The Intersections of Commutative Latin Squares

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Abstract

A latin square of order n is an n×n array such that each of the integers $1,2,\cdots,n$ (or any set of n distinct symbols) occurs exactly once in each row and each column. A latin square $L = [\ell_{i,j}]$ is said to be commutative provided that $\ell_{i,j} = \ell_{j,i}$ for all i and j. Two latin squares, $L = [\ell_{i,j}]$ and $M = [m_{i,j}]$, are said to have intersection k if there are exactly k cells (i,j) such that $\ell_{i,j} = m_{i,j}$.

Let $I[n] = \{0,1,2,\cdots,n^2-9,n^2-8,n^2-6,n^2\}$, $H[n] = I[n] \cup \{n^2-7,n^2-4\}$, and J[n] be the set of all integers k such that there exists a pair of commutative latin squares of order n which have intersection k. In this paper, we prove that $J[n] = I[n]$ for each odd $n \geq 7$, $J[n] = H[n]$ for each even $n \geq 6$, and give a list of J[n] for $n \leq 5$. This totally solves the intersection problem of two commutative latin squares.

1. Introduction and Definitions

The study of the intersections of two latin squares started around ten years ago. T. Webb [4] considered the latin squares which are commutative and idempotent. Since then, several results have been found in this direction. [2,3]

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A latin square of order n is an n×n array such that each of the integers 1,2,···,n occurs exactly once in each row and each column. A latin square $L = [\ell_{i,j}]$ is said to be commutative provided that $\ell_{i,j} = \ell_{j,i}$ for all i and j. L is idempotent if $\ell_{i,i} = i$ for each i, and L is called unipotent if $\ell_{i,i} = c$ for some fixed c. If $\{\ell_{1,1},\ell_{2,2},\cdots,\ell_{n,n}\}$ is just the set $\{1,2,\cdots,n\}$, then L is said to be a diagonal latin square. It is easy to see that a commutative latin square of odd order must be a diagonal latin square.

A partial latin square of order n (briefly PLS(n)) is an n×n array such that each element occurs at most once in each row and each column. A partial commutative latin square of order n can be defined similarly. Two PLS(n) are said to be comparable if the corresponding cells are either both filled or both empty. Two comparable PLS(n), P_1 and P_2, are disjoint if they don't have any entry in common in corresponding cells, and P_1 and P_2 are mutually balanced if for each row (and each column), they contain the same set of entries. Figure 1.1 is an example of two mutually balanced and disjoint PLS(3).

1	2	
2	3	1
	1	3

2	1	
1	2	3
	3	1

Figure 1.1

A PLS(u), $P = [p_{i,j}]$, is said to be embedded in a latin square of order v, if there exists a latin square $L = [\ell_{i,j}]$ of order v such that $\ell_{i,j} = p_{i,j}$ for each filled cell in P. It is well–known that a commutative latin square of odd order u can be embedded in a commutative latin square of order v for each $v \geq 2u$. [1] Figure 1.2 is an example of the embedding of a commutative latin square of order 3 into a commutative latin square of of order 7.

Two latin squares, $L = [\ell_{i,j}]$ and $M = [m_{i,j}]$, are said to

1	3	2	6	4	7	5
3	2	1	5	7	4	6
2	1	3	7	6	5	4
6	5	7	4	2	3	1
4	7	6	2	5	1	3
7	4	5	3	1	6	2
5	6	4	1	3	2	7

Figure 1.2

have intersection k, denoted by $|L \cap M| = k$, if there are exactly k cells (i,j) such that $\ell_{i,j} = m_{i,j}$. In [2], it was shown by Fu that the set of all intersections of two latin squares of order n, $n \geq 5$, is precisely $\{0,1,2,\cdots,n^2-7,n^2-6,n^2-4,n^2\}$. But, for the intersections of two commutative latin squares, the above set is not correct. In this paper, we prove that the set of all possible intersections denoted by J[n], is the set $I[n] = \{0,1,2,\cdots,n^2-9,n^2-8,n^2-6,n^2\}$ for odd $n \geq 7$ and $H[n] = I[n] \cup \{n^2-7,n^2-4\}$ for even $n \geq 6$. Moreover, we use computer to find J[n] for $n \leq 5$, and this totally solves the intersection problem of two commutative latin squares.

2. The Main Results

In what follows, without mention otherwise, we consider only commutative latin squares (or partial latin squares). By observation, there do not exist two disjoint mutually balanced partial latin squares (DMB PLS) with one, two, three or five cells filled only. Thus, we have the following lemma.

Lemma 2.1. $J[n] \subseteq H[n]$ for each n.

As to the case when n id odd, the intersections n^2-7, n^2-4 are not possible any more. We prove it in next lemma.

Lemma 2.2. $J[n] \subseteq I[n]$ for each odd n.

Proof. It suffices to show that n^2-4 and n^2-7 are not in $J[n]$. If $n^2-4 \in J[n]$, then there exists a pair of DMB PLS with four entries only. Since the latin squares are commutative, two of the four entries must be on the diagonal, and they are the same. But this is not possible for a commutative latin square of odd order. Hence we have shown that $n^2-4 \notin J[n]$. On the other case, let $n^2-7 \in J[n]$. Then, there exist two DMB PLS with seven entries only, and they must be in one of the six shapes as shown in Figure 2.1. By looking at their DMB mates in Figure 2.2, it is easy to see that (1), (2) and (3) are not possible, and the x in (4), (5) and (6) can only be d or e. In either case, we have two common elements on the diagonal of a commutative latin square of odd order. It is not possible. This implies that $n^2-7 \notin J[n]$.

a	b	c
b		d
c	d	

(1)

	d	a
d	b	e
a	e	

(2)

	b	c
b		d
c	d	a

(3)

a	d	e
d	b	
e		c

(4)

a	d	
d	b	e
	e	c

(5)

a		d
	b	e
d	e	c

(6)

Figure 2.1

x	d	d
d		y
d	y	

(1)

	a	y
a	x	a
y	a	

(2)

	y	b
y		b
b	b	x

(3)

x	b	c
b	d	
c		e

(4)

d	a	
a	x	c
	c	e

(5)

d		a
e	b	
a	b	x

(6)

Figure 2.2

From Lemma 2.1 and Lemma 2.2, in order to show that $J[n] = I[n]$ for odd n and $J[n] = H[n]$ for even n, it suffices to show that $I[n] \subseteq J[n]$ and $H[n] \subseteq J[n]$ for n is odd and even respectively.

Before we go any further, we need several definitions. Let S_n be the set of all permutations on $\{1,2,\cdots,n\}$, and we use L_σ to denote the latin square obtained by permuting its entries with $\sigma \in S_n$, i.e., if $L = [\ell_{i,j}]$, then $L_\sigma = [\sigma(\ell_{i,j})]$. For example, in Figure 2.3, $L' = L_\sigma$, where $\sigma = (123)$. A bit of reflection, we notice that $|L \cap L_\sigma| = 4$. In what follows, we will use $A + B = \{a+b\colon a \in A \text{ and } b \in B\}$.

<u>Lemma 2.3</u>. If n is odd, $n \geq 7$, and $J[n] = I[n]$, then $J[2n+1] = I[2n+1]$ and $J[2n+3] = I[2n+3]$.

<u>Proof</u>. It is well–known that a commutative latin square of order n (briefly CLS(n)), n is odd, can be embedded

L :

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

L′ :

2	3	1	4
3	2	4	1
1	4	3	2
4	1	2	3

Figure 2.3

in a CLS(2n+1). [1] Let A be a CLS(n) which is embedded in a CLS(2n+1) L. (Figure 2.4) Since A can be replaced by another CLS(n), the entries $1,2,\cdots,n$ in C can be permuted independently, so are the entries $n+1,n+2,\cdots,2n+1$ outside A. Hence we have $J[n] + \{0,n+1,2(n+1),\cdots,(n-2)(n+1),n(n+1)\} + \{0,2n+1,2(2n+1),\cdots,(n-1)(2n+1),(n+1)(2n+1)\} \subseteq J[2n+1]$. By the assumption that $J[n] = I[n]$ and $n \geq 5$ we have $I[2n+1] \subseteq J[2n+1]$. This implies that $J[2n+1] = I[2n+1]$. Similarly, a CLS(n) can be embedded in a CLS(2n+3). [1] By the same idea, we also have $J[2n+3] = I[2n+3]$.

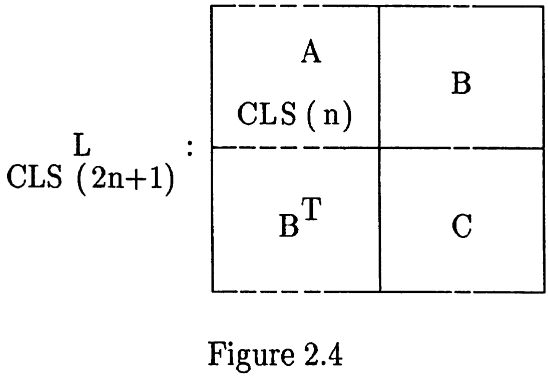

Figure 2.4

With the above lemma, we are able to handle the case when n is odd. Now we need a lemma for the case when n is even.

Lemma 2.4. If n is even, $n \geq 6$, and $J[n] = H[n]$, then $J[2n] = H[2n]$ and $J[2n+2] = H[2n+2]$.

Proof. The lemma follows directly from the fact [3] that a CLS(n), n is even, can be embedded in a CLS(u) when u = 2n or 2n+2, using the idea of the proof in Lemma 2.3.

With the two recursive constructions in Lemma 2.3 and Lemma 2.4, we can prove that $J[n] = I[n]$ for each odd $n \geq 7$, and $J[n] = H[n]$ for each even $n \geq 6$ provided that we can show $J[m] = H[m]$ and $J[k] = I[k]$ for m = 6,8,10 and k = 7,9,11,13. For completeness and convenience, we also find J[n] for $n \leq 5$.

Lemma 2.5. $J[1] = \{1\}$, $J[2] = \{0,4\}$, $J[3] = \{0,3,9\}$, $J[4] = \{0,1,2,3,4,6,8,9,12,16\}$ and $J[5] = I[5] \backslash \{7,14,16,17,19\}$.

Proof. J[1], J[2] and J[3] are easy to obtain. J[4] and J[5] are mainly the results of using the computer. See Appendix A.

Lemma 2.6. $J[6] = H[6]$.

Proof. By Appendix B.

Lemma 2.7. $J[7] = I[7]$.

Proof. Since a CLS(3) can be embedded in a CLS(7), thus with the same idea as in Lemma 2.3, we have $\{0,3,9\} + \{0,4,12\} + \{0,7,14,28\} \subseteq J[7]$, which is $\{0,3,4,7,9,10,\cdots,23,26,27,28,29,31,32,35,37,40,41,43,49\} \subseteq J[7]$. Moreover, we can permute the rows of the rectangle B in L and permute the entries 1,2,3 in C (Figure 2.4), then we have $\{0,3,9\} + \{0,8,24\} + \{0,4,12\} + \{4\} \subseteq J[7]$. This implies that $\{4,7,8,11,12,13,15,16,17,19,21,24,25,27,28,31,32,33,35,37,40,41,43,49\} \subseteq J[7]$. Combine the above two results, we have $\{0,3,4,7,8,\cdots,29,31,32,33,35,37,40,41,43,49\} \subseteq J[7]$. With Appendix C which shows that $\{1,2,5,6,30,34,36,38,39\} \subseteq J[7]$, we have $J[7] = I[7]$.

Lemma 2.8. $J[8] = H[8]$.

Proof. A CLS(4) can be embedded in a CLS(8). (Figure 2.5) Since B can be any latin square based on the set $\{5,6,7,8\}$, hence by using Lemma 2.5, we have $\{0,1,2,3,4,6,8,9,12,16\} + \{0,1,2,3,4,6,8,9,12,16\} + \{0,2,4,6,8,12,16,18,24,32\} \subseteq J[8]$. This implies that $H[8]\setminus\{55,58\} \subseteq J[8]$. Moreover, 55 and 58 can be obtained by using Appendix C, thus we have proved that $J[8] = H[8]$.

L : CLS (8)

A CLS (4)	B LS (4)
B^T	C CLS (4)

Figure 2.5

Lemma 2.9. $J[9] = I[9]$.

Proof. By the result of Webb's thesis [4], we have $\{9+2t\colon t = 0,1,2,\cdots,30,32,36\} \subseteq J[9]$. And if we consider the CLS(9) of the form as in FIgure 2.6, then we obtain $\{0,3,9\} + \{0,3,9\} + \{0,3,9\} + \{0,6,18\} + \{0,6,18\} + \{0,6,18\} \subseteq J[9]$, which is $\{3k\colon k = 0,1,2,\cdots,25,27\} \subseteq J[9]$. With the aid of the Appendix D, we are able to show that $J[9] = I[9]$.

CLS(9) :

A CLS (3)	D LS (3)	E LS (3)
D^T	B CLS (3)	F LS (3)
E^T	F^T	C CLS (3)

Figure 2.6

Lemma 2.10. J[10] = H[10].

Proof. Since a CLS(5) can be embedded in a CLS(10) (as Figure 2.5), hence by using Lemma 2.5 and the intersections of two latin squares [2], we have {0,1,2,···,6,8, ···,13,15,25} + {0,1,···,6,8,···,13,15,25} + {2t: t = 0,1, ···,19,21,25} ⊆ J[10]. This implies that {0,1,···,87,88,90,92, 100} ⊆ J[10]. By Appendix E, {89,91,93,94,96} ⊆ J[10], thus J[10] = H[10].

Lemma 2.11. J[11] = I[11].

Proof. By the idea of the proof of Lemma 2.3, we embed a CLS(5) in a CLS(11) and then we obtain J[5] + {0,6, 12,18,30} + {0,11,22,33,44,66} ⊆ J[11]. From Lemma 2.5, we have {0,1,···,109,111,121} ⊆ J[11]. And by Appendix F, we get {110,112,115} ⊆ J[11]. Moreover, 113 ∈ J[11], which is a result of [4]. Thus J[11] = H[11].

Lemma 2.12. J[13] = I[13].

Proof. By embedding a CLS(3) or CLS(5) into a CLS(13) and Lemma 2.3, and Webb's result, we are able to obtain I[13] \ {7,142,158} ⊆ J[13]. And from Appendix G, we conclude that J[13] ⊆ I[13].

As a direct result of Lemma 2.3 to 2.12, we have our main theorem:

Theorem 2.13. J[n] = I[n] for each odd n ≥ 7 and J[n] = H[n] for each even n ≥ 6.

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