GRAPHIC SUBSEQUENCES

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Abstract. Given a sequence S: $d_1, d_2, \ldots, d_p$ of non-negative integers, we give necessary and sufficient conditions for a subsequence of S with $p-1$ terms to be graphic.

1. Introduction.

Throughout this paper S: $d_1 \geq d_2 \geq \ldots \geq d_p$ denotes a non-increasing sequence of non-negative integers. For $k = 1, 2, \ldots, p$ we define $L(S, k) = \{i: 1 \leq i < k \text{ and } d_i \geq k-1\}$, $R(S, k) = \{i: k < i \leq p \text{ and } d_i \geq k\}$ and $\overline{d}_k = |L(S,k)| + |R(S,k)|$. Following [1], we call the sequence $\overline{S}$: $\overline{d}_1, \overline{d}2, \ldots, \overline{d}_p$ the corrected conjugate sequence of S. If there exists a simple graph G on p vertices, say $v_1, v_2, \ldots, v_p$ such that $\deg_G(v_i) = d_i$, for $i = 1, 2, \ldots, p$ then S is said to be graphic and the graph G is said to be a realization of S. For our convenience we denote the union of two sets A and B by $A + B$. And follow the usual convention that if $m < n$, then $\{i: n \leq i \leq m\} = \emptyset$ and $\sum_{i=n}^{m}(\ldots) = 0$.

A theorem of P. Erdos and T. Gallai [2] characterizes graphic sequences as follows; see Berge [1].

Theorem A. *(Erdos and Gallai): The following statements are equivalent.*

(1) *S is graphic.*

(2) $\sum_{i=1}^{p} d_i$ *is even; and* $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{p} \min(d_j, k)$, *for* $k = 1, 2, \ldots, p$.

(3) $\sum_{i=1}^{p} d_i$ *is even; and* $\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \overline{d}_i$, *for* $k = 1, 2, \ldots, p$.

In [3], Johnson proves that if S is graphic then the sequence $d_1 \geq d_2 \geq \ldots \geq d_p \geq d$ is graphic iff d is a non-negative even integer. In [4], Reid extended this result by giving necessary and sufficient conditions for a sequence $d_1 \geq \ldots \geq d_k \geq d \geq d_{k+1} \geq \ldots \geq d_p$ (obtained from S by inserting d) to be graphic.

In this short note, we give necessary and sufficient conditions for the subsequence $S(j)$: $d_1 \geq \ldots \geq d_{j-1} \geq d_{j+1} \geq \ldots \geq d_p$ of S (obtained from S by deleting d_j) to be graphic.

2. Graphic subsequences.

Notation: In S: $d_1 \geq \ldots \geq d_p$, let m be the largest integer such that $d_m \geq m-1$; so $m \leq d_1 + 1$.

Theorem 1. *Let m be defined as above. If $j \geq m+1$, then $\boldsymbol{S}(j)$ is graphic iff*

(4) *the sum of the terms in $\boldsymbol{S}(j)$ is even;*

(5)

$$\sum_{i=1}^{k} \left(\overline{d}_i - d_i\right) \geq \begin{cases} k, & \text{if} \quad 1 \leq k \leq d_j; \\ d_j, & \text{if} \quad d_j + 1 \leq k \leq j-1; \\ d_{k+1}, & \text{if} \quad j \leq k \leq p-1. \end{cases}$$

Proof: Let $c_i = d_i$ for $i = 1, 2, \ldots, j-1$; and $c_i = d_{i+1}$ for $i = j, j+1, \ldots, p-1$, so that $\boldsymbol{S}(j)$ is the sequence $c_1 \geq c_2 \geq \ldots \geq c_{p-1}$.

By the Erdos-Gallai theorem, $\boldsymbol{S}(j)$ is graphic iff (4) holds and $\sum_{i=1}^{k}(\overline{c}_i - c_i) \geq 0$, for $k = 1, 2, \ldots, p-1$. So, we express $\overline{c}_k$ in terms of $\overline{d}_k$. Since, $j \geq m+1$ and $d_{m+1} \leq m-1$, we first observe that $d_j \leq d_{m+1} \leq m-1 \leq j-2$.

(6) $\overline{d}_k = \overline{c}_k + 1$, if $1 \leq k \leq d_j$.

$$L(\boldsymbol{S}, k) = L(\boldsymbol{S}(j), k), \text{ since } d_i = c_i, \text{ if } 1 \leq i \leq k.$$

Next,

$$\begin{aligned} R(\boldsymbol{S}, k) &= \{i\colon k < i \leq j \text{ and } d_i \geq k\} + \{i\colon j < i \leq p \text{ and } d_i \geq k\} \\ &= \{k+1, \ldots, j\} + \{i\colon j < i \leq p \text{ and } d_i \geq k\}, \quad \text{since } d_j \geq k. \\ R(\boldsymbol{S}(j), k) &= \{i\colon k < i \leq j-1 \text{ and } c_i \geq k\} + \{i\colon j \leq i \leq p-1 \text{ and } c_i \geq k\} \\ &= \{k+1, \ldots, j-1\} + \{i\colon j \leq i \leq p-1 \text{ and } d_{i+1} \geq k\}, \\ &\qquad \text{since } c_{j-1} = d_{j-1} \geq k. \end{aligned}$$

But

$$|\{i\colon j < i \leq p \text{ and } d_i \geq k\}| = |\{i\colon j \leq i \leq p-1 \text{ and } d_{i+1} \geq k\}|.$$

Hence,

$$|R(\boldsymbol{S}, k)| = |R(\boldsymbol{S}(j), k)| + 1,$$

and we have (6).

(7) $\overline{d}_k = \overline{c}_k$, if $d_j + 1 \leq k \leq j-1$.

As in (6),

$$L(\boldsymbol{S}, k) = L(\boldsymbol{S}(j), k).$$

Next,

$$\begin{aligned} R(\boldsymbol{S}, k) &= \{i\colon k < i \le p \text{ and } d_i \ge k\} \\ &= \{i\colon k < i \le j-1 \text{ and } d_i \ge k\}, \text{ since } d_j \le k-1 \\ &= \{i\colon k < i \le j-1 \text{ and } c_i \ge k\} \\ &= \{i\colon k < i \le p-1 \text{ and } c_i \ge k\}, \text{ since } c_j = d_{j+1} \le k-1 \\ &= R(\boldsymbol{S}(j), k)\,. \end{aligned}$$

We, thus, have (7).

(8) $\overline{d}_k = \overline{c}_k$, if $j \le k \le p-1$.

$$\begin{aligned} L(\boldsymbol{S}, k) &= \{i\colon 1 \le i \le j-1 \text{ and } d_i \ge k-1\}, \text{ since } d_j \le j-2 \le k-2 \\ &= L(\boldsymbol{S}(j), k)\,, \text{ since } c_i = d_i \text{ if } 1 \le i \le j-1\,. \end{aligned}$$

Next,

$$R(\boldsymbol{S}, k) = \emptyset = R(\boldsymbol{S}(j), k)\,, \text{ since } c_{k+1} \le d_{k+1} \le d_j \le j-2 \le k-2\,.$$

So, we have (8).

Now, by using (6), (7) and (8) one easily finds that

$$\sum_{i=1}^{k} (\overline{c}_i - c_i) = \begin{cases} \sum_{i=1}^{k} \left(\overline{d}_i - d_i\right) - k, & \text{if } 1 \le k \le d_j; \\ \sum_{i=1}^{k} \left(\overline{d}_i - d_i\right) - d_j & \text{if } d_j + 1 \le k \le j-1; \\ \sum_{i=1}^{k} \left(\overline{d}_i - d_i\right) - d_{s+1}, & \text{if } j \le k \le p-1. \end{cases}$$

Therefore, $\boldsymbol{S}(j)$ is graphic iff (4) and (5) hold. ∎

Theorem 2. *Let m be defined as before. If $j \le m$, then $\boldsymbol{S}(j)$ is graphic iff*

(9) *the sum of the terms in $\boldsymbol{S}(j)$ is even;*

(10)

$$\sum_{i=1}^{k} \left(\overline{d}_i - d_i\right) \ge \begin{cases} k, & \textit{if } 1 \le k \le j-1; \\ k + d_{k+1} - d_j, & \textit{if } j \le k \le m-1; \\ (k-1) + d_{k+1} - d_j, & \textit{if } m \le k \le d_j + 1; \\ d_{k+1}, & \textit{if } d_j + 2 \le k \le p-1. \end{cases}$$

Proof: Define the sequences (c_i) and $(\overline{c}_i)$ as in Theorem 1. Since $j \le m$ we have the following equations.

(11) $\overline{d}_k = \overline{c}_k + 1$, if $1 \le k \le m-1$.

If $k \leq j$, then for every $i(1 \leq i < k)$, $c_i = d_i$ and so $L(\boldsymbol{S}, k) = L(\boldsymbol{S}(j), k)$. If $k \geq j + 1$, then $c_{k-1} = d_k \geq d_{m-1} \geq k$ and so

$$L(\boldsymbol{S}, k) = \{1, 2, \ldots, k-1\} = L(\boldsymbol{S}(j), k) .$$

Next,

$$R(\boldsymbol{S}, k) = \{i: k < i \leq m\} + \{i: m+1 \leq i \leq p \text{ and } d_i \geq k\}, \text{ since } d_m \geq k.$$
$$R(\boldsymbol{S}(j), k) = \{i: k < i \leq m-1\} + \{i: m \leq i \leq p-1 \text{ and } d_{i+1} \geq k\},$$
$$\text{since } c_{m-1} \geq d_m \geq m-1 \geq k.$$

But

$$|\{i: m \leq i \leq p-1 \text{ and } d_{i+1} \geq k\}| = |\{i: m+1 \leq i \leq p \text{ and } d_i \geq k\}|.$$

So,

$$|R(\boldsymbol{S}, k)| = |R(\boldsymbol{S}(j), k)| + 1,$$

and we have (11).

(12) $\overline{d}_m = \overline{c}_m$.

$$L(\boldsymbol{S}, m) = \{1, 2, \ldots, m-1\} = L(\boldsymbol{S}(j), m), \text{ since } c_{m-1} \geq d_m \geq m-1.$$

Next,

$$R(\boldsymbol{S}, m) = \emptyset = R(\boldsymbol{S}(j), m), \text{ since } c_{m+1} = d_{m+2} \leq d_{m+1} \leq m-1.$$

We, thus, have (12).

(13) $\overline{d}_k = \overline{c}_k + 1$, if $m+1 \leq k \leq d_j + 1$.

$$\begin{aligned}
&|L(\boldsymbol{S}(j), k)| \\
&= |\{i: 1 \leq i \leq k-1 \text{ and } c_i \geq k-1\}, \text{ since } c_{k-1} = d_k \leq d_{m+1} \leq m-1 \leq k-2, \\
&= |\{i: 1 \leq i \leq j-1 \text{ and } d_i \geq k-1\}| + |\{i: j \leq i < k-1 \text{ and } d_{i+1} \geq k-1\}| \\
&= |\{i: 1 \leq i \leq j-1 \text{ and } d_i \geq k-1\}| + |\{i: j+1 \leq i < k \text{ and } d_i \geq k-1\}| \\
&= |L(\boldsymbol{S}, k)| - 1, \text{ since } d_j \geq k-1.
\end{aligned}$$

Next,

$$R(\boldsymbol{S}, k) = \emptyset = R(\boldsymbol{S}(j), k), \text{ since } c_{k+1} \leq c_{k-1} \leq k-2$$

as above. So, we have (13).

(14) $\overline{d}_k = \overline{c}_k$, if $d_j + 2 \leq k \leq p-1$.

$$
\begin{aligned}
L(\boldsymbol{S},k) &= \{i\colon 1 \le i \le j-1 \text{ and } d_i \ge k-1\}, \text{ since } d_j \le k-2 \\
&= \{i\colon 1 \le i \le k \text{ and } c_i \ge k-1\}, \text{ since } c_i = d_i, \text{if } 1 \le i \le j-1; \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and } c_j = d_{j+1} \le k-2, \\
&= L(\boldsymbol{S}(j),k)\,.
\end{aligned}
$$

Next,

$$R(\boldsymbol{S},k) = \emptyset = R(\boldsymbol{S}(j),k)\,, \text{ since } c_{k+1} = d_{k+2} \le d_{k+1} \le k-2\,.$$

We, thus, have (14).

The rest of the proof is as in Theorem 1. ∎

3. Relationship between the graphicness of $\boldsymbol{S}$ and $\boldsymbol{S}(j)$.

It is clear from Theorem 1 and Theorem 2 that $\boldsymbol{S}(j)$ $(1 \le j \le p)$ need not be graphic even though $\boldsymbol{S}$ is graphic and d_j is even. And conversely, it is also clear from Theorem 2, that $\boldsymbol{S}$ need not be graphic even though $\boldsymbol{S}(j)$ $(1 \le j \le m)$ is graphic and d_j is even. However, it does easily follow from Theorem 1 that if $\boldsymbol{S}(j)$ is graphic and d_j is even, for some j, $m+1 \le j \le p$, then $\boldsymbol{S}$ is graphic. (This fact is also proved in Theorem 1 of Reid [4].) In this last case, we can actually construct a realization of $\boldsymbol{S}$ from a given realization of $\boldsymbol{S}(j)$ as follows.

We first note that $d_j \le j-2$, because $j \ge m+1$. Let G be an arbitrary realization of $\boldsymbol{S}(j)$ on the vertices $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_p$ such that $\deg(v_i) = d_i$. We give below an algorithm which generates $d_j/2$ independent edges, say $(u_1, u_2), (u_3, u_4), \ldots, (u_{d_j-1}, u_{d_j})$ in G. Once these independent edges are generated it is easy to construct a realization of $\boldsymbol{S}$ from G: delete the edges (u_1, u_2), $\ldots, (u_{d_j-1}, u_{d_j})$, take a new vertex v_j and join v_j with $u_1, u_2, \ldots, u_{d_j}$.

Greedy algorithm to generate $d_j/2$ independent edges in G:

Step 1: Let $n = 1$; let $G_1 := G$.

Step 2: Select an edge say $e_n = (u_{2n-1}, u_{2n})$ from G_n.
Define $G_{n+1} := G_n - \{u_{2n-1}, u_{2n}\}$ and $n := n+1$.

Step 3: If $n \le d_j/2$, then go to Step 2; if $n > d_j/2$, then STOP.

To show that this algorithm does generate $d_j/2$ independent edges it is enough if we show that $E(G_n) \neq \emptyset$ $(1 \le n \le d_j/2)$ in Step 2, so that an edge e_n can indeed be selected from G_n.

G_n is a graph obtained from G by deleting $2n-2$ vertices. So, there are at least $j-2-(2n-2)$ $(>0$, since $2n \le d_j \le j-2)$ vertices from $v_1, v_2, \ldots, v_{j-2}$ of degree at least $d_{j-2}-(2n-2)(\ge d_j - 2n + 2 > 0)$ in G_n. So, $E(G_n) \neq \emptyset$.

References

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