A Sufficient Condition for Hamiltonian Cycles in Digraphs

Lian-Chang Zhao Jing-Hua Meng

Department of Mathematics
Northeast Institute of Technology
Shenyang
PEOPLE'S REPUBLIC OF CHINA

Abstract. Let D be a strict diconnected digraph with n vertices. A common out-neighbor (resp. in-neighbor) of a pair of vertices u and v is a vertex x such that ux and vx (resp. xu and xv) are arcs of D. It is shown that if

$$d^+(u_1) + d^+(v_1) + d^-(u_2) + d^-(v_2) \geq 2n - 1$$

for any pair u_1, v_1 of nonadjacent vertices with a common out-neighbor and any pair u_2, v_2 of nonadjacent vertices with a common in-neighbor, then D contains a directed hamiltonian cycle.

Our notation and terminology are as in [1]. In particular, $D = (V(D), E(D))$ denotes a digraph on n vertices with the vertex set $V(D)$ and the arc set $E(D)$. A digraph is *strict* if it has no loops and multiple arcs, and *diconnected* if, for any two vertices u and v, D contains a path from u to v and a path from v to u. The arc e with head u and tail v is represented by $e = uv$. Define $|uv| = 1$ when $uv \in E(D)$ and $|uv| = 0$ when $uv \notin E(D)$. If $v \in V(D)$ and $S \subseteq V(D)$, we denote the set of arcs from v to S(resp. from S to v) by $E(v, S)$ (resp. $E(S, v)$). Furthermore, we define $d_S^+(v) = |E(v, S)|$, $d_S^-(v) = |E(S, v)|$. Obviously, $d^+(v) = |E(v, V(D))|$ and $d^-(v) = |E(V(D), v)|$. If $S \subseteq V(D)$, an S-path is a directed path of length at least two having exactly its origin and terminus in common with S. A common out-neighbor (resp. in-neighbor) of a pair vertices u and v is a vertex x such that $ux, vx \in E(D)$ (resp. $xu, xv \in E(D)$).

Now we prove the following theorem.

Theorem. *Let D be a strict diconnected digraph with n vertices. If*

$$d^+(u_1) + d^+(v_1) + d^-(u_2) + d^-(v_2) \geq 2n - 1$$

for any pair u_1, v_1 of nonadjacent vertices with a common out-neighbor and any pair u_2, v_2 of nonadjacent vertices with a common in-neighbor, then D contains a directed hamiltonian cycle.

Proof: By contradiction. Suppose that D satisfies the hypotheses of the theorem, but does not contain a directed hamiltonian cycle. Let S be a longest directed cycle in D.

Suppose first there is no $V(S)$-path in D. Since D is diconnected, D contains a directed cycle S_1 having precisely one vertex, say u, in common with S. Let

$S = ux_1x_2\dots x_au$, $S_1 = uy_1y_2\dots y_bu$, $A = \{x_1, x_2, \dots, x_a\}, B = \{y_1, y_2, \dots, y_b\}$ and $F = V(D)\backslash(A \cup B \cup \{u\})$. Clearly, $|A| = a$, $|B| = b$ and $f = |F| = n - (a + b) - 1$. Since there is no $V(S)$-path in D, x_a, y_b and x_1, y_1 are pairs of nonadjacent vertices with a common out-neighbor and in-neighbor, respectively.

Since D contains no $V(S)$-path,

$$d^+_B(x_a) = d^-_B(x_1) = d^+_A(y_b) = d^-_A(y_1) = 0. \tag{1}$$

Also since D contains no $V(S)$-path, there is not a path of the form $x_a\ vy_1\ y_b\ vx_1$, where $v \in F$. Hence,

$$|x_av| + |vy_1| \leq 1, \quad |y_bv| + |vx_1| \leq 1$$

for each $v \in F$. Furthermore,

$$\begin{aligned} &d^+_F(x_a) + d^-_F(y_1) + d^+_F(y_b) + d^-_F(x_1) \\ &\leq \sum_{v\in F}[(|x_av| + |vy_1|) + (|y_bv| + |vx_1|)] \leq 2f. \end{aligned} \tag{2}$$

Clearly,

$$d^+_A(x_a) \leq a-1,\ d^-_A(x_1) \leq a-1,\ d^+_B(y_b) \leq b-1,\ d^-_B(y_1) \leq b-1. \tag{3}$$

Combining (1) - (3), we get

$$\begin{aligned} &d^+(x_a) + d^+(y_b) + d^-(x_1) + d^-(y_1) \\ &\leq 2(a-1) + 2(b-1) + 2f + |x_au| + |y_bu| + |ux_1|) + |uy_1| \\ &= 2(a + b + f + 1) - 2 = 2n - 2, \end{aligned}$$

but this contradicts a hypothesis of the theorem.

Suppose, therefore, that D contains a $V(S)$-path $P = x_ay_1y_2\dots y_bx_1$ where $x_1, x_a \in V(S)$. Let $P_1 = x_1x_2\dots x_a$,‘ $P_2 = x_az_1\dots z_cx_1$ be the directed path on the cycle S such that $S = P_1 \cup P_2$. Let the path P be chosen so that c is minimum. Since S is a longest cycle in D, $c \geq 1$. Let $A = \{x_1, \dots, x_a\}$, $B = \{y_1, \dots, y_b\}$, $C = \{z_1, \dots, z_c\}$ and $F = V\backslash(A \cup B \cup C)$. Clearly, $|A| = a$, $|B| = b$, $|C| = c$ and $f = |F| = n - (a + b + c)$. Because of the minimality of c and $c \geq 1$, y_b, z_c and y_1, z_1 are pairs of nonadjacent vertices with a common out-neighbor and in-neighbor, respectively. By the same reason, we have

$$d^+_C(y_b) = d^-_C(y_1) = d^+_B(z_c) = d^-_B(z_1) = 0. \tag{4}$$

If there exist x_i, $x_{i+1} \in A (i = 1, \dots, a - 1)$ such that $x_iz_1, z_cx_{i+1} \in E$, then $x_iz_1z_2\dots z_cx_{i+1}x_{i+2}\dots x_ay_1y_2\dots y_bx_1x_2\dots x_i$ is a cycle longer than S. This contradiction shows that

$$|x_iz_1| + |z_cx_{i+1}| \leq 1 \quad (i = 1, \dots, a - 1).$$

Hence,

$$\begin{aligned} d_A^+(z_c) + d_A^-(z_1) &= \sum_{i=1}^{a}(|z_c x_i| + |x_i z_1|) \\ &= \sum_{i=1}^{a-1}(|z_c x_{i+1}| + |x_i z_1|) + |z_c x_1| + |x_a z_1| \leq a + 1. \end{aligned} \tag{5}$$

A similar argument yields

$$d_A^+(y_b) + d_A^-(y_1) \leq a + 1. \tag{6}$$

It follows from the maximality of S that there exists no vertex $v \in F$ such that $z_c v, v y_1 \in E$ or $y_b v, v y_1 \in E$. Hence,

$$|z_c v| + |v y_1| \leq 1, \quad |y_b v| + |v z_1| \leq 1$$

for every $v \in F$. Furthermore

$$d_F^+(z_c) + d_F^-(y_1) \leq f, \quad d_F^+(y_b) + d_F^-(z_1) \leq f. \tag{7}$$

Clearly,

$$d_B^+(y_b) \leq b - 1, \; d_B^-(y_1) \leq b - 1, \; d_C^+(z_c) \leq c - 1, \; d_C^-(z_1) \leq c - 1. \tag{8}$$

Combining (4) - (8), we have

$$\begin{aligned} & d^+(y_b) + d^+(z_c) + d^-(y_1) + d^-(z_1) \\ & \quad \leq 2(a + 1) + 2(b - 1) + 2(c - 1) + 2f = 2(a + b + c + f) - 2 = 2n - 2, \end{aligned}$$

this contradiction proves the theorem. ∎

The following example shows that the theorem above is best possible in the sense that it becomes false if the degree condition is relaxed.

Example 1 [3, p. 4]: Let u be a vertex of $K_{n-2}^{\star}$ $(n \geq 5)$, the complete symmetric digraph with $n-2$ vertices. Obtain digraph H_n by adding two new vertices v and w, each of which dominates all $n-2$ vertices of $K_{n-2}^{\star}$ and is dominated by only u.

The pair v, w is the only pair of nonadjacent vertices with a common out-neighbor and in-neighbor, and

$$d^+(v) + d^+(w) + d^-(v) + d^-(w) = 2n - 2.$$

H does not contain a directed hamiltonian cycle.

The second example (Figure 1) shows that, in some sense, the theorem above is not "weaker" than the well-known Meyniel's theorem [**4**].

Example 2: Let $D = (V, E)$ be a digraph with $n = 2k$ ($k \geq 4$) vertices. The pairs a_1, a_2 and b_1, b_2 are the only pairs of nonadjacent vertices with a common out-neighbor and in-neighbor, respectively. Clearly,

$$\begin{aligned} & d^+(a_1) + d^+(a_2) + d^-(b_1) + d^-(b_2) \\ & \quad = (k+1) + k + (k-1) + (k-1) = 4k - 1 = 2n - 1, \end{aligned}$$

and D_n has a directed hamiltonian cycle $S = a_1 b_2 v_1\ a_2 v_2\ v_3 \ldots v_{k-2}\ b_1\ v_{k-1}\ v_k \ldots v_{2k-4}\, a_1$. But D does not satisfy the conditions of Meyniel's theorem.

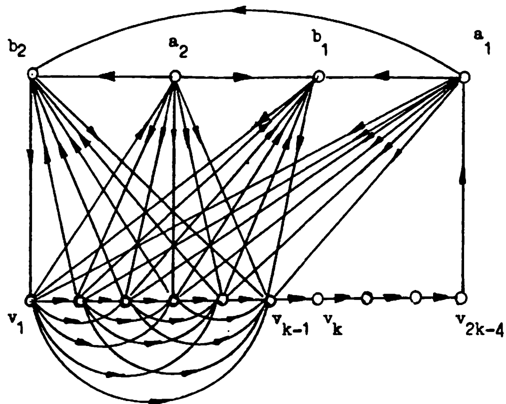

Figure 1

Acknowledgements.

The authors thank the referees for helpful comments and valuable suggestions in revising the paper.

References

1. J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications", Macmillan, New York, 1976.
2. J.A. Bondy and C. Thomassen, *A short proof of Meyniel's theorem*, Discrete Math. **19** (1977), 195-197.
3. J.C. Bermond and C. Thomassen, *Cycles in digraphs — a survey*, J. Graph Theory **5** (1981), 1-43.
4. H. Meyniel, *Une condition suffisante d'existence d'un circuit hamiltonien dans un graphe oriente*, J. Combinatorial Theory **B14** (1973), 137-147.