Chromatic Classes of Certain 2-connected $(n, n+2)$-Graphs

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Abstract. Let $P(G)$ denote the chromatic polynomial of a graph G. Two graphs G and H are chromatically equivalent, written $G \sim H$, if $P(G) = P(H)$. A graph G is chromatically unique if $G \cong H$ for any graph H such that $H \sim G$. Let $\mathcal{G}$ denote the class of 2-connected graphs of order n and size $n+2$ which contain a 4-cycle or two triangles. It follows that if $G \in \mathcal{G}$ and $H \sim G$, then $H \in \mathcal{G}$. In this paper, we determine all equivalence classes in $\mathcal{G}$ under the equivalence relation '$\sim$' and characterize the structures of the graphs in each class. As a by-product of these, we obtain three new families of chromatically unique graphs.

1. Introduction.

Let $P(G)$ denote the chromatic polynomial of a (simple) graph G. Two graphs G and H are chromatically equivalent, in notation : $G \sim H$, if $P(G) = P(H)$. Trivially, the relation '$\sim$' is an equivalence relation on the class of graphs. A graph G is chromatically unique if $G \cong H$ for any graph H such that $H \sim G$; that is, $\langle G \rangle = \{G\}$ (up to isomorphism) where $\langle G \rangle$ denotes the equivalence class determined by G under '$\sim$'.

Chao and Zhao studied in [4] the structures of certain connected graphs of order n and size $n+2$ which contain no end vertices via their chromatic polynomials. Let $\mathcal{G}$ denote the class of 2-connected graphs of order n and size $n+2$ which contain a 4-cycle or two triangles. It follows that (see Lemma 7) if $G \in \mathcal{G}$ and $H \sim G$, then $H \in \mathcal{G}$. By applying certain results of Chao and Zhao [4], we shall determine in this paper all equivalence classes in $\mathcal{G}$ under '$\sim$' and characterize the structures of the graphs in each class. As a by-product of these, we obtain three new families of chromatically unique graphs.

Throughout this paper, all graphs are assumed to be connected. The reader may refer to [2],[6], for all notation and terminology not explained here.

[1] This work was done while the author was visiting the Department of Mathematics, National University of Singapore.

2. Basic Results.

To begin with, we shall state in this section a few basic results which will be useful to us. Of fundamental importance is the following.

Lemma 1. (Reduction Theorem [1]). *Let G be a graph and e an edge of G. Then*

$$P(G) = P(G-e) - P(G\cdot e),$$

where $G-e$ is the graph obtained from G by deleting e, and $G\cdot e$ is the graph obtained from G by contracting the end vertices of e and removing all but one of the multiple edges, if they arise.

The following result of Zykov [9] provides a shortcut for calculating chromatic polynomials. As usual, the symbol 'λ' is used to denote the variable in each chromatic polynomial.

Lemma 2.

Suppose G_1 and G_2 are graphs each containing a complete subgraph K_r with r vertices. Let G be the graph obtained from $G_1 \cup G_2$ by identifying the two subgraphs K_r. Then

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)} = \frac{P(G_1)P(G_2)}{\lambda(\lambda-1)\dots(\lambda-r+1)}.$$

When $r=2$ in Lemma 2 and when G_1 and G_2 are of order at least three, we call the graph G an edge-gluing of G_1 and G_2. By Lemma 2, all edge-gluings of G_1 and G_2 are chromatically equivalent.

Let $G^{(0)}$ be a given graph which is an edge-gluing of some graphs, say G_1 and G_2. Forming another edge-gluing of G_1 and G_2, we obtain a new graph $G^{(1)}$. Note that $G^{(1)}$ may not be isomorphic to $G^{(0)}$. Clearly, $G^{(1)}$ is an edge-gluing of some graphs, say H_1 and H_2. Note that H_1 and H_2 may not be G_1 and G_2. Forming another edge-gluing of H_1 and H_2, we obtain another graph $G^{(2)}$. The process of forming $G^{(1)}$ from $G^{(0)}$ (or $G^{(2)}$ from $G^{(1)}$) is called an elementary operation. A graph G is called a relative of H if G can be obtained from H by applying a finite sequence of elementary operations. It follows from Lemma 2 that if G is a relative of H, then $G \sim H$.

The next result, due to Whitney [8], has profound consequences in the study of chromatic polynomials. In order to understand this result, we need the concept of a broken cycle. Let G be a graph with n vertices and m edges, together with a bijection $\alpha : E(G) \rightarrow \{1,2,\dots,m\}$. Let C be a cycle of G and e an edge of C such that $\alpha(e) \geq \alpha(x)$ for all $x \in E(C)$. Then the path $C-e$ in G is called a broken cycle induced by α.

Lemma 3. (Whitney's Broken Cycle Theorem). *Let G be a graph with n vertices and m edges together with a bijection $\alpha : E(G) \to \{1,2,\dots,m\}$. Then*

$$P(G) = \sum_{i=0}^{n-1}(-1)^i h_i \lambda^{n-i},$$

where h_i is the number of spanning subgraphs of G that have i edges and that contain no broken cycles induced by α.

From the above lemma, one can derive the following easily.

Lemma 4. *Let G be a graph with n vertices and m edges. Then in the polynomial $P(G)$, the coefficient of*

(i) λ^n *is 1;*
(ii) λ^{n-1} *is* $-m$*;*
(iii) λ^{n-2} *is* $\binom{m}{2} - t_1(G)$*, where* $t_1(G)$ *is the number of triangles in* G.

In addition to the above, Farrell [5] provided explicit expressions for the coefficients of the next two terms : λ^{n-3} and λ^{n-4}. However the coefficient of λ^{n-4} is too complicated to be useful.

Lemma 5. *Let G be as in Lemma 4. Then the coefficient of λ^{n-3} in $P(G)$ is*

$$-\binom{m}{3} + (m-2)t_1(G) + t_2(G) - 2t_3(G),$$

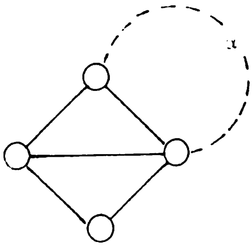
where $t_2(G)$ is the number of pure C_4 (i.e. C_4 without chords), and $t_3(G)$ the number of the complete graph K_4 in G.

If a graph G contains a cut-vertex, then it is easy to see that $(\lambda-1)^2 | P(G)$. Whitehead and Zhao [7] showed that the converse is also true.

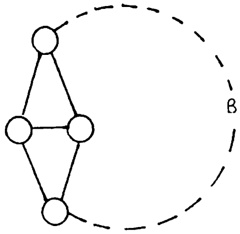
Lemma 6. *A graph G contains a cut-vertex if and only if $(\lambda-1)^2 | P(G)$.*

A graph is 2-connected if it contains no cut-vertices. The following immediate consequence of Lemma 4, 5 and 6 provides some simple necessary conditions for two graphs to be chromatically equivalent.

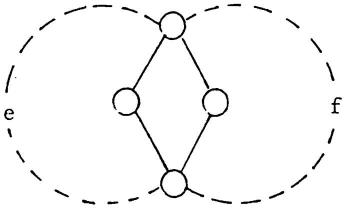
Lemma 7. *Let G and H be two chromatically equivalent graphs. Then G and H have respectively the same number of vertices, edges and triangles. If both G and H do not contain K_4, then they have the same number of pure C_4. Furthermore, G is 2-connected if and only if H is so.*



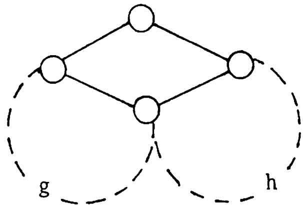
(1) $\alpha \geq 2$



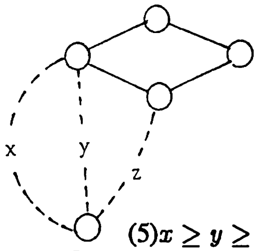
(2) $\beta \geq 2$



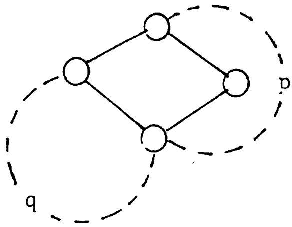
(3) $f \geq e \geq 2$



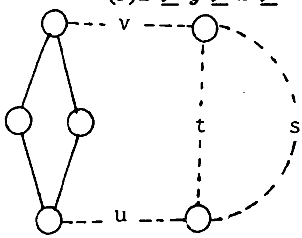
(4) $h \geq g \geq 2$



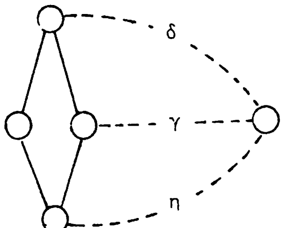
(5) $x \geq y \geq z \geq 1$



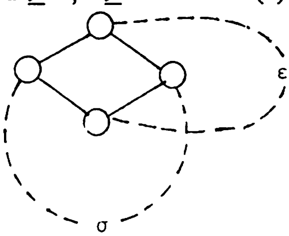
(6) $p, q \geq 2$



(7) $s \geq 2, s \geq t \geq 1, u \geq 1, v \geq 0$



(8) $\delta \geq \eta \geq 1, \gamma \geq 1$



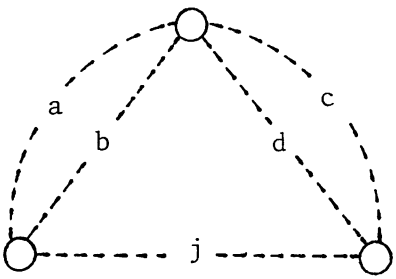
(9) $\sigma \geq \epsilon \geq 2$

3. Classification of Graphs in $\mathcal{G}$.

Recall that $\mathcal{G}$ is the class of 2-connected graphs of order n and size $m = n+2$ which contain either a 4-cycle C_4 or two triangles C_3. Let $G \in \mathcal{G}$. Then the total degrees of vertices in G is $2(n+2)$ and $deg(v) \geq 2$ for each v in G. If G contains K_4 as a subgraph, then it can be shown that $G \cong K_4$, which is chromatically unique. Assume that G does not contain K_4 as a subgraph. Then by exhaustion, it can be checked that G must be one, or a relative of one, of graphs (1) - (9).

For convenience, we denote the graphs (1) - (9) by $G_1(\alpha)$, $G_2(\beta)$, $G_3(e, f)$, $G_4(g,h)$, $G_5(x,y,z)$, $G_6(p,q)$, $G_7(s,t,u,v)$, $G_8(\delta,\gamma,\eta)$ and $G_9(\sigma,\epsilon)$ respectively. From time to time, we may simply use $G_1, G_2, \cdots, G_9$ respectively to denote the above-mentioned graphs if no confusions arise from doing so. We also say that G_i is a graph of type (i).

Note that each of the graphs G_1 and $G_i (i = 3,4,\ldots,7)$ is chromatically equivalent to a graph of the following form (see Chao and Zhao [4]) :



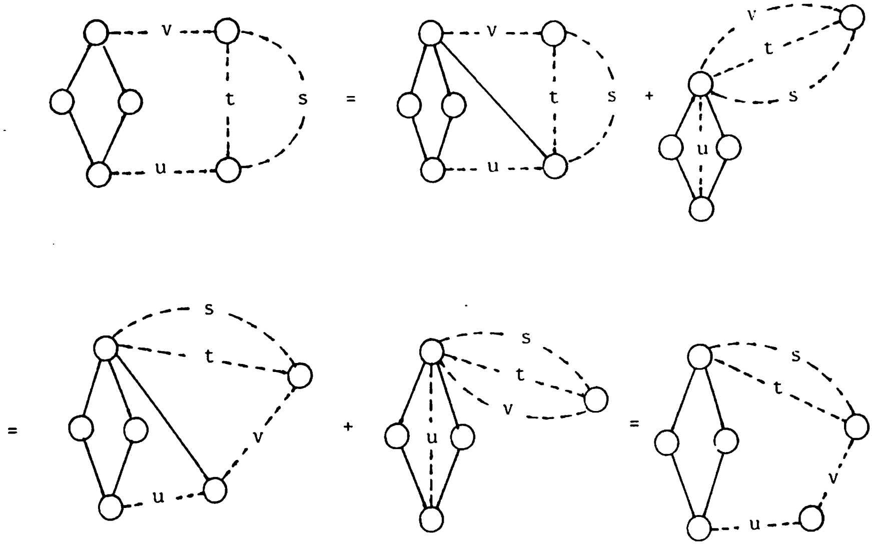
$X(a,b,c,d,j)$

where the internal path lengths a, b, c, d, j are given in the following table (for G_7, see the remark below) :

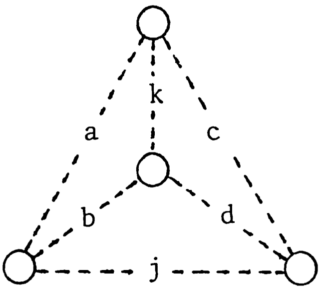
Graph	a	b	c	d	j
G_1	2	1	α	1	1
G_3	e	2	f	2	0
G_4	g	1	h	1	2
G_5	x	y	3	1	z
G_6	p	2	q	1	1
G_7	s	t	2	2	$u+v$

Remark.

The fact that $G_7 = X(s,t,2,2,u+v)$ is not obvious. The following proof is due to Chao and Zhao [4]. By Lemma 1, we have



The graphs G_2, G_8 and G_9 are homeomorphs of K_4, which can be represented as follows :



$K_4(a, b, c, d, k, j)$

where the internal path lengths a, b, c, d, k, j are given in the following table

Graph	a	b	c	d	k	j
G_2	1	β	1	1	1	1
G_8	δ	1	η	1	γ	2
G_9	1	ϵ	σ	1	1	1

Our purpose here is to establish the following result.

Theorem. *For a graph G, let $\langle G\rangle = \{H \mid H$ is a graph and $H \sim G\}$. Then*

(1) $H \in \langle G_1\rangle$ *if and only if H is a relative of G_1.*
(2) *The graph G_2 is chromatically unique (see [3],[4]).*
(3) (i) *The graph $G_3(e,f)$ is chromatically unique provided that $e \neq 3$ or $f \neq 4$;*
(ii) $\langle G_3(3,4)\rangle = \{G_3(3,4), G_5(3,3,1)\}$.
(4) $H \in \langle G_4\rangle$ *if and only if H is a relative of G_4.*
(5) (i) *For $x \neq 3$ or $y \neq 3$ or $z \neq 1$, $H \in \langle G_5\rangle$ if and only if H is a relative of G_5;*
(ii) $\langle G_5(3,3,1)\rangle = \langle G_3(3,4)\rangle$.
(6) $H \in \langle G_6\rangle$ *if and only if H is a relative of G_6.*
(7) $H \in \langle G_7(s,t,u,v)\rangle$ *if and only if $H = G_7(s,t,u',v')$ with $u' + v' = u + v$*
(8) *The graph G_8 is chromatically unique.*
(9) *The graph G_9 is chromatically unique.*

4. Proof of the Main Theorem.

Let H be a graph. Chao and Zhao [4] showed that if H is chromatically equivalent to any one of $G_1, G_3, G_4, \ldots, G_7$, then H must be of the form $X(a,b,c,d,j)$. Hence H must be of type (i) where $i = 1$ or $i = 3,4,\ldots,7$ by Lemma 7. Likewise, if H is chromatically equivalent to one of G_2, G_8, G_9, then H must be of type (2), (8), or (9).

Chao and Zhao also showed that if we first let $w = \lambda - 1$ in the chromatic polynomials of $X(a,b,c,d,j)$ and $K_4(a,b,c,d,k,j)$, and next multiply the polynomials by $(w+1)^2$, then the coefficients of the resulting polynomials can be explicitly expressed in terms of the internal path lengths between the vertices of degree greater than two. For out purpose, only partial sums of the polynomials are given below.

$$\begin{aligned}(w+1)^2 P(X(a,b,c,d,j)) = {} & w^m + \cdots + \sum_{i\in I}(-1)^{m-i} w^{i+2} \\ & + \sum_{i\in J}(-1)^{m-i-1} w^{i+1} + (-1)^m w^3 \\ & + 2(-1)^{m-1} w^2 + (-1)^m w,\end{aligned}$$

where $I = \{a,b,c,d,j\}$ and $J = \{a,b,c,d\}$.

$$(w+1)^2 P(K_4(a,b,c,d,k,j))$$
$$= w^m + \cdots + (-1)^{m-a-d} w^{a+d+1} + (-1)^{m-b-c} w^{b+c+1} + (-1)^{m-j-k} w^{j+k+1} + \sum_{i \in L} (-1)^{m-i} w^{i+2} + \sum_{i \in L} (-1)^{m-i-1} w^{i+1} + (-1)^m w^3 + 3(-1)^{m-1} w^2 + 2(-1)^m w,$$

where $L = \{a,b,c,d,k,j\}$.

In the sequel, we shall use $[w^\ell]P(G)$ to denote the coefficient of $[w^\ell]$ in the polynomial $(w+1)^2 P(G)$ obtained from $P(G)$ in the above manner.

We are now ready to prove our theorem.

Proof of Theorem:

2) The graph G_2 has been shown to be chromatically unique in [3],[4].

1) Let H be a graph such that $H \sim G_1$. By Lemma 7, H is a 2-connected graph of order n and size $n+2$, and with at least two C_3. By 2), H must be a relative of G_1.

3) Let $G = G_3(e,f)$ and let H be a graph such that $H \sim G$. If $e = 2$, then H has at least 3 pure C_4. It is not hard to check that the only ways to arrange 3 pure C_4 in a graph with total degrees $2n+4$ are as in $G_3(2,f), G_6(2,f)$ and $G_9(2,f)$. By considering the broken cycles of length $f+1$ in each graph, it is easy to see that these three graphs are not chromatically equivalent to one another. Hence $G_3(2,f)$ is chromatically unique.

Assume that $f \geq e \geq 3$. Then H is 3-colourable (since $G_3(e,f)$ is) and hence it contains no K_4. By Lemma 7, H has exactly one pure C_4. Thus H must be of type (3), (4), ..., or (7). If $H = G_3(e',f')$, then it is not hard to see by using Whitney's Broken Cycle Theorem that $e = e'$ and $f = f'$; that is $H \cong G_3(e,f)$. We next show that $P(G_3) \not\sim P(G_i)$ for all $i = 4,5,\ldots,7$, except when $e = 3$, $f = 4$.

Since

$$[w^2]P(G_3) = 2(-1)^{m-1} + (-1)^{m-j} = 2(-1)^{m-1} + (-1)^m$$

and

$$[w^2]P(G_4) = 2(-1)^{m-1} + (-1)^{m-b-1} + (-1)^{m-d-1} = 2(-1)^{m-1} + (-1)^m + (-1)^m,$$

$$P(G_3) \not\sim P(G_4).$$

For $[w^2]P(G_7)$ to be equal to $[w^2]P(G_3)$, we must have $t = 1$. In this case, G_7 is chromatically equivalent to G_6.

Now

$$\begin{aligned}[w^3]P(G_3) &= (-1)^m + (-1)^{m-b-1} + (-1)^{m-d-1} \\ &= (-1)^m + (-1)^{m-3} + (-1)^{m-3},\end{aligned}$$

while

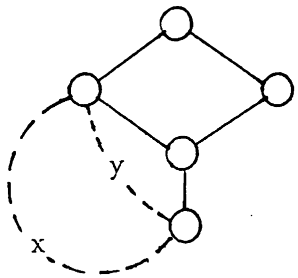
$$\begin{aligned}[w^3]P(G_6) &= (-1)^m + (-1)^{m-d} + (-1)^{m-j} + (-1)^{m-b-1} \\ &= (-1)^m + (-1)^{m-1} + (-1)^{m-1} + (-1)^{m-3}.\end{aligned}$$

Hence $P(G_6) \neq P(G_3)$ and consequently $P(G_3) \neq P(G_7)$.

Now suppose that $P(G_3) = P(G_5)$. For $[w^3]P(G_5)$ to be equal to $[w^3]P(G_3)$, we must have $z = 1$, so that the coefficient is

$$(-1)^m + (-1)^{m-d} + (-1)^{m-z} = (-1)^m + (-1)^{m-1} + (-1)^{m-1}.$$

Thus G_5 must be of the form



$(x \geq y)$

If $y < e$, then the above graph contains a broken cycle (apart from the one formed by C_4) of length y, while all other cycles of G_3 are of length greater than y, which contradicts Whitney's Broken Cycle Theorem. If $y > e$, then G_3 has a broken cycle of length $e + 1$ while the above G_5 does not, which is again a contradiction. Hence we must have $y = e$.

Since G_3 has 2 broken cycles of length $e + 1$, the above G_5 must also have 2 broken cycles of the same length. This implies that $x = y = e$ and $f = e + 1$. We consider two cases.

Case 1: Suppose $x, y > 3$. Then

$$\begin{aligned}[w^4]P(G_5) &= (-1)^{m-c-1} \\ &= (-1)^{m-4}\end{aligned}$$

while

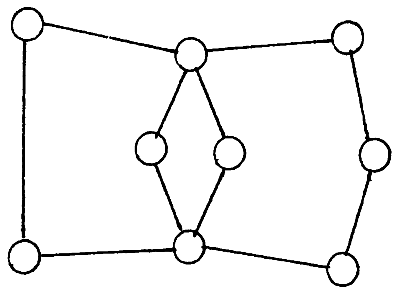
$$[w^4]P(G_3) = (-1)^{m-b} + (-1)^{m-d}$$
$$= (-1)^{m-2} + (-1)^{m-2}.$$

Hence $P(G_3) \neq P(G_5)$, a contradiction.

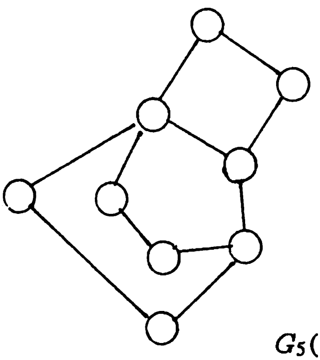
Case 2: Suppose $x = y = 3 = e$ and $f = 4$. Then by the Reduction Theorem (Lemma 1), it can be shown that the chromatic polynomials of both G_3 and G_5 are equal to

$$(w-1)(w^4+w)(w^4-w^3+2w^2-2w+1),$$

where $w = \lambda - 1$. As shown below, it is clear that $G_3 \not\cong G_5$.



$G_3(3,4)$



$G_5(3,3,1)$

4) Let H be a graph such that $H \sim G_4$. We have seen that H is not G_1, G_2 or G_3. If H is of type (4), then it is easy to see by Whitney's Broken Cycle Theorem that HH is a relative of G_4. We next show that $P(G_4) \neq P(G_i)$ for $i = 5,6,7$.

Since

$$[w^2]P(G_4) = 2(-1)^{m-1} + (-1)^{m-b-1} + (-1)^{m-d-1}$$
$$= 2(-1)^{m-1} + (-1)^{m-2} + (-1)^{m-2}$$

and

$$[w^2]P(G_5) = 2(-1)^{m-1} + (-1)^{m-d-1}$$
$$= 2(-1)^{m-1} + (-1)^{m-2},$$

we see that $P(G_4) \neq P(G_5)$. Also

$$[w^2]P(G_6) = 2(-1)^{m-1} + (-1)^{m-d-1}$$
$$= 2(-1)^{m-1} + (-1)^{m-2},$$

implies that $P(G_4) \neq P(G_6)$. Finally

$$[w^2]P(G_7) = \begin{cases} 2(-1)^{m-1} + (-1)^{m-2} & \text{if } t = 1 \\ 2(-1)^{m-1} & \text{if } t \neq 1, \end{cases}$$

which shows that $P(G_4) \neq P(G_7)$.

5) Let $G = G_5(x, y, z)$, where $x \geq y > z \geq 1$. We have seen that $P(G) \neq P(G_i)$, for $i = 1, 2, 4$ and $P(G) \neq P(G_3(e, f))$ except when $e = 3 = x = y, z = 1$ and $f = 4$.

Let $H = G_5(x', y', z')$, where $x' \geq y' > z' \geq 1$. Suppose $H \sim G$. If $y + z < y' + z'$, then $y + z < x' + z'$ and hence $y + z < x' + y'$. Thus $G = G_5(x, y, z)$ contains a broken cycle of length $y + z$ while $H = G_5(x', y', z')$ does not. Hence $P(G) \neq P(H)$, a contradiction. By symmetry, $y + z = y' + z'$. But $x + y + z = x' + y' + z'$. Hence $x = x'$. Now suppose $z < z'$. Then $x + z < x' + z'$ and $x + z < x' + y'$. Hence $G_5(x, y, z)$ contains a broken cycle of length $x + z$ while $G_5(x', y', z')$ does not. Therefore $P(G) \neq P(H)$ which is a contradiction. By symmetry, we must have $z = z'$ and so $y = y'$. That is, $G \cong H$ if $H \sim G$. However G_5 is not chromatically unique as it is an edge-gluing of C_4 and a generalized θ-graph.

Note that $G_5(x, 2, 1) \cong G_6(x, 3)$ for all $x \geq 2$.

For $y > 2$, it can be checked that

$$[w^3]P(G_5(x, y, z)) \neq [w^3]P(G_6).$$

Hence for $y > 2$, $P(G_5(x, y, z)) \neq P(G_6)$.

In order that $[w^2]P(G_5) = [w^2]P(G_7)$, we must have $t = 1$, which implies that G_7 is chromatically equivalent to G_6. Hence if $t \neq 1$ in G_7, then $P(G_5) \neq P(G_7)$.

6) If $t = 1$ in G_7, then clearly $G_7 \sim G_6$. If $t \neq 1$ in G_7, then it is easy to check that $[w^2]P(G_6) \neq [w^2]P(G_7)$. Hence $P(G_7) \neq P(G_6)$. So if $H \sim G_6$, then H must be of type (6). It is now easy to check that H is a relative of G_6.

7) We have seen in Section 3 that if $u + v = u' + v'$, then $G_7(s, t, u, v) \sim G_7(s, t, u', v')$. In fact, on studying the coefficient of each term of $(w + 1)^2 P(G_7)$,

one sees that $G_7(s,t,u,v) \sim G_7(s',t',u',v')$ if and only if $s = s'$, $t = t'$ and $u + v = u' + v'$.

8) and 9) We now consider G_8 and G_9. We recall that G_8 and G_9 are homeomorphs of K_4, which are not chromatically equivalent to $G_i(i = 1,2,\ldots,7)$. In order that $[w^2]P(G_8) = [w^2]P(G_9)$, two of δ, γ, η in G_8 must be equal to 1. Suppose $\delta = \gamma = 1$ or $\gamma = \eta = 1$. Then G_8 contains a triangle while G_9 does not. Thus $\delta = \eta = 1$. In this case, $G_8(1,\gamma,1) \cong G_9(2,\gamma)$. Hence if δ or $\eta \neq 1$, $P(G_8(\delta,\gamma,\eta)) \neq P(G_9)$.

Suppose $G_8(\delta',\gamma',\eta') \sim G_8(\delta,\gamma,\eta)$. Then by considering the coefficients $[w^{\ell+1}]P(G_8(\delta,\gamma,\eta))$ and $[w^{\ell+1}]P(G_8(\delta',\gamma',\eta'))$ for $\ell \in \{a,b,c,d,k,j\}$ (see table 2), we see that $\min\{\delta,\gamma,\eta\} = \min\{\delta',\gamma',\eta'\}$. Moreover $\{\delta,\gamma,\eta\} = \{\delta',\gamma',\eta'\}$ as multisets.

Now consider the coefficients of $w^{a+d+1} = w^{a+2}$, $w^{b+c+1} = w^{c+2}$ and $w^{j+k+1} = w^{k+3}$. We observe that if $\min\{\delta,\gamma,\eta\} = \eta$, then $\eta = \eta'$, and if $\min\{\delta,\gamma,\eta\} = \gamma$, then $\gamma = \gamma'$. By the same token, if $\min\{\delta,\gamma,\eta\} = \eta = \eta'$, then we must have $\gamma = \gamma'$ and $\delta = \delta'$. If $\min\{\delta,\gamma,\eta\} = \gamma = \gamma'$, then it is not hard to see by Whitney's Broken Cycle Theorem that $\eta = \eta'$ and $\gamma = \gamma'$. Hence we conclude that $G_8(\delta',\gamma',\eta') \cong G_8(\delta,\gamma,\eta)$. Finally, suppose $G_9(\sigma',\epsilon') \sim G_9(\sigma,\epsilon)$. Then it is easy to argue by Whitney's Broken Cycle Theorem that $\sigma = \sigma'$ and $\epsilon = \epsilon'$. Hence $G_9(\sigma',\epsilon') \cong G_9(\sigma,\epsilon)$. This completes the proof of our theorem.

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