A Series of BIB Designs

K. Sinha
Department of Statistics
Birsa Agricultural University
Ranchi 834006

M. K. Singh
Department of Mathematics
Ranchi University
Ranchi 834001
INDIA

Abstract. A series of partially balanced incomplete block design yields under certain restrictions, a new series of BIB designs with parameters:

$$v = \binom{2s+1}{2}, \quad b = \frac{1}{2}(s+1)\binom{2s+1}{s+1},$$

$$r = s\binom{2s-1}{s}, \quad k = s^2, \quad \lambda = (s-1)\binom{2s-2}{s-1},$$

where $s \geq 2$ is any positive integer.

1. Introduction.

We shall recall the definition of a BIB design from Mathon and Rosa (1985).

A balanced incomplete block design (BIBD) is a pair (V, B) where V is a v-set and B is a collection of b k-subsets of V called blocks such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks.

Let X be a set of n positive integers and the $v = \binom{n}{2}$ elements be represented by 2-subsets $(ij) = (ji), i, j = 1, 2, \ldots, n, \; i \neq j$. Now any pair of 2-subsets having an integer in common are first associates; and any pair of disjoint 2-subsets are second associates. This defines a two-associate-class triangular association scheme.

A partially balanced incomplete block (PBIB) design with two-associate-classes is an arrangement of v elements (treatments) into b subsets (blocks) such that:

(i) each subset contains exactly $k(< v)$ elements;
(ii) each element occurs in exactly r subsets;
(iii) with respect to any element X, the other $v - 1$ elements can be grouped into two classes such that
  (a) an element in the ith $(i = 1, 2)$ class of size n_i, is called ith associate of X,
  (b) if X is ith associate of Y, then Y is ith associate of X,

(c) for any two elements which are ith associates, the number of elements common to the jth associate of first and kth associate of the second is p^i_{jk} $(i,j,k = 1,2)$ and is independent of the pair of elements;

(iv) any two elements which are ith associates occur together in λ_i subsets.

This definition of partially balanced incomplete block (PBIB) design is obtained by rephrasing the definition in Clatworthy (1973). Here the PBIB designs are based on triangular association scheme.

2. The construction.

Theorem. *There exists a series of two-associate-class triangular partially balanced incomplete block (PBIB) design with parameters;*

$$v = \binom{n}{2}, \quad b = \frac{1}{2}\binom{n}{s}\binom{n-s}{s}, \quad r = \binom{n-2}{s-1}\binom{n-s-1}{s-1}, \quad k = s^2,$$
$$\lambda_1 = \binom{n-3}{s-1}\binom{n-s-2}{s-2}, \quad \lambda_2 = 2\binom{n-4}{s-2}\binom{n-s-2}{s-2}, \tag{1}$$

where $n > s \geq 2$.

Proof: Let X be a set of n positive integers $X = 1,2,\ldots,n$. Let the 2-subsets of X be the treatments of a design and let the s-subsets of X correspond to blocks, for some fixed s such that n $s \geq 2$. A block corresponding to an s-subset consists of two parts A and B. Part A corresponds to the s-subset and part B to another s-subset obtained from the remaining $(n-s)$ integers. Then 2-subsets of a block are formed by taking an integer each from parts A and B. Thus each s-subset corresponds to $\binom{n-s}{s}$ blocks; each of these b, blocks occur twice in this construction. Hence, only one copy of each block is kept. The values of the parameters (1) can easily be verified. ∎

When $n = 2s + 1$ in (1), we have:

Corollary. *There exists a series of BIBD with parameters:*

$$v = \binom{2s+1}{2}, \quad b = \frac{1}{2}(s+1)\binom{2s+1}{s+1}, \quad r = s\binom{2s-1}{s}, \quad k = s^2$$
$$\lambda = (s-1)\binom{2s-2}{s-1}, \quad s \geq 2. \tag{2}$$

When $n = 2,3,4,5$, *we get respectively BIBDs:*

(i) $v = 10,\ b = 15,\ r = 6,\ k = 4,\ \lambda = 2,$

(ii) $v = 21,\ b = 70,\ r = 30,\ k = 9,\ \lambda = 12,$

(iii) $v = 36,\ b = 63.5,\ r = 28.5,\ k = 16,\ \lambda = 12.5,$

(iv) $v = 55,\ b = 99.14,\ r = 45.14,\ k = 25,\ \lambda = 20.14.$

The BIBD (i) is well known. The design (ii) is quasi-multiple by 2 of the known BIBD 'D69', and the design (iii) is quasi-multiple by 5 of the known BIBD 'D338'. The BIBDs with D numbers are from Dipaola et al (1973). Recently, Mathon and Rosa (1985) tabulated BIBDs with $r \leq 41$. *The BIBD (iv) is quasi-multiple by 14 of the unlisted BIBD:*

$$v = 55, \quad b = 99, \quad r = 45, \quad k = 25, \quad \lambda = 20.$$

Illustration: Given below is a plan of a BIBD with parameters:

$$v = 10, \quad b = 15, \quad r = 6, \quad k = 4, \quad \lambda = 2, \quad \text{for } n = 5,$$

where 2-subsets represent elements and parentheses enclose blocks:

(13, 14, 23, 24), (12, 14, 23, 34), (12, 13, 24, 34), (12, 13, 25, 35),
(24, 25, 34, 35), (13, 15, 23, 25), (12, 15, 23, 35), (12, 15, 24, 45),
(12, 14, 25, 45), (23, 25, 34, 45), (14, 15, 24, 25), (14, 15, 34, 35),
(13, 15, 34, 45), (13, 14, 35, 45), (23, 24, 35, 45).

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