On Cycle Graphs

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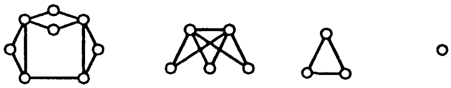
Abstract. The cycle graph $C(G)$ of a graph G has vertices which correspond to the chordless cycles of G, and two vertices of $C(G)$ are adjacent if the corresponding chordless cycles of G have at least one edge in common. If G has no cycle, then we define $C(G) = \emptyset$, the empty graph. For an integer $n \geqq 2$, we define recursively the n-th interated cycle graph $C^n(G)$ by $C^n(G) = C(C^{n-1}(G))$. We classify graphs according to their cycle graphs as follows. A graph G is cycle-vanishing if there exists an integer n such that $C^n(G) = \emptyset$; and G is cycle-periodic if there exist two integers n and $p \geqq 1$ such that $C^{n+p}(G) \cong C^n(G) \neq \emptyset$. Otherwise, G is cycle-expanding. We characterize these three types of graphs, and give some other results on cycle graphs.

1. Introduction

We consider finite graphs which have neither loops nor multiple edges. All definitions and notation not presented here may be found in [1]. Let G be a graph. A *chord* of a cycle of G is an edge of G joining two non-consecutive vertices in the cycle. If a oycle C has no chord, we say that C is *chordless*. The *cycle graph* $C(G)$ of G has vertices which correspond to all the chordless cycles of G, and two vertices of $C(G)$ are adjacent if the corresponding chordless cycles of G have at least one edge in common. Examples of cycle graphs are shown in Figure 1. If G has no cycles, then we define $C(G) = \emptyset$, the graph of order zero. We call $\emptyset$ the *empty graph* and define $C(\emptyset) = \emptyset$.

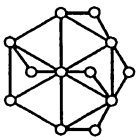


G $\quad$ $C(G)$ $\quad$ $C^2(G)$ $\quad$ $C^3(G)$

Figure 1.

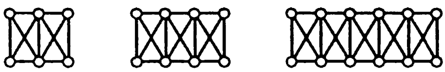
For an integer $n \geqq 2$, we define recursively the n-th interated cycle graph $C^n(G)$ by $C^n(G) := C(C^{n-1}(G))$. We may denote G by $C^0(G)$. Based on the behavior of $C^n(G)$, we can classify graphs as cycle-vanishing, cycle-periodic or cycle-expanding. A graph G is *cycle-vanishing* if there exists an integer $n \geqq 0$ such that $C^n(G) = \emptyset$; and G is *cycle-periodic* if there exist two integers $n \geqq 0$ and $p \geqq 1$ such that $C^{n+p}(G) \cong C^n(G) \neq \emptyset$ (i.e., $C^{n+p}(G)$ is isomorphic to $C^n(G)$). Otherwise, G is *cycle-expanding*. It is easy to see that G is cycle-expanding if and only if

$$\lim_{n\to\infty} |V(C^n(G))| = \infty .$$



FW $\quad$ $C(FW)$

Figure 2. An ff-wheel FW and $C(FW)$.



G $\quad$ $C(G)$ $\quad$ $C^2(G)$

Figure 3. A cycle-expanding graph R.

If G is cycle-periodic, the smallest p such that $C^{n+p}(G) \cong C^n(G)$ is called the *period* of G. Examples of a cycle-periodic graph and a cycle-expanding graph are shown in Figures 2 and 3. The notion of the cycle graph of a graph was introduced by Gervacio [2], and he obtained some results on cycle graphs. In particular, he proved the following theorem, which characterizes cycle-vanishing graphs.

Theorem A. [2] *A graph G is cycle-vanishing if and only if* $C^4(G) = \emptyset$.

Our main theorem gives a more detailed characterization of cycle-vanishing graphs together with characterizations of other graphs. In order to state our main theorem, we give some definitions and notation. A graph with no edges is called a *null graph*, and a null graph of order one is called a *trivial graph*. We denote by W_n the wheel of order $n+1$ ($n \geqq 3$). Then $W_3 = K_4$, the complete graph of order four, and it is clear that $C(W_n) = W_n$. If $n \geqq 4$, the vertex of W_n with degree n is called the *center* and all the other vertices are called *outer vertices*. Note that for W_3, we can choose any vertex as its center, and regard the other three vertices as its outer vertices. The cycle of length n of W_n passing through all the outer vertices is called the *outer cycle*. An edge of W_n joining two outer vertices is called an *outer edge*, and an edge joining the center to an outer vertex is called a *spoke*.

We define a new graph, an *ff-wheel* (*flip-flap-wheel*), as follows. An ff-wheel FW consists of one wheel W_n and some triangles C_3 (cycles of order three) such that W_n and C_3 have exactly one edge in common, and every edge of W_n is contained in at most one C_3. We call C_3 a *flip* or a *flap* according as the common edge is a spoke or an outer edge of W_n. For example, an ff-wheel FW shown in Figure 2 has three flaps and two flips. We can easily verify that an ff-wheel FW is a cycle-periodic graph with period one (if $C(FW) \cong FW$) or two (otherwise) (see Figure 2). We regard a wheel as an ff-wheel without flips or flaps. We now give our main theorem.

Theorem 1. *Let G be a graph. Then*
(i) *G is cycle-vanishing if and only if $C^3(G)$ is a null graph or an empty graph;*
(ii) *G is cycle-periodic if and only if $C^3(G)$ is not a null graph and each non-trivial component of $C^3(G)$ is an ff-wheel. In particular, the period of a cycle-periodic graph is one or two; and*
(iii) *G is cycle-expanding if and only if one of the nontrivial components of $C^3(G)$ is not an ff-wheel.*

Note that Theorem A is an immediate consequence of (i) of Theorem 1, and our proof of (i) of Theorem 1 is essentially different from the proof of Theorem A given in [2]. We also prove the following result.

Theorem 2. *Every non-planar graph is cycle-expanding.*

2. Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a subset X of $V(G)$, we denote by $\langle X\rangle_G$ the subgraph of G induced by X. A subgraph H of G is called an *induced subgraph* if $H = \langle X\rangle_G$ for some subset X of $V(G)$. For a subgraph H' of G, we briefly write $\langle H'\rangle_G$ for $\langle V(H')\rangle_G$. For a vertex subset X of G, we denote by $G - X$ the induced subgraph $\langle V(G)\setminus X\rangle_G$. For two vertices u and v of G, if G has a path P connecting u to v, then P is called a *u–v path*, and the length of P is defined to be the number of edges in P. Let A and B be two paths, two cycles or a path and a cycle of G. Then we denote by $A\cup B$, $A\cap B$ and $A - B$ the edge-induced subgraphs of G with edge sets $E(A)\cup E(B)$, $E(A)\cap E(B)$ and $E(A)\setminus E(B)$, respectively. If $E(A)\cap E(B) = \emptyset$, then we denote $A\cup B$ by $A + B$. Moreover, we define $A\Delta B := (A \cup B) - (A\cap B)$. The next lemma follows immediately from the definition of the cycle graph.

Lemma 1. *Let G be a graph.*

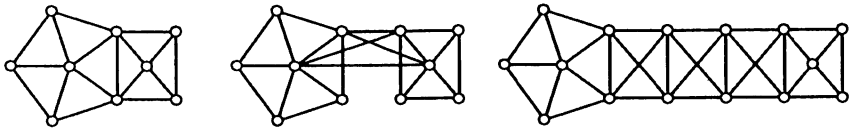
(i) *Then if H is an induced subgraph of G, then $C(H)$ is also an induced subgraph of $C(G)$. In particular, every chordless cycle of H is a chordless cycle of G ;*

(ii) *Let D be a cycle of G and e be an edge of D. Then e is contained in a chordless cycle of $\langle D\rangle_G$;*

(iii) *Let D be a cycle of G, e_1 and e_2 be two edges in D, and D_1 and D_2 be chordless cycles of G containing e_1 and e_2, respectively. Then $D_1 = D_2$ or two vertices D_1 and D_2 of $C(G)$ are joined by a path of $C(G)$ whose internal vertices are contained in $C(\langle D\rangle_G)$.*

Assume that a graph G contains a wheel W_n. Then G contains a wheel W_m $(3 \leqq m \leqq n)$ as an induced subgraph, and so G is not cycle-vanishing by (i) of Lemma 1 and by $C(W_m) = W_m$. We now give some results on graphs containing wheels.

Let $W_n \mid W_m$ $(n,m \geqq 3)$ denote the graph which consists of two wheels W_n and W_m such that an outer edge of W_n coincides with an outer edge of W_m (see Fig. 4). As shown in Figure 4, $C^2(W_n \mid W_m)$ contains the cycle-expanding graph R given in Figure 3. Since every chordless cycle of $C^k(R)$ $(k \geqq 0)$ is a triangle, if a graph H contains R as a subgraph, then $C^k(H) \supseteq C^k(R)$ and so H is cycle-expanding. Furthermore, we can observe that every chordless cycle of $W_n \mid W_m$ except the outer cycles of W_n and W_m is a triangle and that if a graph G contains $W_n \mid W_m$, then for some s and t $(3 \leqq s \leqq n$, $3 \leqq t \leqq m)$, G contains $W_s \mid W_t$ with W_s and W_t as induced subgraphs. Hence the following lemma holds.



K $C(K)$ $C^2(K)$

Figure 4. $K := W_5 \mid W_4$, $C(K)$ and $C^2(K)$.

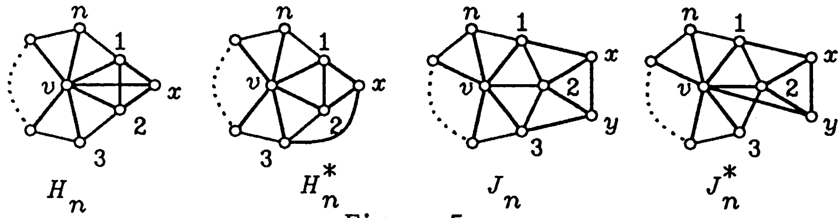


Figure 5.

Lemma 2. *Every* $W_n \mid W_m$ $(n,m \geq 3)$ *is cycle-expanding. Moreover, if a graph* G *contains* $W_n \mid W_m$, *then* G *is cycle-expanding.*

Let H_n, H^*_n, J_n, J^*_n $(n \geq 3)$ denote the graphs shown in Figure 5, each of which contains a wheel W_n as an induced subgraph.

Lemma 3. *If a graph* G *contains one of the graphs* H_n, H^*_n, J_n *and* J^*_n, *then* G *is cycle-expanding.*

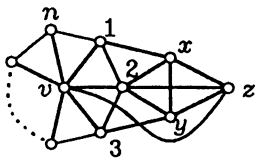


Figure 6. $C(H_n)$.

Proof It is easy to see that if G contains H_n, then for some m $(3 \leq m \leq n)$, G contains H_m such that the wheel of H_m is an induced subgraph of G. Hence we may assume that the wheel of H_n is an induced subgraph of G. Analogously, we may assume that the wheel of each of H^*_n, J_n, J^*_n is also an induced subgraph of G.

Let W_n denote the wheel of H_n, H^*_n, J_n or J^*_n contained in G. Then $C^k(G)$ contains W_n as an induced subgraph. We first assume that G contains H_n. If $n \geq 4$, then $C^2(H_n)$ contains $W_n \mid W_5$, where the center of W_5 is the chordless cycle $v,1,x,y,3,v$ of $C(H_n)$ (see Fig. 6) and the outer vertices of W_5 are the chordless cycles $1,v,2,1$, $2,v,3,2$, $1,2,x,1$, $x,2,y,x$

and $y,2,3,y$ of $C(H_n)$ (see Fig. 6). Hence G is cycle-expanding by Lemma 2. Since $C^2(H_3)$ contains $W_3 \mid W_4$, G is cycle--expanding by Lemma 2.

Assume that G contains H_n^*. We may assume that the vertex x of $V(H_n^*)\backslash V(W_n)$ is not adjacent to the center of W_n since otherwise G contains H_n. Then G contains a chordless cycle $1,x,3,v,1$. Hence, if $n \geq 4$ then $C(G)$ contains $W_n \mid W_4$, where the center of W_4 is $1,x,3,v,1$ and the outer vertices of W_4 are $1,2,x,1$, $x,2,3,x$, $3,2,v,3$, and $v,2,1,v$. It is obvious that $H_3^* = H_3$. Consequently G is cycle-expanding.

We now assume that G contains J_n. Suppose $n \geq 4$. Since we may assume that G contains neither H_n nor H_n^*, the cycle $x,y,3,v,1,x$ of J_n is a chordless cycle of G. Thus $C(G)$ contains $W_n \mid W_5$, where the center of W_5 is $x,y,3,v,1,x$ and the outer vertices of W_5 are $1,2,x,1$, $x,2,y,x$, $y,2,3,y$, $3,2,v,3$ and $v,2,1,v$. Hence G is cycle-expanding by Lemma 2. Since $J_3 = H_4$, G is cycle-expanding when $n = 3$.

We finally assume that G contains J_n^*. Since we may assume that G contains neither H_n nor H_n^*, the cycle $1,x,y,v,1$ of J_n^* is a chordless cycle of G. Hence $C(G)$ contains J_n, where the two vertices x and y of J_n in Figure 5 are obtained from the chordless cycles $1,x,y,v,1$ and $2,v,y,2$ of J_n^*. Consequently G is cycle-expanding.

For a vertex v of a graph G, we denote by $N_G(v)$ the neighbourhood of v (i.e. $N_G(v)$ is the set of vertices of G adjacent to v). For a subset S of $V(G)$, we write

$$N_G(S) := \bigcup_{v \in S} N_G(v).$$

Lemma 4. *Let G be a graph which contains a wheel W_n as an induced subgraph. Let $S = V(W_n)$ and X be the set of vertices of $V(G)\backslash S$ which are adjacent to at least two vertices of S. Assume that one of the following three situations occurs:*

(i) *X contains a vertex which is adjacent to two non-adjacent vertices of S;*

(ii) *X contains a vertex x with $|N_G(x) \cap S| \geq 3$; or*

(iii) *X contains two vertices x and y such that $N_G(x) \cap S = N_G$*

$(y) \cap S$.

Then G is cycle-expanding. In particular, K_m ($m \geq 5$) is cycle-expanding (by (ii) with $n = 3$.)

Proof (i) It is easy to observe that G contains H_n^* or $C(G)$ contains H_n. Hence G is cycle-expanding by Lemma 3.

(ii) Since G contains H_n or G satisfies (i), G is cycle-expanding by Lemma 3.

(iii) By (i) and (ii), we may assume that $N_G(x) \cap S$ consists of the end-vertices of an edge of W_n. If $N_G(x) \cap S$ consists of the end-vertices of a spoke, then $C(G)$ contains $W_n \mid W_3$. If $N_G(x) \cap S$ consists of the end-vertices of an outer edge of G, then $C^2(G)$ contains $W_n \mid W_3$. Hence G is cycle-expanding by Lemma 2.

Lemma 5. *Let G be a graph which contains a wheel W_n as an induced subgraph. Let $S = V(W_n)$. Assume that one of the following three situations occurs:*

(i) two non-adjacent vertices u and v of S are connected by a path P of G such that $V(P) \cap S = \{u, v\}$;

(ii) two adjacent vertices u and v of S are connected by two internally disjoint paths P_1 and P_2 of $G - uv$, $uv \in E(G)$, such that $V(P_i) \cap S = \{u, v\}$ (i=1, 2); or

(iii) $V(G) \setminus S$ contains a vertex x such that G has three internally disjoint paths P_1, P_2 and P_3 connecting x to three distinct vertices v_1, v_2 and v_3 of S with $V(P_i) \cap S = \{v_i\}$ (i=1,2,3).

Then G is cycle-expanding.

Proof Assume (i) holds. We may assume that P is a shortest path satisfying (i). Let e and f be the spoke and an outer edge of the wheel incident with u, and let B_e and B_f be the cycles of G passing through u,e,v,P,u and $u,f,\ldots,v,P,u$, respectively. Let D_e and D_f be chordless cycles of $\langle B_e \rangle_G$ and $\langle B_f \rangle_G$ containing e and f, respectively. Then both D_e and D_f contain the edge of P incident with u since P is a shortest path. Hence the two vertices D_e and D_f of $C(G)$ are adjacent. Therefore $C(G)$ contains J_n^*.

Consequently, G is cycle-expanding.

We next assume (ii) holds. Let B_1 and B_2 be the cycles of G passing through u,v,P_1,u and u,v,P_2,u, respectively. Let D_1 and D_2 be chordless cycles of $\langle B_1\rangle_G$ and $\langle B_2\rangle_G$, respectively, containing the edge uv. Then the two vertices D_1 and D_2 of $C(G)$ satisfy (iii) of Lemma 4. Therefore G is cycle-expanding.

We now assume that (iii) holds. Suppose first $n \geqq 4$. By (i), we may assume that v_1 and v_2 are two adjacent outer vertices of W_n and v_3 is the center of W_n. Let $B_1 := x,P_3,v_3,v_1,P_1,x$ and $B_2 := x,P_3,v_3,v_2,P_2,x$ be cycles of G and let D_i (i=1, 2) be a chordless cycle of $\langle B_i\rangle_G$ containing the edge v_3v_i. Then D_1 and D_2 are connected by a path of $C(G) - V(C(W_n))$ by (iii) of Lemma 1. We observe that $C(W_n) = W_n$ contains two non-adjacent vertices u_1 and u_2 which are adjacent to the vertices D_1 and D_2, respectively. Then u_1 and u_2 satisfy (i), and thus G is cycle-expanding.

For $n = 3$, either $C(G)$ satisfies (iii) of Lemma 4, or there exists a shortest $D_1 - D_2$ path ub $C(G)\backslash C(W_3)$, which implies that $C^3(G)$ contains $W_4 \mid W_5$. In both cases, G is cycle-expanding.

The first statement of the following lemma was essentially proved by Gervacio [2] though it was not stated in this form. We give a short proof.

Lemma 6. *Let G be a graph which contains a subdivision of W_3. Then $C(G)$ contains a wheel as an induced subgraph; in particular, G is not cycle-vanishing. Furthermore, if G is cycle-periodic, then G contains an induced subgraph K such that K is a subdivision of a wheel W_n and $C(K)\cong W_n$.*

Proof. If $C(G)$ contains a wheel W_n as an induced subgraph, then $C^k(G)$ contains W_n as an induced subgraph for every $k \geqq 2$ by (i) of Lemma 1 and by $C(W_n) = W_n$, and so G is not cycle-vanishing. Moreover, if a graph H contains a wheel W_n, then H contains a wheel W_m as an induced subgraph for some m ($3 \leq m \leq n$). Hence, in order to prove the first part of the lemma, it suffices to show that if G contains a subdivision of W_3, then $C(G)$ contains a wheel. We prove this and the second part of the lemma by induction on $|V(G)|$. We consider three cases.

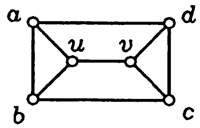


Figure 7. The graph G_1.

Case 1. *G contains a subdivision of the graph G_1 shown in Figure* 7.

Let H be a subdivision of G_1 in G. We denote by u, v, a, b, c, d the vertices of H corresponding to the vertices of G_1 labeled with the same letters. If the u–a path P_1 of H corresponding to the edge ua of G_1 has length greater than one, then for any internal vertex w of P_1, $H-w$ contains a subdivision of W_3. Hence, by the induction hypothesis, $C(G-w)$ contains a wheel, and so does $C(G)$. Moreover, since $G-w$ is not cycle-vanishing, if G is cycle-periodic, then $G-w$ is also cycle-periodic, and so $G-w$ contains a required induced subgraph K by the induction hypothesis. This implies that G also contains K. By the symmetry of G_1, we may now assume that ua, ub, ab, vc, vd, cd are edges of H. Similarly, we may assume that the three cycles of H corresponding to the chordless quadrangles (cycles with order four) of G_1 are chordless cycles of G. Thus all of the five cycles of H corresponding to the chordless cycles of G_1 are chordless cycles of G, and in $C(G)$, they induce a subgraph isomorphic to $C(G_1)$. Since $C(G_1)$ contains W_3, this means that $C(G)$ contains W_3. Also note that since $C(G_1)$ is cycle-expanding by Lemma 4, G is cycle-expanding.

Case 2. *G contains a subdivision of W_4.*

Let H be a subdivision of W_4 in G, and let D denote the cycle of H corresponding to the outer cycle of W_4 and w denote the vertex of H corresponding to the center of W_4. Arguing as in the proof of Case 1, we may assume that the paths corresponding to the spokes of W_4 are edges, and that D is a chordless cycle of G. Hence $\langle H\rangle_G$ is obtained from H by adding all the edges of G joining w to vertices of D. Thus $\langle H\rangle_G$ is a subdivision of a wheel W_n, and $C(G)$ contains $C(\langle H\rangle_G) = C(W_n) \cong W_n$.

Case 3. *Otherwise.*

Let H be a subdivision of W_3 in G. We may assume that each of the six paths of H corresponding to the edges of W_3 is an induced path. We denote by v, a, b, c the vertices of H corresponding to the vertices of W_3. If there exists an edge $e \in E(\langle H\rangle_G) - E(H)$ such that the end-vertices of e are internal vertices of two paths which correspond to two adjacent edges of W_3, then Case 1 occurs. If there exists an edge $e \in E(\langle H\rangle_G) - E(H)$ such that one of the end-vertices of e lie in $\{v,a,b,c\}$, then Case 2 occurs. Hence we may assume that the four cycles D_1, D_2, D_3, D_4 of H which correspond to the four triangles of W_3 are chordless cycles of G, and it is clear that the subgraph of $C(G)$ induced by these four cycles is isomorphic to W_3.

Now suppose that H is not an induced subgraph. Then we can find an edge $e = uw \in E(\langle H\rangle_G) - E(H)$ and a u-w path P of H such that e and P form a chordless cycle D of G. This D shares an edge with at least three D_is, which means that in $C(G)$, D is adjacent to at least three D_is. Thus G is cycle-expanding by Lemma 4. Next suppose that H is an induced subgraph but $V(C(H)) \neq \{D_1,D_2,D_3,D_4\}$. Then, $C(G)$ contains W_3, and for some quadrangle of W_3, the corresponding cycle D of H is a chordless cycle. Again D shares an edge with each D_i, and hence G is cycle-expanding.

Lemma 7. *Let G be a graph containing two chordless cycles D_1, D_2 such that $\langle V(D_1) \cap V(D_2)\rangle_G$ consists of two or more components. Then G is cycle-expanding.*

Proof Let P be a component of $\langle V(D_1)\cap V(D_2)\rangle_G$, and let v be a common vertex of D_1 and D_2 not contained in P. Clearly, P is a path. Let x and y be the end vertices of P. If $x\neq y$, then for each $i=1,2$, let Q_i be the x–v path on D_i not containing P, and R_i be the y–v path on D_i not containing P; if $x=y$ we simply let Q_i, R_i be the two x–v paths on D_i. Let e_i be the edge on Q_i incident with x, and f_i be the edge on R_i incident with y. Since D_1 and D_2 are chordless cycles, we can find in $Q_1\cup P\cup R_2$ a chordless cycle D_3 of G containing e_1, f_2 and P. Similarly, in $Q_2\cup P\cup R_1$, there exists a chordless cycle D_4 containing e_2, f_1 and P, and in $Q_1\cup Q_2$ (resp. $R_1\cup R_2$) there exists a chordless cycle D_5 (resp. D_6) containing e_1 and e_2 (resp. f_1 and f_2). Then in $C(G)$, D_1, D_3, D_4, D_5, D_6 form a wheel W_4 with center D_1, and D_2 is adjacent to D_3, D_4, D_5, D_6. Therefore G is cycle-expanding by Lemma 4 (ii).

Lemma 8. *Let G be a cycle-periodic graph containing an induced subgraph H such that H is a subdivision of W_n and $C(H)\cong W_n$. Then the block (2-connected component) of $C(G)$ containing $C(H)$ is an ff-wheel.*

Proof Let X be the set of vertices of $C(G)$ adjacent to a vertex of $C(H)$. Since each edge of H is contained in two chordless cycles of H, each vertex in X has at least two neighbors in $C(H)$. Moreover, if there exists a path P joining two vertices $u, v\in X$ $(u\neq v)$ such that $V(P)\cap(X\cup V(C(H)))=\{u,v\}$, then G is cycle-expanding by (iii) of Lemma 5, a contradiction. Therefore, in view of Lemma 4, this implies that $\langle X\cup V(C(H))\rangle_{C(G)}$ is an ff-wheel. Thus $\langle X\cup V(C(H))\rangle_{C(G)}$ is a block.

3. Proofs of Theorems

We begin with some lemmas, which will be used in the proof of Theorem 1.

Lemma 9. *Let G be a graph which contains a subdivision K of W_3 such that each of the six paths of K corresponding to the* *edges of W_3 is an induced path of G. Then the cycle graph $C(\langle K\rangle_G)$ is 2-connected.*

Proof Let $P_1, \dots, P_6$ be the induced paths of K corresponding to the edges of W_3, and let $C_1, \dots, C_4$ be the cycles of K corresponding to the triangles of W_3. For each i, $1 \leqq i \leqq 4$, let S_i be the set of chordless cycles of $\langle C_i\rangle_G$. Since each P_k is an induced path, we see that for every cycle D of $\langle K\rangle_G$, $V(D)$ is not contained in any one of the P_k. Thus $S_i \cap S_j = \emptyset$ for all i, j with $i \neq j$. Let T be the set of those chordless cycles of $\langle K\rangle_G$ which share an edge with K. We first prove that in $C(G)$, T is contained in a block of $C(\langle K\rangle_G)$. Let $D_1, D_2 \in T$, $D_1 \neq D_2$, and take $e_1 \in E(D_1) \cap E(K)$ and $e_2 \in E(D_2) \cap E(K)$. We want to show that in $C(\langle K\rangle_G)$ there exist two internally disjoint paths connecting D_1 and D_2. We here consider only the case where e_1 and e_2 lie on paths of K which correspond to two adjacent edges of W_3 (other cases can be handled in a similar fashion). At the cost of relabeling, we may assume that e_1 is contained in C_1 and C_4, and e_2 is contained in C_2 and C_4. Then by (iii) of Lemma 1, we can find, in $C(\langle K\rangle_G)$, a D_1-D_2 path all of whose internal vertices lie in $S_1 \cup S_2$, as well as a D_1-D_2 path all of whose internal vertices lie in S_4. Since these two paths are clearly internally disjoint, this show that T is contained in a single block B. Now let D be a chordless cycle of $\langle K\rangle_G$ not belonging to T. Then D has an edge e not contained in K. We may assume that e is in $\langle C_1\rangle_G$. Then, by (iii) of Lemma 1, in each of the two cycles formed by e and an "arc" of C_1, there exist chordless cycles which form a path in $C(\langle K\rangle_G)$ connecting D to a cycle belonging to T. Thus D belongs to B and, since D was arbitrary, this means that $C(\langle K\rangle_G)$ is 2-connected.

Lemma 10. *Let G be a graph which is not cycle-expanding, and let $f = v_1v_2$ be an edge of $C(G)$. Suppose that f is contained in a chordless cycle $C = v_1, v_2, \dots, v_k, v_1$ of $C(G)$, where $v_i \in V(C(G))$ and $k \geqq 3$, and for each i, let D_i denote the cycle of G*

corresponding to v_i. *If* $k = 3$, *assume further that* $D_1 \cap D_2 \cap D_3 = \emptyset$ *and* $D_3 \neq D_1 \Delta D_2$. *Then there exists an induced subgraph* H *of* G *such that* H *is a subdivision of a wheel* W_n, $C(H) \cong W_n$, *and* $C(H)$ *contains the edge* f; *in particular,* G *is cycle-periodic.*

Proof Set $P = D_1 \cap D_2$. By Lemma 7, P is a path and $V(D_1) \cap V(D_2) = V(P)$. If $k \geq 4$, then since C is chordless, we have $P \cap D_i = \emptyset$ and $D_i \neq D_1 \Delta D_2$ for all i, $3 \leq i \leq k$, and the same also holds when $k = 3$ because of the assumption. Now supppose that there exists D_i $(3 \leq i \leq k)$ which shares two or more vertices with P. Then since $P \cap D_i = \emptyset$, D_1 and D_i satisfy the hypothesis of Lemma 7 unless $V(D_1) \cap V(D_i) = V(D_1 - P)$; and if $V(D_1) \cap V(D_i) = V(D_1 - P)$, then since $D_i \neq D_1 \Delta D_2$, D_2 and D_i satisfy the hypothesis of Lemma 7. In either case, Lemma 7 implies that G is cycle-expanding, which contradicts our assumption. Thus for each i, $3 \leq i \leq n$, D_i contains at most one vertex of P. From this we see that $D_i - V(P)$ is connected for all i, $1 \leq i \leq k$, and $(V(D_i) \cap V(D_{i+1})) - V(P) \neq \emptyset$ for all i, $2 \leq i \leq k$ (we take $D_{k+1} = D_1$). Hence in $G - V(P)$, there exists a path Q which joins a vertex x of $D_1 - V(P)$ to a vertex y of $D_2 - V(P)$. We may assume that $V(Q) \cap V(D_1) = \{x\}$, $V(Q) \cap V(D_2) = \{y\}$ and Q is an induced path. Then the graph K obtained by adding Q to $D_1 \cup D_2$ is a subdivision of W_3. This in particular implies that v_1 and v_2 have degree at least 3 in $C(\langle K \rangle_G)$ by (ii) of Lemma 1. By Lemma 6, $\langle K \rangle_G$ contains an induced subgraph H such that H is a subdivision of a wheel W_n and $C(H) \cong W_n$. Also note that K satisfies the hypothesis of Lemma 8. Thus by Lemmas 8 and 9, $C(\langle K \rangle_G)$ is an ff-wheel, and since v_1 and v_2 have degree at least 3, they must lie in $C(H)$, the "wheel part" of $C(\langle K \rangle_G)$.

The following Corollary is an immediate consequence of the above Lemma 10.

Corollay 1.[2] *Every chordless cycle of the cycle graph of a cycle-vanishing graph is a triangle.*

Lemma 11. *Supose that a cycle-periodic graph G contains a graph* G_2 *shown in Figure* 8. *For each i, let* D_i *denote the triangle* $x_i, y, z, x_i,$ *and set* $K = \langle D_1, D_2, D_3, D_4 \rangle_{C(G)} \cong W_3$. *then K is a block of* $C(G)$.

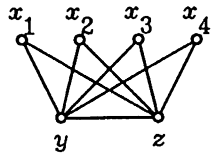


Figure 8. The graph G_2.

Proof By way of contradiction, suppose that there exists a cycle C of $C(G)$ which intersects K at precisely two vertices. We may assume $V(C) \cap V(K) = \{D_1, D_2\}$. We may also assume that C is a chordless cycle. If C satsifies the hypotheses of Lemma 10, they by Lemmas 10 and 8, the block of $C(G)$ which contain D_1 and D_2 is an ff-wheel, whose wheel W comes from a wheel of G. Since the wheel W and the wheel K cannot be the same wheel, this is a contradiction. Thus C does not satisfy the hypotheses of Lemma 10, which implies that C is a triangle, and the cycle D of G which correspond to the vertex of $V(C) \setminus V(K)$ contains the edge yz as $D \neq D_1 \Delta D_2$ and $D \cap D_1 \cap D_2 \neq \emptyset$. This implies that D_1, D_2, D_3, D_4 and D induces a complete graph of order 5 in $C(G)$, which is a contradiction by Lemma 4.

We shall prove only (i) and (ii) of Theorem 1 since (iii) of Theorem 1 follows immediately from (i) and (ii).

Proof of (i) and (ii) of Theorem 1. Let G be a graph which is not cycle-expanding, and let B be a block of $C^2(G)$ such that $C(B)$ is not a null graph. To prove (i) and (ii) of Theorem 1, it suffices to show that B is an ff-wheel under this assumption.

If B contains a chordless cycle of length greater than or equal to 4, then we immediately obtain the desired conclusion by applying Lemmas 10 and 8 to $C(G)$. Therefore we may assume that all the chordless cycles of B are triangles. Let u be an arbitrary vertex of B, and let C be the corresponding cycle of $C(G)$. Assume for the moment that C has length at least 4. Let v be a vertex of B adjacent two u, and let D be the corresponding cycle of $C(G)$. By Lemma 10, there exists an induced wheel W of $C(G)$ containing a common edge of C and D. Then each vertex of u and v is either contained in $C(W)$ or adjacent to two vertices of $C(W)$. Hence $C(W)$ is contained in B, and the desired conclusion follows from Lemma 8. Since u was arbirtrary, this means we may assume that all the chordless cycles of $C(G)$ corresponding to the vertices of B are triangles.

Now from the assumption that $C(B)$ is not a null graph, it follows that B has two triangles C_1 and C_2 which share an edge. Let $V(C_1)=\{v_1, v_3, v_4\}$ and $V(C_2)=\{v_2, v_3, v_4\}$, and let D_i and denote the triangle of $C(G)$ that corresponds to v_i. Write $V(D_3)=\{x_1, x_3, x_4\}$ and $V(D_4)=\{x_2, x_3, x_4\}$. If x_1 and x_2 are adjacent, we can again apply Lemma 8 to get the desired conclusion. Thus we may assume that x_1 and x_2 are not adjacent. Since D_1 and D_2 must share an edge with both D_3 and D_4, both D_1 and D_2 contain $\{x_3, x_4\}$. Therefore, in $C(G)$, the triangles D_i $(1 \leq i \leq 4)$ form a subgraph isomorphic to G_2 in Figure 8, and we can obtain the desired conclusion from Lemma 11.

Proof of Theorem 2. Let G be a non-planar graph. By Kuratowski's theorem [3], G contains a subgraph which is contractible to K_5 or $K_{3,3}$. Let R be a minimal such subgraph of G, and let $H=\langle R\rangle_G$. It suffices to show that H is cycle-expanding. Suppose that H is not cycle-expanding. Since H contains a subdivision of a wheel W_3, if follows from Lemma 6 that H is cycle-periodic and contains an induced subgraph K such that K is a subdivision of a wheel W_n and $C(K)\cong W_n$. Thus the block B of $C(H)$ containing $C(K)$ is an ff-wheel by Lemma 8. On the other hand, by our choce of R, there exists a vertex x in $V(H)\backslash V(K)$ such that H has three paths P_1, P_2 and P_3 connecting x to three distinct vertices of K such that $V(P_i) \cap V(P_j) = \{x\}$ for all $i \neq j$. Hence the block B is not an ff-wheel. This is a contradiction, and we can conclude that G is cycle-expanding.

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