Stability, total vertices and hamiltonian cycles

A. Pawel Wojda and Mariusz Woźniak
Instvtut Matematyki
Akademia G6rniczo-Hutnicza
Al. Mickiewicza 30
30–059 Krak6w, Poland

Abstract. A known theorem of Bigalke and Jung says that the only nonhamiltonian, tough graph G with $\alpha(G) \leq \mathcal{H}(G) + 1$, where $\mathcal{H}(G) \geq 3$, is the Petersen graph.
In this paper we characterize all nonhamiltonian, tough graphs having k total vertices (*i. e.* adjacent to all others) with $\alpha(G) \leq k + 2$ (Theorem 3).

1. Terminology

We consider only finite undirected graphs without loops or multiple edges. For the sake of completeness we recall some definitions.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. $\omega(G)$ denotes the number of components of G. The graph G is tough if $|S| \geq \omega(G \setminus S)$ for any $S \subset V$ with $\omega(G \setminus S) > 1$. We shall denote by $\alpha = \alpha(G)$ the cardinality of a maximum set of independent vertices of G (stability) and by $\mathcal{H}(G)$ the connectivity of G. A vertex $v \in V(G)$ is called total iff v is adjacent to all remaining vertices of G.

A complete graph with n vertices is denoted by K_n as usual. Given graphs G and H, $H \subset G$ means that H is a subgraph of G, *i. e.* $V(H) \subset V(G)$ and $E(H) \subset E(G)$. If at the same time $V(H) = V(G)$, then H is a factor of G.

The star * denotes the operation of join on vertex disjoint graphs, with the convention that

$$F * G * H = (F * G) \cup (G * H),$$

where $\cup$ denotes the ordinary union of (not necessarily disjoint) graphs.

$G \dot{\bigcup} H$ stands for vertex disjoint union of the graphs G and H.

2. Results

Our work was motivated by the following theorem of Bigalke and Jung, [3].

Theorem 1. *Let G be a tough graph. If $\alpha(G) \leq \mathcal{H}(G) + 1$ and $\mathcal{H}(G) \geq 3$ then either G is hamiltonian or $\mathcal{H}(G) = 3$ and G is the Petersen graph.*

The analogous theorem for $\mathcal{H}(G) = 2$ is not true; however the class of nonhamiltonian, tough graphs with $\mathcal{H}(G) = 2$ and $\alpha(G) = 3$ can be easily characterised (Theorem 2). In order to formulate Theorem 2, as well as Theorem 3, we shall define some classes of graphs.

Definition of class $\mathcal{G}^0$: A graph $G \in \mathcal{G}^0$ iff there exist three integers n_1, n_2, n_3; $n_i \geq 3$, such that G is obtained from $K_{n_1} \dot{\bigcup} K_{n_2} \dot{\bigcup} K_{n_3}$ in the following way: in each graph K_{n_i} we choose two distinct vertices a_i, b_i and add the edges $a_i a_j$ and $b_i b_j$, $1 \leq i < j \leq 3$.

Definition of class $\mathcal{J}$: A graph $G \in \mathcal{J}$ iff there exist three integers p_1, p_2, p_3; $p_i \geq 2$, such that G is constructed from $K_{p_1} \dot{\bigcup} K_{p_2} \dot{\bigcup} K_{p_3}$ by choosing one vertex a_i in each K_{p_i} and adding the edges $a_i a_j$, $1 \leq i < j \leq 3$.

Definition of class $\mathcal{G}^1$: A graph $G \in \mathcal{G}^1$ iff G is of the form $G = K_1 * J$ for some $J \in \mathcal{J}$.

Definition of class $\mathcal{G}^k$, k>1: A graph $G \in \mathcal{G}^k$ iff $G = (K_{n_1} \dot{\bigcup} \cdots \dot{\bigcup} K_{n_{k-1}}) * K_k * J$ for some integers n_i, $i = 1, \cdots, k-1$, and some $J \in \mathcal{J}$.

Remark: The classes $\mathcal{G}^0$ and $\mathcal{G}^k$ already occur in the literature cf. [2]-[7].

Now we can formulate

Theorem 2. *Let G be a tough graph with $\mathcal{H}(G) = 2$ and $\alpha(G) = 3$. Then either G is hamiltonian or $G \in \mathcal{G}^0$ or G is a factor of a graph $G' \in \mathcal{G}^1$.*

This theorem is an immediate consequence of a theorem of Jung [4,Theorem 1].

Our main result is the following

Theorem 3. *Let G be a tough graph having k total vertices, $k \geq 1$. If $\alpha(G) \leq k+2$ then either G is hamiltonian or $G \in \mathcal{G}^k$.*

Remark: The following graph $\tilde{G}$ of Tietz shows that the condition in Theorem 3 that G has k total vertices cannot be replaced by the weaker condition that $\mathcal{H}(G) \geq k$. Let G_p be the Petersen graph and let $x \in V(G_p)$. Denote by x_1, x_2, x_3 the vertices adjacent to x. $\tilde{G}$ is constructed from G_p by replacing the vertex x by K_3 and the edges xx_i, $i = 1, 2, 3$, by the edges ax_1, bx_2, cx_3 where $\{a, b, c\} = V(K_3)$.

Let us mention a related result of D. Amar, I. Fournier and A. Germa, [1].

Theorem 4. *If $\mathcal{H}(G) \geq 2$ and $\alpha(G) = \mathcal{H}(G) + 2$, then there is a longest cycle C of G such that $\alpha(G \setminus V(C)) \leq 2$.*

3. Lemmas

Let $P = [a, b]$ be a path of a graph $G = (V, E)$ with ends a and b. We denote by $\overrightarrow{P}$ the orientation of P from a to b. This orientation defines the relation of order in $V(P)$ (denoted by $\leq$). Let x, y be two vertices on P such that $x \leq y$. We denote by $x \overrightarrow{P} y$ the consecutive vertices on P from x to y and by $y \overleftarrow{P} x$ the same vertices but in reverse order. We use also the notation x^+ and x^- for the successor and the predecessor of x on P with respect to $\overrightarrow{P}$ (if it exists).

A path P is said to be complete if the subgraph of G induced by $V(P)$ is complete. In particular, for any $a \in V$, a is considered as a complete path.

For a connected graph G we denote by $s(G)$ the minimum number of disjoint paths of G covering G. Let $\mathcal{M}(G)$ be the set of all path-coverings $(P_1, \cdots, P_s)$ of G with minimal number of elements *i. e.* $s = s(G)$. We put

$$M = \{(p_1 \cdots, p_s) \in N^s : \text{ there exists } (P_1, \cdots, P_s) \in \mathcal{M} \text{ with } p_i = |P_i|,\ i = 1, \cdots, s\}.$$

The path-covering $(P_1, \cdots, P_s)$ is said to be extremal iff $(|P_1|, \cdots, |P_s|)$ is a maximal element of M with respect to the standard lexicographic order in N^s.

Lemma 5. *Let $(P_1, \cdots, P_s)$ be an extremal path-covering of a connected graph $G = (V, E)$, $P_i = [a_i, b_i]$, $i = 1, \cdots, s$. If $s \geq 2$ then the vertices a_1 and b_1 are not adjacent.*

Proof: It is easily seen that $|V(P_1)| \geq 3$. Suppose that $a_1 b_1 \in E$. Since G is connected, there exists an edge $xy \in E$ with $x \in V(P_1)$ and $y \in V(P_i)$, $i \neq 1$. If, e. g. $x \neq a_i$ and $y \neq b_i$ then the paths P_1, P_i can be replaced by $P_1' = a_i \overrightarrow{P}_i y x \overrightarrow{P}_1 b_1 a_1 \overrightarrow{P}_1 x^-$ and $P_i' = y^+ \overrightarrow{P}_i b_i$ with $|P_1'| > |P_1|$, a contradiction. The remaining cases we leave to the reader. ∎

Corollary 6. *If $(P_1 \cdots P_s)$ is an extremal path-covering of a connected graph G and $s \geq 2$ then there is no cycle with vertex set $V(P_1)$.* ∎

Lemma 7. *Let $(P_1, \cdots, P_s)$ be an extremal path-covering of a connected graph G. $P_i = [a_i, b_i]$, $i = 1, \cdots, s$. If $s \geq 2$ then the set $\{a_1, b_1, a_2, \cdots, a_s\}$ is independent.* ∎

Corollary 8. *For a connected graph G we have: If $s(G) \geq 2$ then $\alpha(G) \geq s(G) + 1$. If $\alpha(G) = s(G)$ then $\alpha(G) = 1$ i. e. G is complete.* ∎

Lemma 9. *Let G be a connected graph with $\alpha(G) = s(G) + 1$, $s(G) \geq 2$ such that $\omega(G \setminus \{v\}) \leq s(G)$ whenever $v \in V(G)$. Then $\alpha(G) = 3$ and $G \in \mathcal{J}$.*

Proof: Let $(P_1, \cdots, P_s)$ be an extremal path-covering of G, $P_i = [a_i, b_i]$, $i = 1, \cdots, s$. By Lemma 7 the set $A = \{a_1, b_1, a_2, \cdots, a_s\}$ is independent and contains $s + 1 = \alpha(G)$ elements. Thus, for $i \neq 1$, the vertex b_i must be connected by an edge with A (if $b_i \notin A$). By the extremality of the covering we have only one possibility i. e. $a_i b_i \in E$. Suppose now that there exist vertices $x \in P_i$ and $y \in P_j$, $1 < i < j \leq s$, such that $xy \in E$. Proceeding similarly as in the proof of Lemma 5 and using the fact that the vertices a_i, b_i are adjacent, we get a contradiction. Hence

$$\text{for } 1 < i < j \leq s \text{ the paths } P_i, P_j \text{ are not connected by an edge.} \tag{1}$$

Let x, y be two vertices on P_i, $i \neq 1$, and suppose that $xy \notin E$. Then, by (1) and the maximality of $|V(P_1)|$ the set $\{a_1, b_1, a_2, \cdots, a_{i-1}, x, y, a_{i+1}, \cdots, a_s\}$ is independent and has $s+2$ elements, a contradiction. Thus

$$\text{for } i \neq 1 \text{ the paths } P_i \text{ are complete.} \tag{2}$$

Let $x, y \in V(P_1)$, $x > a_1^+$, $y < b_1^-$. We shall show that if $a_1 x \in E$ then $a_1 x^- \in E$ and if $y b_1 \in E$ then $y^+ b_1 \in E$. Indeed, let $a_1 x \in E$. By Corollary 6 the edge $b_1 x^- \notin E$; otherwise $x a_1 \overrightarrow{P}_1 x^- b_1 \overleftarrow{P}_1 x$ would be a cycle with vertex set $V(P_1)$. The vertex x^- is not adjacent to any vertex a_i, $i \neq 1$; otherwise the path $a_i x^- \overleftarrow{P}_1 a_1 x \overrightarrow{P}_1 b_1$ would be longer than P_1. Since $\alpha(G) = s+1$, the set $\{a_1, x^-, b_1, a_2, \cdots, a_s\}$ is not independent, implying that x^- must be adjacent to the vertex a_1. By symmetry, $y^+ b_1 \in E$ whenever $y b_1 \in E$.

Denote by a_0 the last vertex on P_1 adjacent to a_1 and by b_0 the first vertex on P_1 adjacent to b_1 (with respect to the orientation $\overrightarrow{P}$). Then $a_1 x \in E$ for $a_1 < x \leq a_0$ and $y b_1 \in E$ for $b_0 \leq y < b_1$. Proceeding similarly as in the proof of (2) and using Corollary 6 it is easy to show that

$$\begin{aligned} &a_0 \leq b_0 \text{ and for all vertices } x \text{ and } y \text{ of } P_1 \text{ we have} \\ &\text{if } a_1 \leq x < a_0 \text{ and } b_0 < y \leq b_1 \text{ then } xy \notin E \end{aligned} \tag{3}$$

Now we shall show that

$$\begin{aligned} &\text{if } a_1 \leq x < a_0 \text{ then there is no edge between } x \text{ and } P_i,\ i \neq 1, \\ &\text{if } b_0 < y \leq b_1 \text{ then there is no edge between } y \text{ and } P_i,\ i \neq 1. \end{aligned} \tag{4}$$

Indeed, suppose e. g. that $xz \in E$, where $z \in V(P_i)$, $i \neq 1$. Recall that, by (2), P_i is complete. Then the paths P_1, P_i can be replaced by one path $z^+ \overrightarrow{P}_i b_i a_i \overrightarrow{P}_i z x \overleftarrow{P}_1 a_1 x^+ \overrightarrow{P}_1 b_1$ (if $z \neq b_i$), contradicting the minimality of the number of paths.

Suppose now that $a_0 = b_0$. We know from (1) that there is no edge between P_i and P_j for $2 \leq i < j \leq s$. Since G is connected, the paths P_i with $i \neq 1$ must be connected with P_1 and by (4) only the vertex a_0 can be connected with some path P_i. As is seen from (3), in this case the graph $G \setminus \{a_0\}$ has $s+1$ components. From the assumptions of the lemma, we conclude that $a_0 \neq b_0$, i. e., by (3),

$$a_0 < b_0. \tag{5}$$

Let c be the last vertex on P_1 connected by an edge, cy say, with some P_i, $i \neq 1$, $y \in V(P_i)$. By (4) we have $a_0 \leq c \leq b_0$. Assume first that $c^- \neq a_1$. We shall show that c^- is not connected with any path P_j, $j \neq 1$. Suppose that there exists a vertex y_1 such that $c^- y_1 \in E$ with $y_1 \in V(P_j)$. If $i = j$ and e. g. $y \neq y_1$, then the path $a_1 \overrightarrow{P}_1 c^- y_1 y c \overrightarrow{P}_1 b_1$ is longer then P_1. If $i \neq j$ then the paths P_1, P_i, P_j may be replaced by two paths with vertex sets $a_1 \overrightarrow{P}_1 c^- \cup V(P_j)$ and $c \overrightarrow{P}_1 b_1 \cup V(P_i)$. In both cases we obtain a contradiction with the definition of an extremal path-covering.

However the vertex c^- must be adjacent to at least one vertex of the independent set $\{a_1, b_1, a_2, \cdots, a_s\}$. We obviously have $a_1 c^- \in E$ and, by definition of a_0, $c^- \leq a_0$. From the definition of the vertex c and from (4) it follows that there exists only one vertex on P_1 (namely c) connected with each P_i, $i \neq 1$. Moreover $a_0 \leq c \leq a_0^+$ and for reasons of symmetry $b_0^- \leq c \leq b_0$. Suppose, e. g. $c = a_0$. Since the number of components of the graph $G \setminus \{c\}$ must be $\leq s$, it exists an edge, xy say, with $a_1 \leq x < c$ and $b_0 \leq y \leq b_1$. The case $y > b_0$ is impossible by (3). If $y = b_0$ and e. g. $x \neq a_1, x \neq a_0$ then the path $b_1 \overleftarrow{P}_1 b_0 x \overleftarrow{P}_1 a_1 x^+ \overrightarrow{P}_1 c z$, with $z \in P_2$ is longer than P_1, a contradiction. Thus, by symmetry, $a_0^+ = c = b_0^-$, and the above argument can be used to show that $a_0 b_0 \in E$ since $a_0 b_0$ is the only possible edge connecting $a_1 \overrightarrow{P} a_0$ and $b_0 \overrightarrow{P} b_1$. Similarly one may prove that the case $c^- = a_1$ is impossible.

Now it is easy to show that $s = 2$ $(\alpha = 3)$. Otherwise the three paths P_1, P_2, P_3 could be replaced by two paths: a path with vertex set $V(P_2) \cup \{c\} \cup V(P_3)$ and $a_1 \overrightarrow{P}_1 a_0 b_0 \overrightarrow{P}_1 b_1$. Finally G is a factor of a graph $G' \in \mathcal{J}$ and, since $\alpha(G) = 3$, G and G' must coincide. This completes the proof. ∎

4. Proof of Theorem 3

Let G be a nonhamiltonian, tough graph with $\alpha(G) \leq k+2$ and let $X = \{x_1, \cdots, x_k\}$ denote the set of total vertices of G, $k \geq 1$.

We evidently have $\mathcal{H}(G) \geq k$. If $\mathcal{H}(G) > k$ then we have

$$\alpha(G) \leq k+2 < \mathcal{H}(G) + 2$$

and, by Theorem 1, either $\mathcal{H} = 3, \alpha = 4$ and G is the Petersen graph (which is impossible) or $\mathcal{H} = 2, \alpha = 3$ and $k = 1$. In this case, on the basis of Theorem 2, $G \in \mathcal{G}^1$.

Thus we can assume that $\mathcal{H}(G) = k$, $k \geq 2$ and $\alpha(G) = k+2$. Then X is a cut-set of G. Let us denote by $A_1, \cdots, A_r$ the components of $G \setminus X$.

Since G is tough we have

$$r \leq k. \qquad (*)$$

Let $s_i = s(A_i)$ denote the minimal number of paths covering A_i and let $s = s_1 + \cdots + s_r$. Thus we have s paths $P_j = [a_j, b_j]$, $j = 1, \cdots, s$, covering $\cup_{i=1}^{r} A_i$. Let us observe that if $s \leq k$ then we are able to define a hamiltonian cycle C in G as follows

$$C = x_1 a_1 \overrightarrow{P}_1 b_1 x_2 a_2 \overrightarrow{P}_2 b_2 x_3 \cdots x_s a_s \overrightarrow{P}_s b_s x_{s+1} x_{s+2} \cdots x_k x_1.$$

Thus $s = \sum_1^r s_i \geq k + 1$.

Let $\alpha_i = \alpha(A_i), i = 1, \cdots, r$. By Corollary 8 we have

$$k + 2 = \alpha(G) = \sum_1^r \alpha_i \geq \sum_1^r s_i \geq k + 1.$$

If $\sum_1^r s_i = k + 2$ then $\alpha_i = s_i$ for $i = 1, \cdots, r$ and, by Corollary 8, $\alpha_i = s_i = 1$. Thus $\sum_1^r 1 = r = k + 2$, a contradiction with (*). If $\sum_1^r s_i = k + 1$ then $\sum_1^r(\alpha_i - s_i) = 1$ and we can assume, without loss of generality, that

$$\alpha_i = s_i \text{ for } i = 1, \cdots, r - 1 \text{ and}$$
$$\alpha_r = s_r + 1.$$

By Corollary 8, $\alpha_i = s_i = 1$ and the subgraphs A_i are complete for $i = 1, \cdots, r - 1$. Moreover, the graph A_r satisfies the assumptions of Lemma 9. First observe that $k + 1 = \sum_{i \neq r} s_i + s_r = r - 1 + s_r$; thus, by (*), $s_r \geq 2$. Next, suppose there exists a vertex v of A_r such that $\omega(A_r \setminus \{v\}) \geq s + 1$. Then

$$\omega(G \setminus \{x_1, \cdots, x_k, v\}) \geq r - 1 + s_r + 1 = k + 2$$

which is impossible since G is tough. By Lemma 9, $A_r \in \mathcal{J}$. In particular we have $s_r = 2, \alpha_r = 3$; thus $r = k$. This completes the proof of Theorem 3. ∎

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