frac{1}{4}f_1} \\ & 1\dots1 & & \\ & & 1\dots1 & \\ & & & 1\dots1 \end{bmatrix}$$

Hence (1.11) and (1.12) give

$$\frac{1}{4}\frac{f_1}{\lambda\rho - \rho + 1} + \frac{f_4}{\lambda\rho - 4\rho + 4} = \mathrm{BIP}(1,1) = 1 + \rho$$
$$\frac{f_4}{\lambda\rho - 4\rho + 4} = \mathrm{BIP}(1,2) = \rho$$

So, $\frac{f_1}{\lambda\rho - \rho + 1} = 4$. By (1.4)

$$\frac{\lambda(1+\rho)^2 - \rho}{\lambda\rho} = \sum_{j=1}^{n} \frac{1}{k_j - \lambda} = \frac{f_1}{\lambda\rho - \rho + 1} + \frac{f_4}{\lambda\rho - 4\rho + 4} = \rho + 4$$

which implies $\lambda - 2\lambda\rho - \rho = 0$. Contradiction since $\rho > 1$.

(ii) $f_0 = f_2 = f_4 = 0$. Since a λ-design has at least two column sums, we must have $f_1 > 0$, $f_3 > 0$. Hence $\ell_1^* - \ell_3^* = 2\rho \in Z$. Then $\lambda\rho \in Z$ since $r_2 = (\lambda - 4)\rho + (\lambda + 4) \in Z$. Therefore $\rho \in Z$ since $\ell_1^* = \lambda + \lambda\rho - \rho \in Z$.

(iii) $f_1 = f_2 = f_4 = 0$. $A = \begin{bmatrix} A_0 & A_3 \\ B_0 & B_3 \end{bmatrix}$. By (1.10)

$$A_3 = \begin{bmatrix} \overbrace{0\dots0}^{\frac{1}{4}f_3} & \overbrace{1\dots1}^{\frac{1}{4}f_3} & \overbrace{1\dots1}^{\frac{1}{4}f_3} & \overbrace{1\dots1}^{\frac{1}{4}f_3} \\ 1\dots1 & 0\dots0 & 1\dots1 & 1\dots1 \\ 1\dots1 & 1\dots1 & 0\dots0 & 1\dots1 \\ 1\dots1 & 1\dots1 & 1\dots1 & 0\dots0 \end{bmatrix}$$

As in (i), we have

$$\frac{3}{4}\frac{f_3}{\lambda\rho - 3\rho + 3} = \mathrm{BIP}(1,1) = 1 + \rho$$
$$\frac{1}{2}\frac{f_3}{\lambda\rho - 3\rho + 3} = \mathrm{BIP}(1,2) = \rho$$

Hence $\rho = 2$.

(iv) $f_1 = f_3 = 0$. $A = \begin{bmatrix} A_0 & A_2 & A_4 \\ B_0 & B_2 & B_4 \end{bmatrix}$. We have

$$f_0 + f_2 + f_4 = n \tag{2.4}$$

$$2f_2 + 4f_4 = 4r_1 \tag{2.5}$$

$$\frac{f_0}{\lambda\rho} + \frac{f_2}{\lambda\rho - 2\rho + 2} + \frac{f_4}{\lambda\rho - 4\rho + 4} = \frac{\lambda(1+\rho)^2 - \rho}{\lambda\rho} \tag{2.6}$$

where (2.5) comes by counting the 1's in $[A_0 A_2 A_4]$, (2.6) is (1.4). Solve the above system to get

$$f_0 = \left(1 - \frac{\rho}{2}\right)\lambda - \rho \tag{2.7}$$

$$f_2 = 3(\lambda\rho - 2\rho + 2) > 0 \tag{2.8}$$

$$f_4 = \left(\rho - \frac{1}{2}\right)(\lambda\rho - 4\rho + 4) \tag{2.9}$$

In (2.8), $f_2 > 0$ since $\lambda \geq 2$ ($\lambda \geq \rho > 1$). At least one of f_0 and f_4 is nonzero since a λ-design has at least two column sums. Hence $\ell_2^* - \ell_4^*$ (or $\ell_0^* - \ell_2^*$) $= 2\rho \in Z$. Then $\rho = \frac{t}{2}$ where $t \geq 3$ is an integer. Putting $\rho = \frac{t}{2}$ in (2.7) and noticing $f_0 \geq 0$, we see that $t = 3$, namely $\rho = \frac{3}{2}$. $\forall\, 1 \leq i \leq 4$, $1 \leq j \leq n-4$, by (1.3)

$$\frac{a(j)}{\lambda\rho} + \frac{b(i,j)}{\lambda\rho - 2(\rho - 1)} = \mathrm{BIP}(j+4, \underline{i}) = \frac{1}{\rho} \tag{2.10}$$

where $a(j)$ is the j^{th} row sum of B_0, $b(i,j)$ is the inner product of the j^{th} row of B_2 and the complement of the i^{th} row of A_2. Multiplying (2.10) by $\lambda\rho$, we have $\frac{\lambda\rho}{\lambda\rho - 2(\rho-1)} b(i,j) \in Z$, or $\frac{2(\rho-1)}{\lambda\rho - 2(\rho-1)} b(i,j) \in Z$. Hence

$$\frac{b(i,j)}{\lambda\rho - 2(\rho-1)} = \frac{k}{2(\rho-1)}, \quad k \in Z \tag{2.11}$$

Now $\frac{k}{2(\rho-1)} \leq \frac{1}{\rho}$ (by (2.10) and (2.11)) and $\rho = \frac{3}{2}$ forces $k = 0$. Hence $b(i,j) = 0$ for all $1 \leq i \leq 4$, $1 \leq j \leq n-4$. Then $B_2 = 0$. Noticing that B_2 must occur (since $f_2 > 0$), we have

$$\ell_2^* = \lambda + \lambda\rho - 2\rho = 0$$

which implies $\lambda = \frac{6}{5}$. Contradiction. ∎

Lemma 2.2. *If* $a, b, c, a_1, b_1, c_1, \frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c} \in Z$, *then*

$$\frac{b_1}{b} = \frac{k}{d} \tag{2.12}$$

where $d = [(a,b),(b,c)]$ *and* $k \in Z$. ((.,.) denotes the greatest common divisor, [.,.] denotes the least common multiple.)

Proof: Let $d_1 = (a,b)$, $d_2 = (b,c)$, then

$$a = sd_1, \quad b = td_1, \quad (s,t) = 1$$
$$b = t'd_2, \quad c = s'd_2, \quad (s',t') = 1$$

Multiplying $\frac{a_1}{a} + \frac{b_1}{b} + \frac{c_1}{c}$ by $ss'd$, we have $\frac{ss'db_1}{td_1} \in Z$. Hence $\frac{s'db_1}{td_1} \in Z$, namely, $\frac{s'db_1}{t'd_2} \in Z$. So, $\frac{db_1}{t'd_2} = \frac{db_1}{b} \in Z$. ∎

Lemma 2.3. $f_1 = f_2 = 0$.

Proof: $\forall\, 1 \le i \le j \le 4$, by (1.14)

$$\frac{f_0}{\lambda\rho} + \frac{a(i,j)}{\lambda\rho - (\rho-1)} + \frac{b(i,j)}{\lambda\rho - 2(\rho-1)} = \mathrm{BIP}(\underline{i},\underline{j}) = \frac{\lambda-\rho}{\lambda\rho} \tag{2.13}$$

where $a(i,j)$ is the number of common zeros of the i^{th} and the j^{th} rows of A_1, $b(i,j)$ is the same number for A_2. Rewrite (2.13) as

$$\frac{f_0 - (\lambda-\rho)}{\lambda\rho} + \frac{a(i,j)}{\lambda\rho - (\rho-1)} + \frac{b(i,j)}{\lambda\rho - 2(\rho-1)} = 0 \in Z \tag{2.14}$$

and apply Lemma 2.2, we have

$$\frac{a(i,j)}{\lambda\rho - (\rho-1)} = \frac{k}{\rho-1}, \quad k \in Z \tag{2.15}$$

$$\frac{b(i,j)}{\lambda\rho - 2(\rho-1)} = \frac{m}{2(\rho-1)}, \quad m \in Z \tag{2.16}$$

(In obtaining (2.15), notice that $[(\lambda\rho, \lambda\rho-(\rho-1)), (\lambda\rho-(\rho-1), \lambda-2(\rho-1))] \mid (\rho-1)$. Similar for (2.16).) From (2.14) and (2.15), we have $\frac{k}{\rho-1} \le \frac{\lambda-\rho}{\lambda\rho}$. This forces $k = 0$. Hence $a(i,j) = 0$ for all $1 \le i < j \le 4$. So $f_1 = 0$. From (2.14), $b(i,j) = b$ is independent of i and j. Hence A_2 must be of the form

$$A_2 = \begin{bmatrix} \overbrace{0\ldots0}^{b} & \overbrace{0\ldots0}^{b} & \overbrace{0\ldots0}^{b} & \overbrace{1\ldots1}^{b} & \overbrace{1\ldots1}^{b} & \overbrace{1\ldots1}^{b} \\ 0\ldots0 & 1\ldots1 & 1\ldots1 & 0\ldots0 & 0\ldots0 & 1\ldots1 \\ 1\ldots1 & 0\ldots0 & 1\ldots1 & 0\ldots0 & 1\ldots1 & 0\ldots0 \\ 1\ldots1 & 1\ldots1 & 0\ldots0 & 1\ldots1 & 0\ldots0 & 0\ldots0 \end{bmatrix}$$

Hence

$$b = \frac{f_2}{6} \tag{2.17}$$

By (2.14) and (2.16), $\frac{m}{2(\rho-1)} \leq \frac{\lambda-\rho}{\lambda\rho}$. This forces $m = 0$ or 1. If $m = 0$, then $b = 0$ and $f_2 = 0$. We are done.

Now assume $m = 1$. Then one can find that

$$f_0 = \lambda - \rho - \frac{\lambda\rho}{2(\rho-1)} \tag{2.18}$$

$$f_2 = \frac{3}{\rho-1}\left[\lambda\rho - 2(\rho-1)\right] > 0 \tag{2.19}$$

$$f_3 = 4\frac{\rho-2}{\rho-1}\left[\lambda\rho - 3(\rho-1)\right] \tag{2.20}$$

$$f_4 = \frac{1}{2(\rho-1)}\left[2(\lambda-4)\rho^3 + (-6\lambda+32)\rho^2 + (7\lambda-52)\rho + 28\right] \tag{2.21}$$

((2.18) comes from (2.13) and (2.16); (2.19) comes from (2.17) and (2.16); (2.20) and (2.21) come from $f_0 + f_2 + f_3 + f_4 = n$ and $2f_2 + 3f_3 + 4f_4 = 4r_1$.) $\forall$ $1 \leq i \leq 4, 1 \leq j \leq n-4$, let $u(j), v(j)$ be the j^{th} row sums of B_0 and B_2, $w(i,j)$ be the inner product of the j^{th} row of B_2 and the complement of the i^{th} row of A_2, $x(i,j)$ be the same inner product for B_3 and A_3. By (1.13)

$$\frac{u(j)}{\lambda\rho} + \frac{w(i,j)}{\lambda\rho - 2(\rho-1)} + \frac{x(i,j)}{\lambda\rho - 3(\rho-1)} = \text{BIP}\,(j+4, \underline{i}) = \frac{1}{\rho} \tag{2.22}$$

By (2.22) and Lemma 2.2,

$$\frac{w(i,j)}{\lambda\rho - 2(\rho-1)} = \frac{p}{2(\rho-1)}, \quad p \in Z \tag{2.23}$$

$$\frac{x(i,j)}{\lambda\rho - 3(\rho-1)} = \frac{q}{3(\rho-1)}, \quad q \in Z \tag{2.24}$$

$\frac{p}{2(\rho-1)} \leq \frac{1}{\rho}$ forces $p = 0$ or 1. If $v(j) = 0$, then $w(i,j) = 0$. Hence $p = 0$. By (2.22), (2.23) and (2.24),

$$u(j) = \lambda - q\frac{\lambda\rho}{3(\rho-1)}$$

$0 \le u(j) \le f_0 = \lambda - \rho - \frac{\lambda\rho}{2(\rho-1)}$ forces $q = 2$. Hence

$$u(j) = \lambda\frac{\rho-3}{3(\rho-1)} = u_1, \quad \text{if } v(j) = 0 \tag{2.25}$$

If $v(j) \neq 0$, then $w(i,j) \neq 0$ for some i. Hence $p = 1$ in (2.23). By (2.22), (2.23), and (2.24),

$$u(j) = \lambda - \frac{\lambda\rho}{2(\rho-1)} - q\frac{\lambda\rho}{3(\rho-1)}.$$

$0 \le u(j) \le f_0 = \lambda - \rho - \frac{\lambda\rho}{2(\rho-1)}$ forces $q = 1$. Hence

$$u(j) = \lambda\frac{\rho-6}{6(\rho-1)} = u_2, \quad \text{if } v(j) \neq 0 \tag{2.26}$$

Write A as

$$\left[\begin{array}{c|c|c|c} A_0 & A_2 & A_3 & A_4 \\ \hline B_0^{(1)} & 0 & & \\ \hline B_0^{(2)} & B_2^{(2)} & & \end{array}\right]$$

where $B_0^{(1)}$ has row sum u_1, $B_0^{(2)}$ has row sum u_2. B_2 must occur since $f_2 > 0$ by (2.19). If $B_2 = 0$, then $\ell_2^* = \lambda + \lambda\rho - 2\rho = 0$, namely, $\lambda = \frac{2\rho}{1+\rho} < 2$. Contradiction. So, $B_2 \neq 0$. Hence u_2 must occur, and $\rho \ge 6$ by (2.26). Counting the 1's of the submatrix of $B_0^{(2)}$ corresponding to the 1's of a column of $B_2^{(2)}$, we have

$$(\lambda + \lambda\rho - 2\rho)u_2 = f_0\lambda$$

which yields

$$(8-\rho)\lambda = 4\rho + 6 \tag{2.27}$$

Hence $\rho \le 7$. If $\rho = 7$, $\lambda = 34$ by (2.27). Then by (2.26)

$$u_2 = \lambda\frac{\rho-6}{6(\rho-1)} = \frac{34}{36} \notin Z$$

Contradiction. If $\rho = 6$, $\lambda = 15$ by (2.27). From (2.18) through (2.21), $f_0 = 0$; f_2 and f_3 are even; f_4 is odd. By (1.7)

$$\begin{aligned}(\det A)^2 &= \frac{\lambda(1+\rho)^2}{\rho}\prod_{j=1}^{n}(k_j - \lambda) \\ &= \frac{\lambda(1+\rho)^2}{\rho}(\ell_2-\lambda)^{f_2}(\ell_3-\lambda)^{f_3}(\ell_4-\lambda)^{f_4}\end{aligned}$$

Hence

$$\frac{\lambda}{\rho}(\ell_4 - \lambda) = \frac{\lambda}{\rho}[\lambda\rho - 4(\rho-1)] = 5^2 \times 7$$

must be a square of a rational number. Contradiction. ∎

Theorem 2.4. $\lambda = 3$.

Proof: Since $f_1 = f_2 = 0$, we can find that

$$f_0 = \lambda - \rho \tag{2.28}$$
$$f_3 = 4[\lambda\rho - 3(\rho - 1)] > 0 \tag{2.29}$$
$$f_4 = (\lambda - 4)\rho^2 + (-2\lambda + 12)\rho - 8 \tag{2.30}$$

((2.28) comes from (1.14); (2.29) and (2.30) come from $f_0 + f_3 + f_4 = n$ and $3f_3 + 4f_4 = 4r_1$.) In (2.29), $f_3 > 0$ since $\rho \leq \lambda$. $\forall\, 1 \leq i \leq 4, 1 \leq j \leq n-4$, let $u(j), v(j)$ be the j^{th} row sums of B_0 and B_3, $w(i,j)$ be the inner product of the j^{th} row of B_3 and the complement of the i^{th} row of A_3. By (1.13)

$$\frac{u(j)}{\lambda\rho} + \frac{w(i,j)}{\lambda\rho - 3(\rho - 1)} = \text{BIP}(j+4, \underline{i}) = \frac{1}{\rho}. \tag{2.31}$$

Hence $w(i,j) = w(j)$ is independent of i. But

$$v(j) = w(1,j) + w(2,j) + w(3,j) + w(4,j) = 4w(j).$$

So $w(j) = \frac{v(j)}{4}$. Now (2.31) becomes

$$\frac{u(j)}{\lambda\rho} + \frac{\frac{v(j)}{4}}{\lambda\rho - 3(\rho - 1)} = \frac{1}{\rho} \tag{2.32}$$

where $\frac{v(j)}{4} \in Z$. In the same way as (2.11) was obtained from (2.10), we have

$$\frac{\frac{v(j)}{4}}{\lambda\rho - 3(\rho - 1)} = \frac{t}{3(\rho - 1)}, \quad t \in Z. \tag{2.33}$$

From (2.32) and (2.33), we have $\frac{1}{\rho} \geq \frac{t}{3(\rho-1)} \geq \frac{1}{\rho} - \frac{f_0}{\lambda\rho} = \frac{1}{\rho} - \frac{\lambda-\rho}{\lambda\rho} = \frac{1}{\lambda} > 0$. This forces $t = 1$ or 2. When $t = 1$,

$$u(j) = \lambda\frac{2\rho - 3}{3(\rho - 1)} = u_1 \tag{2.34}$$
$$v(j) = 4\frac{\lambda\rho - 3(\rho - 1)}{3(\rho - 1)} = v_1. \tag{2.35}$$

When $t = 2$,

$$u(j) = \lambda\frac{\rho - 3}{3(\rho - 1)} = u_2 \tag{2.34'}$$
$$v(j) = 8\frac{\lambda\rho - 3(\rho - 1)}{3(\rho - 1)} = v_2. \tag{2.35'}$$

Write A as

$$s\{\left[\begin{array}{c|c|c} A_0 & A_3 & A_4 \\ \hline B_0^{(1)} & B_3^{(1)} & \\ \hline B_0^{(2)} & B_3^{(2)} & \end{array}\right]$$

where $B_0^{(1)}$, $B_3^{(1)}$ have row sums with $t = 1$, $B_0^{(2)}$, $B_3^{(2)}$ have row sums with $t = 2$. Counting the 1's of $\begin{bmatrix} B_0^{(1)} \\ B_0^{(2)} \end{bmatrix}$, we have

$$su_1 + (n - 4 - s)u_2 = f_0(\rho + 1)\lambda.$$

Solve to get

$$s = (5 - \rho)(\rho + 1)\lambda + \rho^2 - 11\rho. \tag{2.36}$$

B_3 must occur since $f_3 > 0$ by (2.29). Let x be the sum of any column of $B_3^{(1)}$. Counting the 1's of the submatrix of $\begin{bmatrix} B_0^{(1)} \\ B_0^{(2)} \end{bmatrix}$ corresponding to the 1's of the column of $\begin{bmatrix} B_3^{(1)} \\ B_3^{(2)} \end{bmatrix}$, we have

$$xu_1 + (\lambda + \lambda\rho - 3\rho - x)u_2 = f_0\lambda$$

which yields

$$x = 5\lambda - \lambda\rho - 6. \tag{2.37}$$

Now counting the 1's of $B_3^{(1)}$, we have

$$xf_3 = sv_1$$

which leads to

$$2(\rho - 2)(\rho - 5)\lambda = -(\rho - 2)(\rho + 9). \tag{2.38}$$

By (2.38), $\rho \leq 4$. Hence $\rho = 2, 3, 4$. When $\rho = 4$, (2.38) gives $\lambda = \frac{13}{2}$. Contradiction. When $\rho = 3$, $\lambda = 3$ by (2.38), and we are done. When $\rho = 2$, $f_4 = 0$ by (2.30). $\forall\ 1 \leq i < j \leq n - 4$, by (1.15)

$$\frac{\alpha(i,j)}{2\lambda} + \frac{\beta(i,j)}{2\lambda - 3} = \text{BIP}(\underline{i + 4}, \underline{j + 4}) = 2 - \frac{1}{\lambda}$$

where $\alpha(i,j)$ is the number of common zeros of the i^{th} and the j^{th} rows of B_0, $\beta(i,j)$ is the same number for B_3. From the above equation,

$$\frac{\alpha(i,j) + 2}{2\lambda} + \frac{\beta(i,j)}{2\lambda - 3} = 2.$$

Hence $\frac{\alpha(i,j)+2}{2\lambda} = \frac{k}{3}, k \in Z$, or $\alpha(i,j) = \frac{2k\lambda}{3} - 2$. Now $0 \le \alpha(i,j) \le f_0 = \lambda - 2$ forces $k = 1$. So

$$\alpha(i,j) = \alpha = \frac{2}{3}\lambda - 2. \tag{2.39}$$

Since $u_2 < 0$ (by (2.34′)) cannot occur,

$$u(j) = u_1 = \frac{1}{3}\lambda \text{ for all } 1 \le j \le n-4. \tag{2.40}$$

From (2.39), (2.40) and (2.28),

$$\alpha(i,j) = f_0 - u(j) \text{ for all } 1 \le i < j \le n-4. \tag{2.41}$$

Recalling the definitions of $\alpha(i,j)$ and $u(j)$, one can see that for (2.41) to be true, B_0 has to be of the form:

$$\begin{bmatrix} 1\ldots1 & 0\ldots0 \\ 1\ldots1 & 0\ldots0 \\ \ldots & \ldots \\ 1\ldots1 & 0\ldots0 \end{bmatrix}$$
$$\qquad \underbrace{}_{u_1} \quad \underbrace{}_{\alpha}$$

So $f_0 \le 1$ since the incidence matrix of a λ-design is nonsingular. Also $f_0 > 0$ since a λ-design has at least two column sums. So $f_0 = 1$; and $\lambda = 3$ by (2.28). ∎

Acknowledgement.

The author wishes to express his gratitude to Professor W. G. Bridges for his help and encouragement.

References

1. W.G. Bridges, *Some results on λ-designs*, J. Comb. Theory **8** (1970), 350–360.
2. E.S. Kramer, "On λ-designs", Ph.D. Dissertation, University of Michigan, June, 1969.
3. E.S. Kramer, *On λ-designs*, J. Comb. Theory Ser. A **16** (1974), 57–75.
4. H.J. Ryser, *An extension of a theorem of de Bruijn and Erdős on combinatorial designs*, J. Algebra **10** (1969), 246–261.
5. M.N. Singhi and S.S. Shrikhande, *On the λ-design conjecture*, Utilitas Mathematica **9** (1976), 301–318.
6. D.R. Woodall, *Square λ-linked designs*, Proc. London Math. Soc. (3)**20** (1970), 669–687.
7. Qiu-rong Wu, *Results on λ-designs*, Ars Comb. (to appear).