The isomorphic factorization of complete bipartite graphs into trees

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Abstract. We study the isomorphic factorization of complete bipartite graphs into trees. It is known that for complete bipartite graphs, the divisibility condition is also a sufficient condition for the existence of isomorphic factorization. We give necessary and sufficient conditions for the divisibility, that is, necessary and sufficient conditions for a pair $[m, n]$ such that mn is divisible by $(m + n - 1)$, and investigate structures of the set of pairs $[m, n]$ satisfying divisibility. Then we prove that the divisibility condition is also sufficient for the existence of an isomorphic tree factor of a complete bipartite graph by constructing the tree dividing $K(m, n)$.

1. Introduction

A *factor* of a graph G is a spanning subgraph of G and a *factorization* of G is a decomposition of G into an edge disjoint union of factors. A factorization in which all of the factors are isomorphic to each other is called an *isomorphic factorization*. Isomorphic factorizations are extensively studied by F.Harary and others (see references in F.Harary and R.W.Robinson[3]). If a graph G is decomposed into t isomorphic factors, then a trivial necessary condition is that t divides the size of G. This condition is called the *divisibility condition*. It is shown in [4,5] that if G is a complete graph or a complete bipartite graph, then the divisibility condition means the existence of isomorphic factors. If any topological condition is required on isomorphic factors, then the situation differs. In [3], it is pointed out that characterizing isomorphic factors, which are trees, of a complete graph or a complete multipartite graph is an open problem.

In this paper, we investigate the isomorphic factorization of complete bipartite graphs into trees. We characterize the divisibility condition, that is, we give necessary and sufficient conditions for a pair $[m, n]$ of positive integers so that mn is divisible by $m + n - 1$. We classify pairs $[m, n]$ satisfying divisibility, and show that there is a binary tree structure in the set of such pairs. Then, we give a construction algorithm for the isomorphic factorization of a complete bipartite graph into trees and prove that the divisibility condition is also a sufficient condition for the existence of an isomorphic factorization of a complete bipartite graph into trees.

For any integers m and n, (m, n) stands for the nonnegative greatest common divisor of m and n. If m divides n, then we write $m \mid n$, and otherwise we write $m \nmid n$. A pair $[m, n]$ of positive integers is called a *factor pair* (abbreviated FP), if $(m+n-1)$ divides mn. Since a complete bipartite graph $K(m, n)$ has mn edges and any spanning tree has $m+n-1$ edges, the divisibility condition is equivalent to the divisibility of mn by $m + n - 1$.

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A *graph* $G = (V, E)$ consists of a nonempty finite set V of vertices and a finite set E of edges, each of which is associated with two vertices.

For graph theoretic terminology and notation, we follow Bondy and Murty[2] and for any terminology and notation for isomorphic factorization, we follow Harary et al[3,4,5]. For number theoretic terminology and notation, we refer to Shapiro[7].

2. Necessary and sufficient conditions for divisibility

In this section, we give necessary and sufficient conditions for positive integers m and n so that $m+n-1$ divides mn.

Theorem 2.1. *For any integers m and n,*
$(m+n-1, mn) = (m, n-1)(m-1, n)$.

Proof. Let c be any divisor of $(m+n-1, mn)$. Since $c \mid mn$, there exist c_1, c_2 such that $c_1 \mid m$, $c_2 \mid n$ and $c = c_1c_2$. Since $c \mid m+n-1$ and $c_1 \mid c$, we have $c_1 \mid (m+n-1, m)$. Similarly, we have $c_2 \mid (m+n-1, n)$. Therefore $c \mid (m+n-1, (m+n-1, m)(m+n-1, n))$.
Next, we assume that c divides $(m+n-1, (m+n-1, m)(m+n-1, n))$. Then there exist integers c_1, c_2 such that $c_1 \mid (m+n-1, m)$, $c_2 \mid (m+n-1, n)$ and $c = c_1c_2$. Then $c_1 \mid m+n-1$, $c_1 \mid m$, $c_2 \mid m+n-1$ and $c_2 \mid n$, we have $c \mid (m+n-1, mn)$ since $c = c_1c_2$ divides $m+n-1$.
Therefore we have $(m+n-1, mn) = (m+n-1, (m+n-1, m)(m+n-1, n))$. Since $(m+n-1, m) = (n-1, m)$, $(m+n-1, n) = (m-1, n)$, $(n-1, m) \mid m+n-1$, $(m-1, n) \mid m+n-1$ and $((n-1, m), (m-1, n)) = 1$, we have $(m+n-1, (m+n-1, m)(m+n-1, n)) = (m+n-1, (n-1, m)(m-1, n))$ $= (n-1, m)(m-1, n)$. □

Theorem 2.2. *For any positive integers m, n, a pair $[m, n]$ is an FP if and only if $(m, n-1)(m-1, n) = m+n-1$.*

Proof. It is obvious that $(m+n-1)$ divides mn if and only if $(m+n-1, mn) = m+n-1$. Combining this with Theorem 2.1 yields the Theorem. □

From this theorem, we can deduce many necessary conditions and sufficient conditions.

Proposition 2.3.

1. *If $m \geq n = 1$, then $(m+n-1) \mid mn$.*
2. *If $m = n > 1$, then $(m+n-1) \nmid mn$.*
3. *If $m = k(2k+1)$, $n = k(2k-1)$ for some $k \geq 1$, then $(m+n-1) \mid mn$.*
4. *If $m = k(n-1)$, $n \geq 2$, $(k+1) \mid n$ and $k \geq 1$, then $(m+n-1) \mid mn$.*
5. *If $m = k^2$, $n = (k-1)^2$ and $k \geq 2$, then $(m+n-1) \mid mn$.*
6. *$m = kn+1$, $n \geq 1$, $(k+1) \mid (n-1)$ and $k \geq 0$, then $(m+n-1) \mid mn$.*
7. *If $m = 2k$, $n = 2k-1$ and $k \geq 1$, then $(m+n-1) \mid mn$.*

8. *If n is a prime number and $(m+n-1) \mid mn$, then $m = kn+1$ and $(k+1) \mid (n-1)$.*
9. *If $m-1$ is a prime number, $m \geq n$ and $(m+n-1) \mid mn$, then $n = k(m-1)$ and $(k+1) \mid m$.*
10. *If m is a prime number and $m \geq n$, then $(m+n-1) \nmid mn$.*

Proof. omitted. □

To investigate the structure of factor pairs, we introduce several numbers.

Let $d_m = (m, n-1) = (m, m+n-1)$,
$d_n = (m-1, n) = (m+n-1, n)$,
$m = d_m\alpha$, $n-1 = d_m\alpha'$ and
$n = d_n\beta$ and $m-1 = d_n\beta'$.

Then $(\alpha,\alpha') = (\beta,\beta') = (\alpha,\beta') = (\alpha',\beta) = (d_m, d_n) = 1$.

Theorem 2.4. *For integers $m \geq n \geq 1$,*
a) a pair $[m,n]$ is an FP if and only if $d_m = \beta + \beta'$,
b) a pair $[m,n]$ is an FP if and only if $d_n = \alpha + \alpha'$.

Proof. a) First we assume that $[m,n]$ is an FP.
Then $\beta' n = \beta' d_n \beta = (m-1)\beta$.
Hence if $m \geq 2$, then

$$n = \frac{\beta}{\beta'}(m-1). \tag{1}$$

From Theorem 2.2,

$$d_m d_n = m+n-1. \tag{2}$$

Substituting from (1) into (2), we obtain

$$d_m d_n = m + \frac{\beta}{\beta'}(m-1) - 1 = \frac{\beta+\beta'}{\beta'}(m-1).$$

Then $d_m d_n \beta' = (\beta+\beta')(m-1)$.
Substituting from $d_n\beta' = m-1$, we obtain

$$d_m(m-1) = (\beta+\beta')(m-1).$$

Since $m \geq 2$, we obtain $d_m = \beta+\beta'$.
If $m = 1$, then $n = 1$, $d_m = d_n = \beta = 1$ and $\beta' = 0$.
Therefore $d_m = \beta + \beta'$.

For the converse, we assume that $d_m = \beta+\beta'$.
By multiplying by d_n both sides of the equation, we obtain

$$d_m d_n = d_n(\beta+\beta') = n + (m-1) = m+n-1.$$

Hence $(m+n-1) \mid mn$.
b) Similarly proved. □

Corollary 2.5. *For any integers $m \geq n \geq 1$,*

a) a pair [m,n] is an FP if and only if $(\beta+\beta') \mid m$,
b) a pair [m,n] is an FP if and only if $(\alpha+\alpha') \mid n$. □

Theorem 2.6. *For any integers* m, n *such that* $m > n \geq 1$ *or* $m = n = 1$,
a) a pair [m,n] is an FP if and only if $\alpha\beta - \alpha'\beta' = 1$,
b) a pair [m,n] is an FP if and only if $\alpha\beta' - \alpha'\beta = m - n$.

Proof. a) Let us assume that $[m, n]$ is an FP.
From the definition of α, α', β and β',

$$(m-1)(n-1) = d_m d_n \alpha'\beta'.$$

Then $\quad mn - (m+n-1) = d_m d_n \alpha'\beta'$.
Substituting m and n from $m = d_m\alpha$ and $n = d_n\beta$, we obtain

$$d_m d_n \alpha\beta - d_m d_n \alpha'\beta' = m+n-1.$$

Then $\quad d_m d_n(\alpha\beta - \alpha'\beta') = m+n-1$.
By Theorem 2.2, $\alpha\beta - \alpha'\beta' = 1$.

For the converse, by multiplying by $d_m d_n$ both sides of $\alpha\beta - \alpha'\beta' = 1$, we obtain

$$d_m d_n(\alpha\beta - \alpha'\beta') = d_m d_n.$$

Then $\quad mn - (m-1)(n-1) = d_m d_n$.
Then $\quad m+n-1 = d_m d_n$.
By Theorem 2.2, $(m+n-1) \mid mn$.
b) Let us assume that $[m, n]$ is an FP.

$$m(m-1) - n(n-1) = d_m d_n \alpha\beta' - d_m d_n \alpha'\beta$$

Then $\quad (m-n)(m+n-1) = d_m d_n(\alpha\beta' - \alpha'\beta)$
If $m > n$, by Theorem 2.2, $m - n = \alpha\beta' - \alpha'\beta$.
If $m = n = 1$, then $\alpha = \beta = 1$ and $\alpha' = \beta' = 0$.
Hence $m - n = \alpha\beta' - \alpha'\beta$.

For the converse, we consider two cases.
i) $m > n \geq 1$. By multiplying by $d_m d_n$ both sides of $\alpha\beta' - \alpha'\beta = m - n$, we obtain

$$d_m d_n(\alpha\beta' - \alpha'\beta) = d_m d_n(m-n).$$

Then $\quad m(m-1) - n(n-1) = d_m d_n(m-n)$.
Then $\quad (m-n)(m+n-1) = d_m d_n(m-n)$.
Since $m > n$, we obtain $m+n-1 = d_m d_n$.
Hence $(m+n-1) \mid mn$.
ii) $m = n = 1$. In this case, $d_m = \alpha = 1$, $\alpha' = 0$, $d_n = \beta = 1$ and $\beta' = 0$.
Therefore $\alpha\beta' - \alpha'\beta = 0 = m - n$, and we have $(m+n-1) \mid mn$.

Theorem 2.7. *For any integers* $m,\ n \geq 1$,
a) a pair [m,n] is an FP if and only if

$$(\beta+\beta')\alpha' + 1 \equiv 0 \ (mod\ \alpha+\alpha'),$$

b) a pair [m,n] is an FP if and only if

$$(\alpha+\alpha')\beta'+1 \equiv 0 \ (mod\ \beta+\beta'),$$

c) a pair [m,n] is an FP if and only if

$$(\beta+\beta')\alpha-1 \equiv 0 \ (mod\ \alpha+\alpha'),$$

d) a pair [m,n] is an FP if and only if

$$(\alpha+\alpha')\beta-1 \equiv 0 \ (mod\ \beta+\beta').$$

Proof. a) Let us assume that $[m,n]$ is an FP.
From Theorem 2.6,

$$\begin{aligned}(\alpha+\alpha')(\beta+\beta') &= \alpha\beta+\alpha\beta'+\alpha'\beta+\alpha'\beta' \\ &= (\alpha'\beta'+1)+\alpha\beta'+\alpha'\beta+\alpha'\beta' \\ &= (\alpha+\alpha')\beta'+(\beta+\beta')\alpha'+1.\end{aligned}$$

Therefore, $(\beta+\beta')\alpha'+1 \equiv 0 \ (mod\ \alpha+\alpha')$.

For the converse, let us assume that

$$(\beta+\beta')\alpha'+1 \equiv 0 \ (mod\ \alpha+\alpha').$$

Then there exists an integer s such that

$$(\beta+\beta')\alpha'+1 = (\alpha+\alpha')s. \qquad (1)$$

From definition,

$$m = d_m\alpha = d_n\beta'+1, \qquad (2)$$

$$n = d_m\alpha'+1 = d_n\beta. \qquad (3)$$

Adding (2) to (3) yields

$$d_m(\alpha+\alpha') = d_n(\beta+\beta').$$

Since $(d_m, d_n) = 1$, we obtain

$$d_m \mid (\beta+\beta') \text{ and } d_n \mid (\alpha+\alpha').$$

Let

$$t = \frac{\beta+\beta'}{d_m} = \frac{\alpha+\alpha'}{d_n}.$$

Then

$$\beta+\beta' = td_m \text{ and } \alpha+\alpha' = td_n. \qquad (4)$$

Substituting from (4) into (1), we obtain

$$td_m\alpha'+1 = td_n s.$$

Therefore

$$t(d_n s - d_m \alpha') \;=\; 1. \tag{5}$$

Since $t > 0$, we obtain $t = 1$ and $d_n s - d_m \alpha' = 1$.
Then from (4) we obtain $\beta + \beta' = d_m$ and $\alpha + \alpha' = d_n$.
By Theorem 2.4, $[m, n]$ is an FP.
b), c) and d) are similarly proved. □

3. Classification of factor pairs

In this section, we investigate the structure of the set of factor pairs and classify factor pairs.

Let S be the set of factor pairs of positive integers. Let $\equiv_\alpha$ and $\equiv_\beta$ be binary relations on S defined by

$$[m_1, n_1] \equiv_\alpha [m_2, n_2] \rightleftharpoons \frac{m_1}{d_{m_1}} = \frac{m_2}{d_{m_2}} \text{ and } \frac{n_1 - 1}{d_{m_1}} = \frac{n_2 - 1}{d_{m_2}},$$

$$[m_1, n_1] \equiv_\beta [m_2, n_2] \rightleftharpoons \frac{n_1}{d_{n_1}} = \frac{n_2}{d_{n_2}} \text{ and } \frac{m_1 - 1}{d_{n_1}} = \frac{m_2 - 1}{d_{n_2}}.$$

Then $\equiv_\alpha$ (resp. $\equiv_\beta$) is an equivalence relation on S and S is partitioned into equivalence classes with respect to $\equiv_\alpha$ (resp. $\equiv_\beta$). We denote those equivalence classes as follows.

$$S(\alpha, \alpha') = \left\{ [m, n] \in S \mid \frac{m}{d_m} = \alpha \text{ and } \frac{n-1}{d_m} = \alpha' \right\},$$

$$S'(\beta, \beta') = \left\{ [m, n] \in S \mid \frac{n}{d_n} = \beta \text{ and } \frac{m-1}{d_n} = \beta' \right\}.$$

For $[m_i, n_i] \in S$, we write

$$d_{m_i} = (m_i, n_i - 1), \quad d_{n_i} = (m_i - 1, n_i),$$

$$m_i = d_{m_i} \alpha_i, \quad n_i - 1 = d_{m_i} \alpha'_i,$$

$$n_i = d_{n_i} \beta_i \text{ and } m_i - 1 = d_{n_i} \beta'_i.$$

Lemma 3.1. *If FP's $[m_1, n_1]$ and $[m_2, n_2]$ are members of $S(\alpha, \alpha')$, then*

$$d_{m_1} \equiv d_{m_2} \pmod{\alpha + \alpha'}.$$

If FP's $[m_1, n_1]$ and $[m_2, n_2]$ are members of $S'(\beta, \beta')$, then

$$d_{n_1} \equiv d_{n_2} \pmod{\beta + \beta'}.$$

Proof. First suppose that $\alpha' > 0$. By Theorem 2.7,

$$(\beta_1 + \beta'_1)\alpha' + 1 \equiv 0 \pmod{\alpha + \alpha'}, \tag{1}$$

$$(\beta_2 + \beta_2')\alpha' + 1 \equiv 0 \pmod{\alpha + \alpha'}. \quad (2)$$

By subtracting (2) from (1), we have

$$(\beta_1 + \beta_1')\alpha' \equiv (\beta_2 + \beta_2')\alpha' \pmod{\alpha + \alpha'}.$$

Since $(\alpha', \alpha + \alpha') = 1$, we obtain

$$(\beta_1 + \beta_1') \equiv (\beta_2 + \beta_2') \pmod{\alpha + \alpha'}.$$

By Theorem 2.4, $d_{m_1} \equiv d_{m_2} \pmod{\alpha + \alpha'}$.
If $\alpha' = 0$, then we have $\alpha = 1$. Therefore $\alpha + \alpha' = 1$ and $d_{m_1} \equiv d_{m_2} \pmod{\alpha + \alpha'}$.
The remaining half is similarly proved. □

Lemma 3.2. *For FP's $[m_1, n_1]$, $[m_2, n_2] \in S(\alpha, \alpha')$, there is an integer k such that*

$$\beta_2 = \beta_1 + k\alpha' \text{ and } \beta_2' = \beta_1' + k\alpha.$$

Similarly, for FP's $[m_1, n_1]$, $[m_2, n_2] \in S'(\beta, \beta')$, there is an integer k' such that

$$\alpha_2 = \alpha_1 + k'\beta' \text{ and } \alpha_2' = \alpha_1' + k'\beta.$$

Proof. By Lemma 3.1,

$$(\beta_1 + \beta_1') \equiv (\beta_2 + \beta_2') \pmod{\alpha + \alpha'}.$$

Then there exists an integer k such that

$$\beta_2 + \beta_2' = (\beta_1 + \beta_1') + k(\alpha + \alpha'). \quad (1)$$

By Theorem 2.6,

$$\alpha_1\beta_1 - \alpha_1'\beta_1' = \alpha\beta_1 - \alpha'\beta_1' = 1 \quad (2)$$

$$\alpha_2\beta_2 - \alpha_2'\beta_2' = \alpha\beta_2 - \alpha'\beta_2' = 1. \quad (3)$$

From (1),(2) and (3), we obtain

$$\begin{aligned} \alpha\beta_2 - \alpha'\beta_2' &= \alpha\beta_2 - \alpha'\{(\beta_1 + \beta_1') + k(\alpha + \alpha') - \beta_2\} \\ &= \alpha\beta_1 - \alpha'\beta_1'. \end{aligned}$$

Then $(\alpha + \alpha')\beta_2 = (\alpha + \alpha')\beta_1 + k\alpha'(\alpha + \alpha')$.
Since $\alpha + \alpha' > 0$, we have $\beta_2 = \beta_1 + k\alpha'$.
Hence by (1), $\beta_2' = \beta_1' + k\alpha$.
The remaining half is similarly proved. □

Now we will introduce an order relation in $S(\alpha, \alpha')$.

Suppose that an FP $[m, n]$ is a member of $S(\alpha, \alpha')$.
Then let $\alpha_1 = \alpha$, $\alpha_1' = \alpha'$, $\beta_1 = \beta + \alpha'$, $\beta_1' = \beta' + \alpha$,
$d_{m_1} = \beta_1 + \beta_1' = \alpha + \alpha' + \beta + \beta'$, $d_{n_1} = \alpha_1 + \alpha_1' = \alpha + \alpha'$,
$m_1 = d_{m_1}\alpha_1 = \alpha(\alpha + \alpha' + \beta + \beta')$ and $n_1 = d_{n_1}\beta_1 = (\alpha + \alpha')(\beta + \alpha')$.
Since $[m, n] \in S(\alpha, \alpha')$, $m = d_m\alpha = \alpha(\beta+\beta') = d_n\beta'+1 = \beta'(\alpha+\alpha')+1$ and $n = d_n\beta = \beta(\alpha + \alpha') = d_m\alpha' + 1 = \alpha'(\beta + \beta') + 1$.
Therefore $m_1 + n_1 - 1 = \alpha(\alpha + \alpha' + \beta + \beta') + (\alpha + \alpha')(\beta + \alpha') - 1$
$= \alpha(\alpha + \alpha' + \beta + \beta') + \alpha'(\alpha + \alpha') + \beta(\alpha + \alpha') - 1$
$= \alpha(\alpha + \alpha' + \beta + \beta') + \alpha'(\alpha + \alpha') + \alpha'(\beta + \beta')$
$= (\alpha + \alpha')(\alpha + \alpha' + \beta + \beta') \mid m_1 n_1$.
Therefore, $[m_1, n_1]$ is an FP.

Similarly, let $\alpha_2 = \alpha$, $\alpha_2' = \alpha'$, $\beta_2 = \beta - \alpha'$, $\beta_2' = \beta' - \alpha$,
$d_{m_2} = \beta_2 + \beta_2' = \beta + \beta' - \alpha - \alpha'$, $d_{n_2} = \alpha_2 + \alpha_2' = \alpha + \alpha'$,
$m_2 = d_{m_2}\alpha_2 = \alpha(\beta + \beta' - \alpha - \alpha')$ and $n_2 = d_{n_2}\beta_2 = (\alpha + \alpha')(\beta - \alpha')$.
If $m_2, n_2 \geq 1$, then $[m_2, n_2]$ is also an FP.
Therefore, elements in $S(\alpha, \alpha')$ are linearly ordered with respect to the natural order of $\beta + \beta'$.

Similarly, elements in $S'(\beta, \beta')$ are linearly ordered with respect to the natural order of $\alpha + \alpha'$.
Since, $\alpha, \beta > 0$ and $\alpha', \beta' \geq 0$, every equivalence class has the least element.

Lemma 3.3. *An FP $[m, n]$ is the least element of $S(\alpha, \alpha')$ if and only if $(\beta + \beta')$ is the least solution of the equation $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$).*

Proof. Let us assume that $[m, n]$ is the least element of $S(\alpha, \alpha')$. From the assumption $\beta_1 = \beta - \alpha' \leq 0$ or $\beta_1' = \beta' - \alpha < 0$.
If $\beta_1 = \beta - \alpha' \leq 0$, then $\beta_1' = \beta' - \alpha < 0$, since $\alpha(\beta - \alpha') - \alpha'(\beta - \alpha) = 1$ and $\alpha'(\beta' - \alpha) = \alpha(\beta - \alpha') - 1 < 0$.
Therefore $\beta + \beta' < \alpha + \alpha'$. From Theorem 2.7,
$(\beta + \beta')\alpha' + 1 \equiv 0$ (mod $\alpha + \alpha'$). Since $[m, n]$ is an FP, $\beta + \beta' > 0$ and $(\alpha',\ \alpha + \alpha') = 1$, $(\beta + \beta')$ is the least solution of $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$).

If $\beta_1' = \beta' - \alpha < 0$, then two cases arise.
i) $\alpha' > 0$. In this case, the same result is obtained.
ii) $\alpha' = 0$. In this case, $\alpha = \beta = 1$ and $\alpha' = \beta' = 0$.

Then any integer is a solution of $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$). In this case $\beta + \beta' = 1$ and this mean $\beta + \beta'$ is the least solution of $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$).

For the converse, suppose that $(\beta + \beta')$ is the least solution of $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$).
Then $0 < \beta + \beta' \leq \alpha + \alpha'$, and therefore $[m, n]$ is the least element of $S(\alpha, \alpha')$. □

A similar result is valid for $S'(\beta, \beta')$.

Lemma 3.4. *An FP $[m, n]$ is the least element of both $S(\alpha, \alpha')$ and $S'(\beta, \beta')$ if and only if $m = n = 1$.*

Proof. If $m = n = 1$, then $\alpha = \beta = 1$, $\alpha' = \beta' = 0$ and $\alpha + \alpha' = \beta + \beta' = 1$.
Therefore $(\beta + \beta')$ is the least solution of $\alpha' x + 1 \equiv 0$ (mod $\alpha + \alpha'$) and

$(\alpha+\alpha')$ is the least solution of $\beta' y + 1 \equiv 0 \ (mod\ \beta+\beta')$.
Therefore $[m,n]$ is the least element of $S(\alpha,\alpha')$ and $S'(\beta,\beta')$.

For the converse, let us assume that $[m,n]$ is the least element of both $S(\alpha,\alpha')$ and $S'(\beta,\beta')$.
Then $\alpha+\alpha' = \beta+\beta'$. Since $(\beta+\beta')\alpha'+1 \equiv 0 \ (mod\ \alpha+\alpha')$ and $\alpha+\alpha' = \beta+\beta'$, we have $1 \equiv 0 \ (mod\ \alpha+\alpha')$. Therefore $\alpha+\alpha' = 1$. Since $\alpha > 0$ and $\alpha' \geq 0$, $\alpha = 1$ and $\alpha' = 0$. By the same reason, $\beta = 1$ and $\beta' = 0$. Then $d_m = \beta+\beta' = 1$ and $d_n = \alpha+\alpha' = 1$.
Hence $m = d_m\alpha = 1$ and $n = d_n\beta = 1$. □

Lemma 3.5. *The equivalence classes $S(\alpha,\alpha')$ and $S'(\beta,\beta')$ have at most one element in common.*

Proof. omitted.

Lemma 3.6. *Every FP $[m,n]$ is the least element of $S(\alpha,\alpha')$ or of $S'(\beta,\beta')$, or of both.*

Proof. From Theorem 2.7, α, α', β and β' satisfy the following equations,

$$(\beta+\beta')\alpha'+1 \equiv 0 \ \ (mod\ \alpha+\alpha'),$$

$$(\alpha+\alpha')\beta'+1 \equiv 0 \ \ (mod\ \beta+\beta').$$

Therefore, $\beta+\beta' < \alpha+\alpha'$ or $\alpha+\alpha' < \beta+\beta'$ or $\alpha+\alpha' = \beta+\beta'$.
Hence $(\beta+\beta')$ is the least solution of the equation $\alpha' x+1 \equiv 0 \ (mod\ \alpha+\alpha')$ or $(\alpha+\alpha')$ is the least solution of the equation $\beta' y+1 \equiv 0 \ (mod\ \beta+\beta')$ or both. Therefore, the statement is proved. □

By Lemma 3.4, if an FP $[m,n]$ is the least element of both $S(\alpha,\alpha')$ and $S'(\beta,\beta')$, then $m = n = 1$. Therefore, any other FP is the least element of exactly one of $S(\alpha,\alpha')$ and $S'(\beta,\beta')$.

With the preparation up to here, we define a graph on S. Let $G(S)$ be a directed graph with vertex set S and there is a directed edge from v to v' if v' is a next element of v in $S(\alpha,\alpha')$ or in $S'(\beta,\beta')$ with respect to the order introduced above.

Theorem 3.7. *$G(S)$ is an infinite binary tree with the root* $[1,1]$.

Proof. From the definition, every vertex of $G(S)$ is of outdegree 2. From Lemma 3.4 and 3.6, every vertex other than [1,1] is of indegree 1. Therefore, if there is a directed edge from v to v' and v' is the next element of v in $S(\alpha,\alpha')$(resp. $S'(\beta,\beta')$), then we call v' the left son (resp. right son) of v, and v the father of v'.

Let $[m,n]$ be any vertex of $G(S)$. If $[m,n] \neq [1,1]$, then either $[m,n]$ is the least element of $S(\alpha,\alpha')$ or the least element of $S'(\beta,\beta')$ for some α, α', β and β'. Without loss of generality, we assume that $[m,n]$ is the least element of $S(\alpha,\alpha')$. Then we can trace to the least element of $S'(\beta,\beta')$. Then it is an element of $S(\alpha_1,\alpha_1')$ for some α_1, α_1'. If it is the least element of $S(\alpha_1,\alpha_1')$, then by Lemma 3.4, it is the element $[1,1]$. If it is not the least element of $S(\alpha_1,\alpha_1')$, then we can trace to the least element of $S(\alpha_1,\alpha_1')$. At each step, the value of $\alpha+\alpha'$ or $\beta+\beta'$ decreases keeping positivity. Therefore, we can arrive at the element which is the least element of both $S(\alpha, \alpha')$ and $S'(\beta, \beta')$ for some α, α', β and β'. But it has to be $[1,1]$, Therefore, any element is joined from the element $[1,1]$ by a unique directed path. Since $[1,1]$ has not a father and S is an infinite set, $G(S)$ is an infinite binary tree with root $[1,1]$. □

A part of the left half of the binary tree $G(S)$ on S is shown in Fig. 1. Since $G(S)$ is symmetric, the right half of $G(S)$ is obtained from the left half by reversing.

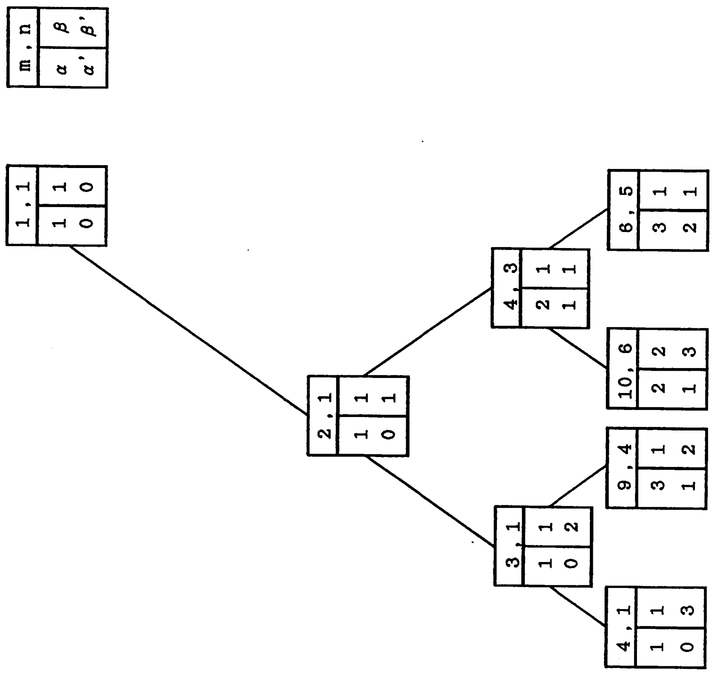
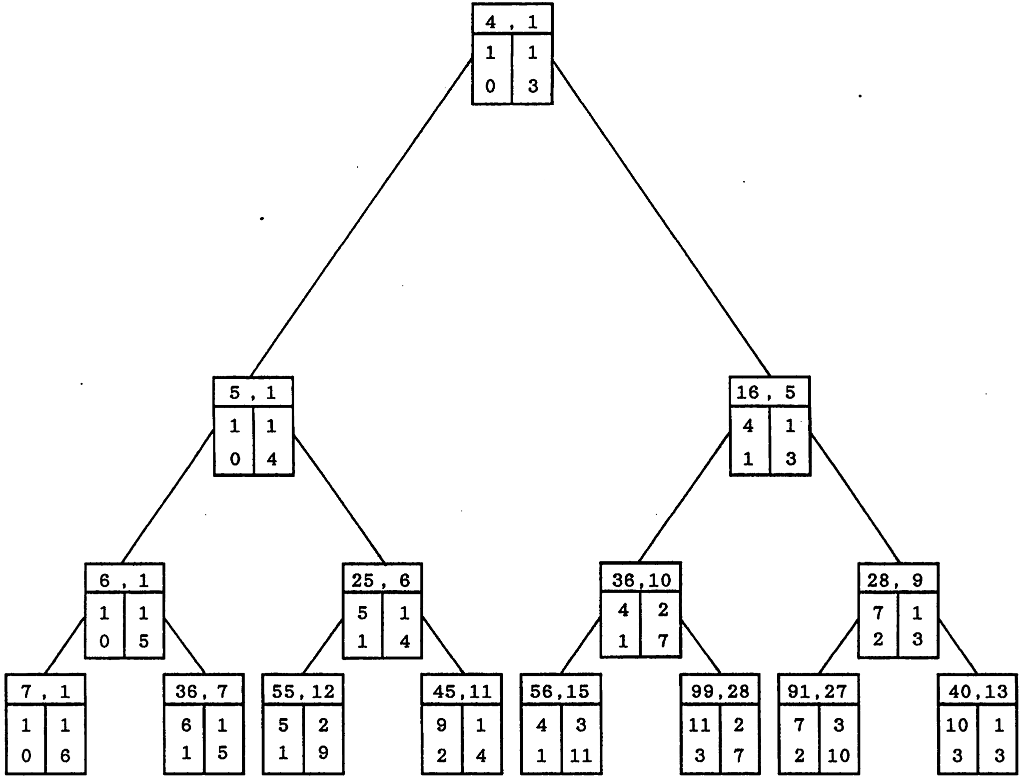


Fig. 1-1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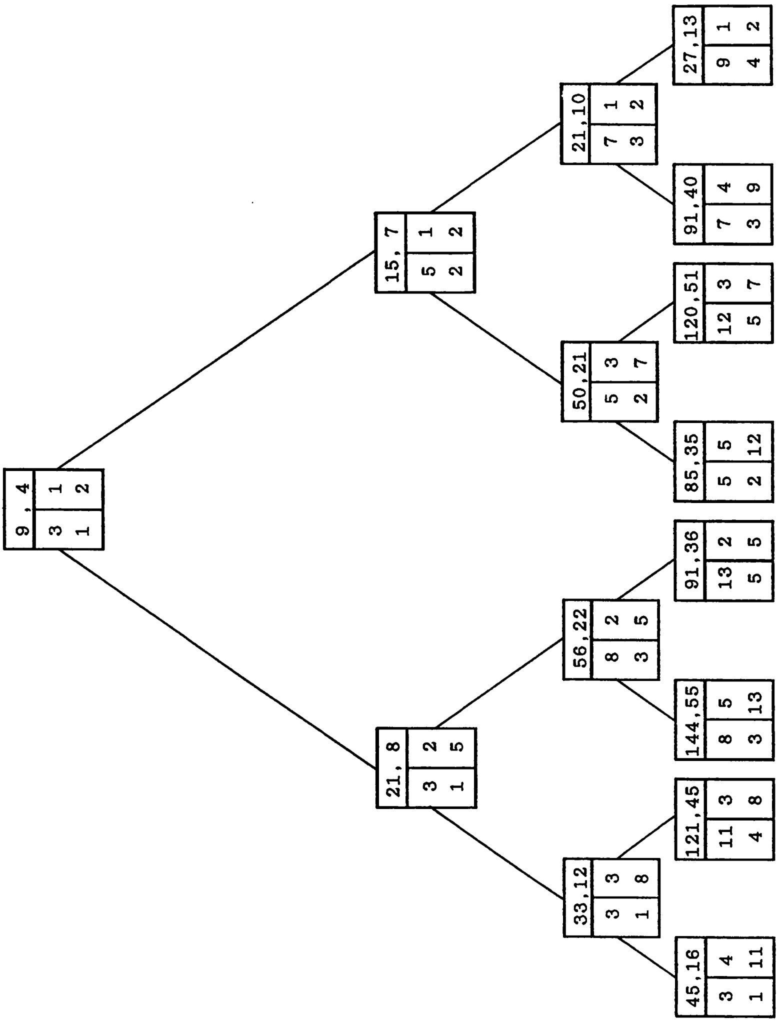
1 1
0 3
5 , 1
1 1
0 4
16 , 5
4 1
1 3
6 , 1
1 1
0 5
25 , 6
5 1
1 4
36,10
4 2
1 7
28, 9
7 1
2 3
7 , 1
1 1
0 6
36, 7
6 1
1 5
55,12
5 2
1 9
45,11
9 1
2 4
56,15
4 3
1 11
99,28
11 2
3 7
91,27
7 3
2 10
40,13
10 1
3 3

Fig. 1-2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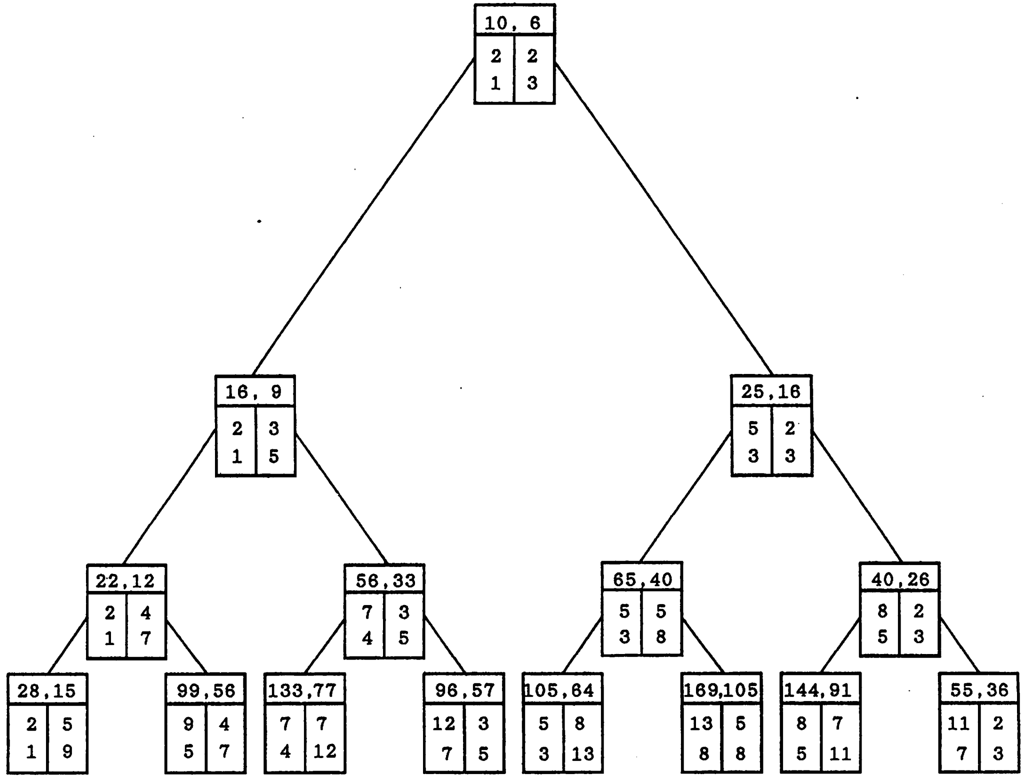
1 2
21, 8
3 2
1 5
15, 7
5 1
2 2
33,12
3 3
1 8
56,22
8 2
3 5
50,21
5 3
2 7
21,10
7 1
3 2
45,16
3 4
1 11
121,45
11 3
4 8
144,55
8 5
3 13
91,36
13 2
5 5
85,35
5 5
2 12
120,51
12 3
5 7
91,40
7 4
3 9
27,13
9 1
4 2

Fig. 1-3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2 3
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7 3
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65,40
5 5
3 8
40,26
8 2
5 3
28,15
2 5
1 9
99,56
9 4
5 7
133,77
7 7
4 12
96,57
12 3
7 5
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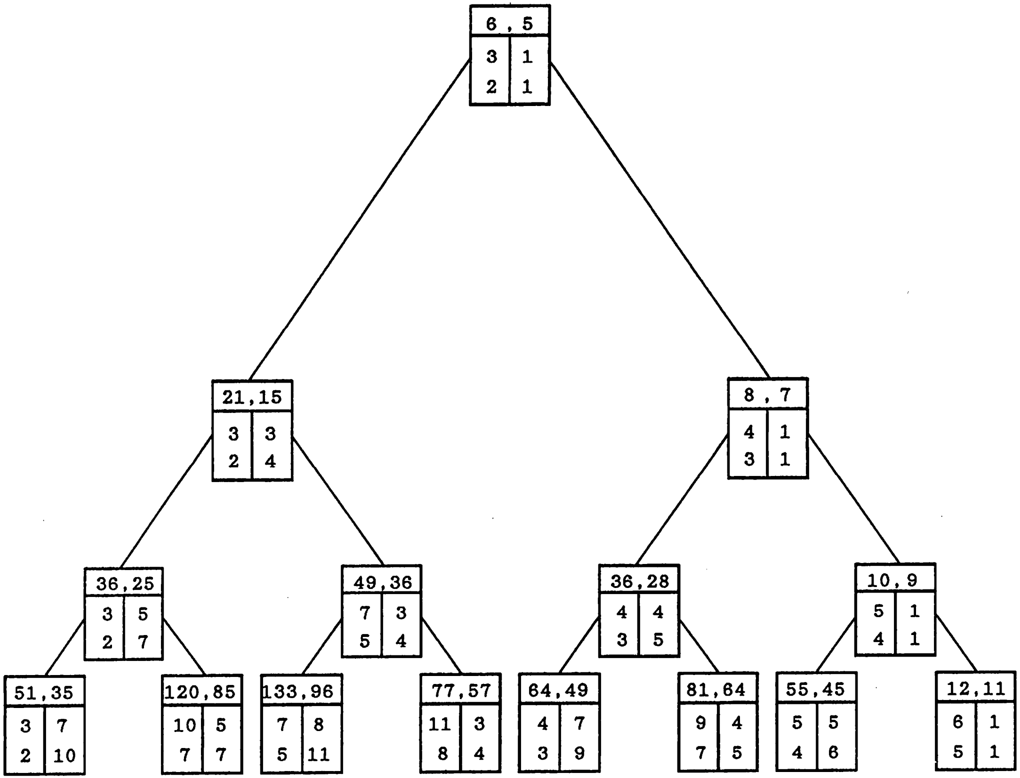
169,105
13 5
8 8
144,91
8 7
5 11
55,36
11 2
7 3

Fig. 1-4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3 3
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8 , 7
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3 1
36,25
3 5
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7 3
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36,28
4 4
3 5
10,9
5 1
4 1
51,35
3 7
2 10
120,85
10 5
7 7
133,96
7 8
5 11
77,57
11 3
8 4
64,49
4 7
3 9
81,64
9 4
7 5
55,45
5 5
4 6
12,11
6 1
5 1

Fig. 1-5

4. Isomorphic tree factors of a complete bipartite graph

In this section, we investigate the structure of isomorphic tree factors of $K(m,n)$ based on the results in Section 2. In many papers dealing with isomorphic factorizations, specific permutations are adopted to realize isomorphic factorizations and we follow this method.

Let the partite sets of the complete bipartite graph $K(m,n)$ be $U = \{u_0, u_1, \cdots, u_{m-1}\}$ and $V = \{v_0, v_1, \cdots, v_{n-1}\}$. Let σ and τ be permutations on U and V, respectively, such that $\sigma = \gamma_0\gamma_1\cdots\gamma_{d_m-1}$ and $\tau = \phi_0\phi_1\cdots\phi_{d_n-1}$, where γ_i $(0 \le i \le d_m - 1)$ and ϕ_j $(0 \le j \le d_n - 1)$ are disjoint permutations of length α and β, respectively. Let Γ_i $(0 \le i \le d_m - 1)$ and Φ_j $(0 \le j \le d_n - 1)$ be defined as follows;

$$\Gamma_i = \{u \in U \mid \gamma_i(u) \ne u\},$$

$$\Phi_j = \{v \in V \mid \phi_j(v) \ne v\}.$$

Then

$$\Gamma_i \cap \Gamma_j = \phi \quad (i \ne j),$$

$$\Phi_i \cap \Phi_j = \phi \quad (i \ne j),$$

$$\mid \Gamma_i \mid = \alpha \quad (0 \le i \le d_m - 1),$$

$$\mid \Phi_j \mid = \beta \quad (0 \le j \le d_n - 1),$$

and U is partitioned into a disjoint union of Γ_i's and V is partitioned into a disjoint union of Φ_j's.

Let G be a bipartite graph with partite sets U and V, and with edge set $E(G)$. Since we consider isomorphic factorizations under the permutations σ and τ, we define a graph G_{ij} as follows. G_{ij} is a bipartite graph with partite sets U and V, and with edge set

$$E(G_{ij}) = \{\sigma^i(u)\tau^j(v) \mid uv \in E(G),\ u \in U,\ v \in V\}.$$

Then $G \cong G_{ij}$ $(0 \le i \le \alpha - 1,\ 0 \le j \le \beta - 1)$ and $G_{00} = G$. If $\cup_{i,j} E(G_{ij})$ is a partition of $E(K(m,n))$, then we say that G is an *isomorphic factor* of $K(m,n)$ under σ and τ, or G *divides* $K(m,n)$ under σ and τ.
Let $E_{ij} = \{uv \mid u \in \Gamma_i,\ v \in \Phi_j\}$, $(0 \le i \le d_m - 1,\ 0 \le j \le d_n - 1)$. Then $\cup_{i,j} E_{ij}$ is a partition of $E(K(m,n))$.
Lemma 4.1. *A bipartite graph G with partite sets U and V divides $K(m,n)$ under σ and τ if and only if $\mid E(G) \cap E_{i,j} \mid = 1$ for $0 \le i \le d_m - 1$, $0 \le j \le d_n - 1$. If G divides $K(m,n)$ under σ and τ, then* $\mid E(G) \mid = m + n - 1$.
Proof. First, assume that G divides $K(m,n)$ under σ and τ. Then, G has just one edge in every E_{ij}, since any edge in E_{ij} is transformed to any other edge in E_{ij} under σ and τ. The converse is also true. There

are d_m Γ_i's and d_n Φ_j's, and this means that there are $d_m d_n$ E_{ij}'s. This shows that $| E(G) | = d_m d_n = m + n - 1$. □

Corollary 4.2. *If G divides $K(m,n)$ under σ and τ, then*

$$\sum_{u \in \Gamma_i} deg(u) = d_n \quad (0 \leq i \leq d_m - 1),$$

$$\sum_{v \in \Phi_j} deg(v) = d_m \quad (0 \leq j \leq d_n - 1). \quad \square$$

Let $N_G(v)$ be the set of vertices of G adjacent with v.

Corollary 4.3. *If G divides $K(m,n)$ under σ and τ, then*

$$N_G(u) \cap N_G(u') = \phi, \ u \neq u', \ u, u' \in \Gamma_i, \ 0 \leq i \leq d_m - 1,$$

$$N_G(v) \cap N_G(v') = \phi, \ v \neq v', \ v, v' \in \Phi_j, \ 0 \leq j \leq d_n - 1. \quad \square$$

If G divides $K(m,n)$ under σ and τ and if G is a tree, then G is an isomorphic tree factor of $K(m,n)$. Hence our purpose in the remaining of this paper is to construct such a tree.

Now, we investigate the structure of isomorphic tree factors of $K(m,n)$. We assume that $m \geq n \geq 2$ and $m + n - 1 \mid mn$. We give a labelling to vertices of G as follows.

$$\Gamma_i = \{u_i^0, u_i^1, \cdots, u_i^{\alpha-1}\}, \quad (0 \leq i \leq d_m - 1),$$

$$\Phi_j = \{v_j^0, v_j^1, \cdots, u_j^{\beta-1}\}, \quad (0 \leq j \leq d_n - 1),$$

where $\gamma_i = (u_i^0, u_i^1, \cdots, u_i^{\alpha-1})$ and $\phi_j = (v_j^0, v_j^1, \cdots, u_j^{\beta-1})$.

Definition 4.4. A graph G dividing $K(m,n)$ under σ and τ is said to be *interlaced* if

$$N_G(\Phi_j) = \cup_{v \in \Phi_j} N_G(v) = \{u_i^j \mid 0 \leq i \leq d_m - 1\}, \quad (0 \leq j \leq \alpha - 1). \quad \square$$

If G is an interlaced graph, then

$$\cup_{0 \leq j \leq \alpha-1} N_G(\Phi_j) = \cup_{0 \leq i \leq d_m - 1} \Gamma_i = U.$$

Let $G = (U \cup V, E)$ be an interlaced tree. A bipartite graph G_1 is defined from G as follows. Bipartition of $V(G_1)$ is $U_1 = \cup_{0 \leq j \leq \alpha-1} \Phi_j$ and $V_1 = \cup_{\alpha \leq j \leq d_n - 1} \Phi_j$ ($U_1 \cup V_1$ is a partition of V. $| V | = | U_1 | + | V_1 | = n$). Vertices $u \in U_1$ and $v \in V_1$ are joined by an edge if and only if $N_G(u) \cap N_G(v) \neq \phi$ in G.

Lemma 4.5. *Let G be an interlaced tree, then G_1 is a tree such that*

$$\sum_{v \in \Phi_j} deg_{G_1}(v) = d_m, \quad (\alpha \leq j \leq d_n - 1).$$

Proof. Since G is interlaced, each v in V_1 is adjacent with some vertex u in U_1. If $| N_G(v) \cap N_G(u) | \geq 2$, then G has a cycle and this contradicts the assumption. Therefore, $| N_G(v) \cap N_G(u) | = 1$. Then $deg_G(v) = deg_{G_1}(v)$. Therefore, $\sum_{v \in \Phi_j} deg_{G_1}(v) = d_m$, $(\alpha \leq j \leq d_n - 1)$.

$$| E(G_1) | = d_m(d_n - \alpha) = \alpha' d_m = n - 1.$$

$$| V(G_1) | = | U_1 | + | V_1 | = (\cup_{0 \leq j \leq d_n - 1} \Phi_j) = n.$$

Then $| E(G_1) | = | V(G_1) | - 1$.
Finally, in order to show that G_1 is acyclic, let us assume contrary that G_1 has a cycle. Let the cycle be $C : u_1, v_1, u_2, v_2, \cdots, u_k, v_k, u_1$.
Since $| N_G(u_1) \cap N_G(v_1) | = | N_G(v_1) \cap N_G(u_2) | = \cdots = | N_G(v_k) \cap N_G(u_1) | = 1$, we can construct a cycle in G from C. This contradicts the assumption. Hence G_1 is acyclic. Therefore G_1 is a tree. □

In the remaining, we construct an interlaced tree from a tree with partite sets U_1 and V_1, and satisfying $\sum_{v \in \Phi_j} deg_{G_1}(v) = d_m$, $(\alpha \leq j \leq d_n - 1)$.

Let G_1 be a tree with partite sets U_1 and V_1 such that $U_1 = \cup_{0 \leq j \leq \alpha - 1} \Phi_j$ and $V_1 = \cup_{\alpha \leq j \leq d_n - 1} \Phi_j$, and such that $\sum_{v \in \Phi_j} deg_{G_1}(v) = d_m$, $(\alpha \leq j \leq d_n - 1)$.

Let us define a graph G_2 from G_1 as follows.
Let edges of G_1 be $e_0, e_1, \cdots, e_{\alpha' d_m - 1}$. Subdivide each edge of G_1 and let w_i be the added vertex on the edge e_i. The resulting graph is G_2.

Next, we construct a graph $K(G_1)$ from G_2 as follows.

Let $K(G_1)$ be a graph such that

$V(K(G_1)) = \{w_i \mid 0 \leq i \leq \alpha' d_m - 1\}$ and

$$E(K(G_1)) = \left\{ ww' \;\middle|\; \begin{array}{l} w, w' \in N_{G_2}(\Phi_j), \quad \alpha \leq j \leq d_n - 1. \\ \text{or} \\ w \in N_{G_2}(v), \; w' \in N_{G_2}(v'), \\ v, v' \in \Phi_j, \; v \neq v', \; 0 \leq j \leq \alpha - 1 \end{array} \right\}$$

Now we consider colorings of $K(G_1)$ such that adjacent vertices are not colored by the same color. If the vertices of $K(G_1)$ are colored, then the same coloring is possible for vertices $w_0, w_1, \cdots, w_{\alpha' d_m - 1}$ of G_2. Since $| N_{G_2}(\Phi_j) | = d_m$, $\alpha \leq j \leq d_n - 1$, chromatic number $\chi(K(G_1))$ satisfies $\chi(K(G_1)) \geq d_m$.

Lemma 4.6. *If the vertices $w_0, w_1, \cdots, w_{\alpha' d_m - 1}$ of G_2 are colored with the same colors as vertices in $K(G_1)$, then the coloring satisfies the following properties;*
i) *for $\alpha \leq j \leq d_n - 1$, d_m vertices of $N_{G_2}(\Phi_j)$ are colored with different colors,*
ii) *for $0 \leq j \leq \alpha - 1$, and for $v, v' \in \Phi_j$, $v \neq v'$, each vertex in $N_{G_2}(v)$ is given a color different from the colors given to vertices in $N_{G_2}(v')$, and vice versa.*

Proof. omitted.

When $\chi(K(G_1)) = d_m$, we construct a graph G_3 from G_2 as follows.

For a given d_m-coloring of $K(G_1)$, give the same color to vertices $w_0, w_1, \cdots, w_{\alpha' d_m - 1}$ of G_2.
Let the set of d_m colors be $C = \{c_0, c_1, \cdots, c_{d_m - 1}\}$.
For each vertex $v \in \cup_{0 \leq j \leq \alpha-1} \Phi_j$, let $S_i(v)$ be the set of vertices in $N_{G_2}(v)$ with color c_i.
If $S_i(v) \neq \phi$, delete vertices in $S_i(v)$ and incident edges from G_2 and add a vertex $s_i(v)$ and edges joining $s_i(v)$ and vertices which were adjacent with $S_i(v)$ in G_2. Execute this operation for all vertices in $\cup_{0 \leq j \leq \alpha-1} \Phi_j$ and for all colors $c_0, c_1, \cdots, c_{d_m-1}$.
The resulting graph is G_3.

Lemma 4.7. *If $\chi(K(G_1)) = d_m$, then G_3 is a tree such that*
i) *for $\alpha \leq j \leq d_n - 1$, $| N_{G_3}(\Phi_j) | = d_m$ and vertices in $N_{G_3}(\Phi_i)$ are given different colors,*
ii) *for $0 \leq j \leq \alpha - 1$, and for $v, v' \in \Phi_j$, $v \neq v'$, each vertex in $N_{G_3}(v)$ is given a color different from colors given to vertices in $N_{G_3}(v')$, and vice versa.*
iii) *for $0 \leq j \leq \alpha - 1$, $| N_{G_3}(\Phi_j) | \leq d_m$.*

Proof. From the construction method, G_3 is connected.
i) Since d_m vertices in $N_{G_2}(\Phi_j)$, $\alpha \leq j \leq d_n - 1$, are given different colors, we have $| N_{G_3}(\Phi_j) | = d_m$, $\alpha \leq j \leq d_n - 1$, and vertices in $N_{G_3}(\Phi_j)$, $\alpha \leq j \leq d_n - 1$, are given different colors.
ii) For $\alpha \leq j \leq d_n - 1$ and $v, v' \in \Phi_j$ ($v \neq v'$), each vertex in $N_{G_2}(v)$ is given a color different from colors given to each vertex in $N_{G_2}(v')$. Therefore each vertex in $N_{G_3}(v)$ is given a color different from colors given to each vertex in $N_{G_3}(v')$, and vice versa.
iii) From ii), $| N_{G_3}(\Phi_j) | \leq d_m$ for $0 \leq j \leq \alpha - 1$.

Since the number of edges in G_2 minus the number of edges in G_3 is equal to the number of vertices in G_2 minus the number of vertices in G_3, and since G_3 is connected, G_3 is a tree.

When $\chi(K(G_1)) = d_m$, we have the following key theorem.

Theorem 4.8. *If $\chi(K(G_1)) = d_m$, then G_3 is extensible to an interlaced tree.*

Proof. Let the color class of d_m colors be $C = \{c_0, c_1, \cdots, c_{d_m-1}\}$. For $v \in \cup_{0 \leq j \leq \alpha-1} \Phi_j$, let $CS(v)$ be the set of colors given to vertices in $N_{G_3}(v)$. Then from Lemma 4.7, $CS(v) \cap CS(v') = \phi$ for $v, v' \in \Phi_j$, $v \neq v'$, $0 \leq j \leq \alpha - 1$.
Since $| N_{G_3}(\Phi_j) | \leq d_m$, $0 \leq j \leq \alpha - 1$, the color class C is partitioned into $C_j^0 \cup C_j^1 \cup \cdots \cup C_j^{\beta-1}$ so that $C_j^i \supseteq CS(v_j^i)$, $0 \leq i \leq \beta - 1$.
For each $i, j (0 \leq i \leq \beta - 1,\ 0 \leq j \leq \alpha - 1)$, add $| C_j^i - CS(v_j^i) |$ vertices to G_3, join those vertices with v_j^i and give different colors in $C_j^i - CS(v_j^i)$ to those vertices.
For each $i, j (0 \leq i \leq d_m - 1,\ 0 \leq j \leq d_n - 1)$, give the label u_i^j to the vertex in $N_{G_3}(\Phi_j)$ with color c_i.

The resulting tree is G. Then G divides $K(m,n)$ under σ and τ, and satisfies

$$N_G(\Phi_j) = \{u_i^j \mid 0 \le i \le d_m - 1\}, \quad (0 \le j \le \alpha - 1).$$

Therefore, G is an interlaced tree and this completes the proof. □

5. Existence of interlaced trees

In the previous section, we have shown that if there exists a tree G_1 with partite sets U_1 and V_1 satisfying $\sum_{v \in \Phi_j} deg_{G_1}(v) = d_m$, $\alpha \le j \le d_n - 1$ and $\chi(K(G_1)) = d_m$, then we can find an interlaced tree. Then, the remaining task is to show the existence of such a tree G_1.

Construction Algorithm of G_1

Step 1. Determine $a_0, a_1, a_2, \cdots, a_{\alpha'\beta}$ so that

(1) $a_0 = 0$,

(2) $a_r \ge 2$ are integers, $(1 \le r \le \alpha'\beta)$,

(3) $\sum_{j=1}^{\beta} a_{i\beta+j} = d_m = \beta + \beta'$, $(0 \le i \le \alpha' - 1)$.

Step 2. Construct graphs H_i $(0 \le i \le \alpha'\beta - 1)$.

H_i is a complete bipartite graph $K(1, a_{i+1})$ with partite sets $U_i = \{v_{\alpha\beta+i}\}$ and $V_i = \{v_j \mid \sum_{k=0}^{i} a_k - i \le j \le \sum_{k=0}^{i+1} a_k - (i+1)\}$.

Step 3. $G_1 = \cup_{0 \le i \le \alpha'\beta-1} H_i$. □

Note that existence of integers $a_0, a_1, a_2, \cdots, a_{\alpha'\beta}$ satisfying conditions in Step 1 is shown as follows. If $a_r = 2$ $(1 \le r \le \alpha'\beta)$, then $\sum_{j=1}^{\beta} a_{i\beta+j} = 2\beta \le \beta + \beta'$ $(0 \le i \le \alpha' - 1)$. Therefore, we can determine a_r $(1 \le r \le \alpha'\beta)$ satisfying (2) and (3) in Step 1.

Lemma 5.1. *The graph G_1 obtained by the above algorithm satisfies $\chi(K(G_1)) = d_m$.*

Proof. Subdivide each edge of G_1 and give a label u_{i+j} for the vertex added on the edge $v_{\alpha\beta+i}v_j$. The resulting graph is G_2.

Let $\Phi_j = \{v_{j\beta}, v_{j\beta+1}, \cdots, v_{j\beta+(\beta-1)}\}$, $(0 \le j \le d_n - 1)$.

Then, for $0 \le j \le \alpha' - 1$, $N_{G_2}(\Phi_{\alpha+j}) = \{u_{jd_m}, u_{jd_m+1}, \cdots, u_{jd_m+(d_m-1)}\}$.

Let j be the residue of i mod d_m and give the color c_j to the vertex u_i. Then $N_{G_2}(\Phi_j)$, $(\alpha \le j \le d_n - 1)$, consists of d_m vertices of different colors. Also, vertices in $N_{G_2}(\Phi_j)$, $(0 \le j \le \alpha - 1)$, are given different colors. Therefore, $\chi(K(G_1)) = d_m$.

Now we should conclude this paper.

Starting from the above algorithm, we can construct an interlaced tree for m, n such that $m + n - 1 \mid mn$. Therefore, this means the following theorem.

Theorem 5.2. *A complete bipartite graph $K(m,n)$ is divisible by a tree if and only if $m + n - 1 \mid mn$.* □

6. Example

In this section, we show an example to construct an isomorphic tree factor of $K(m,n)$. Here we use the FP $[m,n] = [21,15]$. Then, $d_m = 7$, $d_n = 5$, $\alpha = 3$, $\alpha' = 2$, $\beta = 3$, $\beta' = 4$. Graphs are constructed according to the algorithm and results in Sections 4 and 5.

1) $\mathbf{H}_i$'s

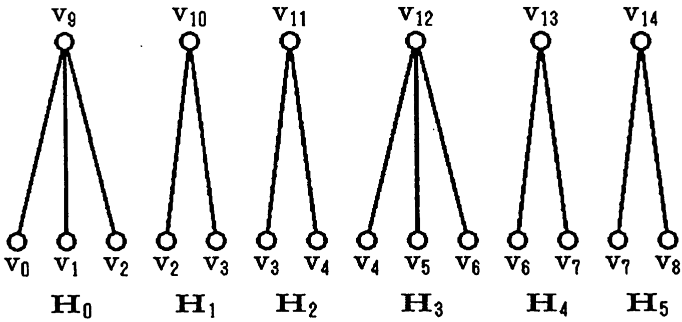

Fig. 2-1

2) $\mathbf{G}_1$

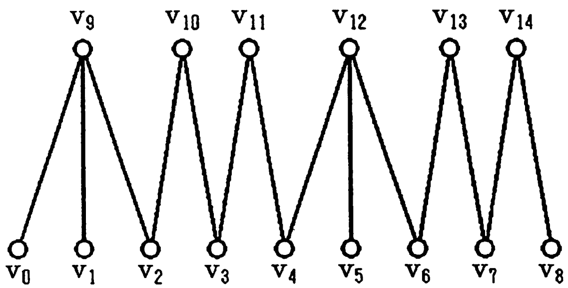

Fig. 2-2

3) G_2

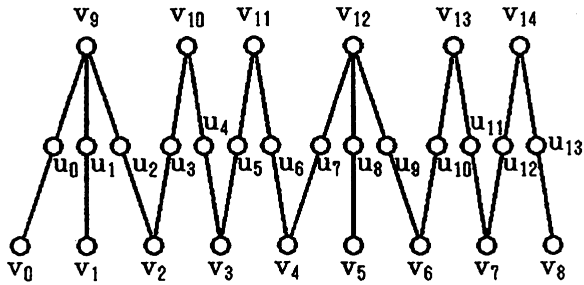

Fig. 2-3

4) Coloring of G_2

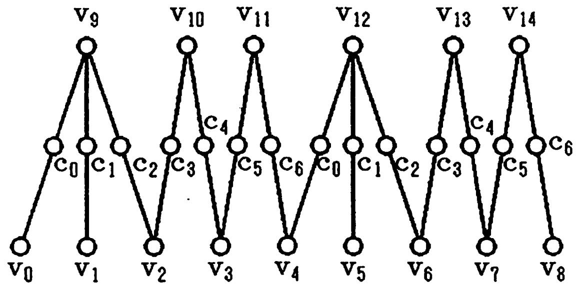

Fig. 2-4

5) G_3

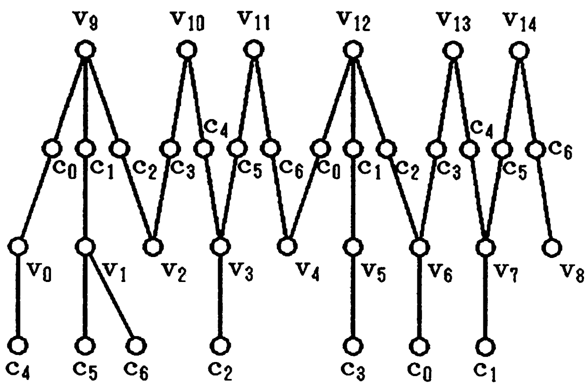

Fig. 2-5

6) G

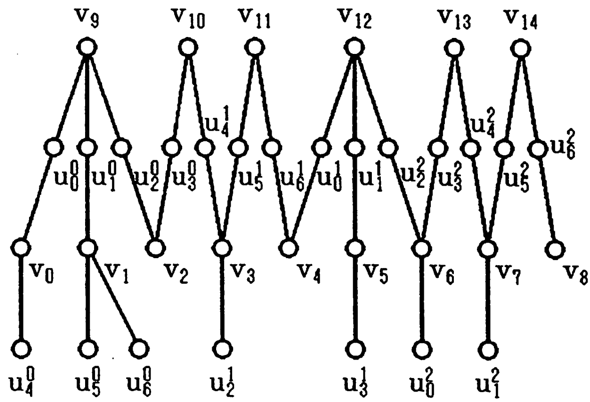

Fig. 2-6

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