On Edge-Coloring Graphs

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1. Introduction

It is well known [15] that every graph G has edge-chromatic number $\triangle(G)$ or $\triangle(G) + 1$. In the former (latter) case we say G is of class 1 (class 2).

Our goal in this paper is to investigate the edge coloring problem for three important classes of graphs. In Section 2, we consider outerplanar graphs and give a new proof of a theorem, first proved by Fiorini [3], that every outerplanar block other than an odd cycle is class 1. Fiorini's proof used a major lemma of Vizing [16] whereas our proof relies on two simple properties of outerplanar blocks. (Another proof of the theorem is reported in Fiorini and Wilson [4], but appears incomplete since the inductive assumption does not necessarily hold when $\triangle(G) = 3$.) In Section 3 we consider 2-degenerate graphs, and show in particular that every 2-degenerate graph G with $\triangle(G) \geq 4$ is class 1. We also suggest that determining the class of 2-degenerate graphs with $\triangle \leq 3$ is probably very difficult. In Section 4 we investigate the problem of edge coloring planar graphs. We give a rather striking conjecture which states that there is an easy way to characterize the class of any planar graph. This conjecture in turn implies a number of important results about edge-coloring planar graphs (e.g., the Four Color Theorem, Vizing's planar graph conjecture, the critical graph conjecture for planar graphs and the existence of a polynomial algorithm to determine the class of a planar graph.)

2. Outerplanar Graphs

An outerplanar graph is a graph which can be embedded in the plane with all vertices on the border of the outer region. Clearly any outerplanar block except K_1 and K_2 contains a Hamiltonian cycle C. An *outerpath* in an outerplanar block is a path on C such that the terminal vertices are adjacent vertices of degree larger than 2 and all the interior vertices have degree 2. The *length* of a chord xy is the length of a shortest xy path on C.

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Lemma 1. *Let G be an outerplanar block with $\triangle(G) \geq 3$. If all outerpaths have length 2 and all endvertices of outerpaths have degree $\triangle(G)$, then $\triangle(G) = 3$ or 4.*

Proof: Suppose $\triangle(G) \geq 5$. Then there exists a chord xy of length larger than 2 which is as small possible. Let $x, v_1, v_2, \ldots, v_t, y, t \geq 2$, be a shortest xy path on C. Then deg $v_1 \leq 4$, for each i, by the minimality of chord xy. Thus no v_1 is the endvertex of an outerpath, which is impossible. ∎

Lemma 2. *If G is an outerplanar block with all outerpaths of length 2 and all endvertices of outerpaths have degree 4, then there is a vertex which is the endvertex of two outerpaths.*

Proof: Without loss of generality let v_1, v_2, v_3, v_4 be consecutive vertices of C such that v_1, v_2, v_3 is an outerpath and the fourth neighbor of v_3 is v_j. Furthermore, assume the vertices have been chosen so that the chord $v_3 v_j$ is as small as possible. If $j = 5$, then v_3 is the required vertex. So assume $j > 5$. Then $v_3, v_4, \ldots, v_j$ contains an outerpath, which by hypothesis must have length 2. Then its endvertices have degree 4, and we have a chord smaller than v_3, v_j, which is impossible. ∎

Theorem 3. *If G is an outerplanar block and not an odd cycle, then G is class 1.*

Proof: We use induction on the order n of G and observe the result is true for small n and for even cycles. Let H be any outerplanar block of order $n+1$ with $\triangle(H) \geq 3$. If H has an outerpath P of length at least three, then remove the interior vertices of P from H. By the induction assumption the result is edge colorable with $\triangle(H)$ colors, and we easily extend that coloring to H. Thus we assume that all outerpaths have length 2. If endvertex u of outerpath $P: u, x, v$ has degree less than $\triangle(H) = t$, then we extend any t-edge-coloring of $H - x$ by coloring xv and then xu with available colors. Thus we assume that all endvertices of outerpaths have degree $\triangle(H)$. By Lemna 1, $\triangle(H) = 3$ or 4.

If $\triangle(H) = 3$, identify vertices u, x, and v of any outerpath. We 3-edge-color the result and then extend that coloring to H.

If $\triangle(H) = 4$, then by Lemma 2 there is a vertex v as shown in Figure 1. Then 4-edge-color $H - y$. If four colors are used on the edges incident with v and z, we can easily extend the coloring to H. So suppose only colors 1, 2, and 3 are used as shown in Fig. 1. If edge ux is not colored 4, then recolor edge vx with 4 and we are back in an earlier case. Thus assume ux is colored 4. Now interchange colors 2 and 4 at u, and we are again back in the previous case. ∎

Since a connected graph with more than one block can easily be edge colored by suitably using a minimum number of colors on each block we have

Corollary 4. *A connected outerplanar graph is class 1 iff it is not an odd cycle.*

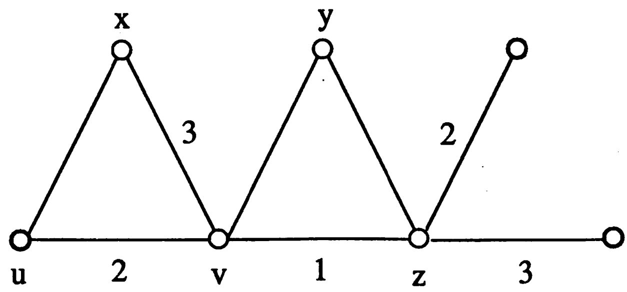

Figure 1

3. 2-Degenerate Graphs

The concept of k-degenerate graphs was introduced in [12] and has been investigated in a number of papers including [11] and [13] Specifically a graph is called k-degenerate if each subgraph has a vertex of degree at most k. Thus all outerplanar graphs are 2-degenerate. However, many 2-degenerate graph are not outerplanar. For example, the graph obtained by subdividing each edge of K_n, $n \geq 4$, is 2-degenerate but not outerplanar. Since 2-degenerate graphs are a generalization of outerplanar graphs, it is natural to attempt to determine which 2-degenerate graphs are class 1.

Theorem 5. *If G is 2-degenerate with $\Delta(G) \geq 4$, then G is class* 1.

Proof: Suppose H is a counterexample of smallest order after observing that the theorem is true for small graphs. It follows then that H has a vertex of degree 2. Then we have the following two properties.

(1) Each vertex x of degree 2 has both neighbors of degree Δ. Otherwise, we can easily extend any Δ-edge-coloring of $H - x$ to H.

(2) No two vertices of degree 2 have the same neighbors. In order to see this suppose degree 2 vertices w_1 and w_2 have common neighbors x_1 and x_2. Then there are two colors available at each of x_1 and x_2 and these colors can be used to color the edges $w_1x_1, w_1x_2, w_2x_1, w_2x_2$.

Now remove all vertices of degree 2 from H. The result H' is a 2-degenerate graph and thus has a vertex w of degree at most 2. We may assume w has degree 2, for otherwise, we put one or two of the removed vertices in H' to make w have degree 2. However, by (1), w has degree Δ in H. Let $w_1, \ldots, w_t$, $t = \Delta - 2$, be vertices of $H - H'$ which are adjacent to w. For $1 \leq i \leq t$, let $x_i \neq w$ be a neighbor of w_i. According to (2) all x_i are distinct and from (1) we know each has degree Δ in H.

Now let 1 and 2 be the colors used on the edges of H' incident with w. Each edge w_ix_i, $1 \leq i \leq t$, has an available color. Suppose that some x_iw_i has either 1 or 2 available, then without loss of generality assume 1 is available at x_tw_t. Use the available colors on each x_iw_i. Then successively color $ww_1, \ldots, ww_t$

with available colors. Since color 1 is used on edges incident with both w and w_t, there is a color available for ww_t. Thus we have Δ-edge-colored H, which is impossible. Thus we assume 1 and 2 are unavailable at each x_iw_i.

If the set of colors available at the various x_iw_i has order at least 2, let these available colors be $C_1, \ldots, C_r, r \geq 2$. Without loss of generality we may assume that C_i is available at x_iw_i, $1 \leq i \leq r$. Color each w_ix_i, $1 \leq i \leq t$ with the available color. Color ww_i with C_{i+1} for $1 \leq i \leq r$ and ww_r with C_1. Now there are available colors for (consecutively) $ww_{r+1}, \ldots, ww_t$. Thus we can Δ-edge-color H in this case which is impossible. Thus only one color, say 3, is available for each x_iw_i, $1 \leq i \leq t$. Find the maximum 1–3 path P from w. Such a path contains at most one of the x_i. However $t > 1$, so there is a vertex $x_j, j \neq i$. Interchange colors 1 and 3 on P. The result is an edge coloring of H' with colors 2 and 3 used at w and colors 1 and 3 or just color 3 available for the various x_iw_i. Either way this returns us to an earlier case and completes the proof. ∎

In order to see that the above theorem cannot be improved we observe that any cubic connected graph with one edge subdivided is 2-degenerate and class 2. In order to see this let G be the graph which results when one edge of a cubic graph is subdivided. Clearly G is 2-degenerate with $2k + 1$ vertices and $3k + 1$ edges. If G were class 1, then one of its color classes would contain $k + 1$ edges. But then one of the $2k + 1$ vertices would be incident with 2 edges in the same color class which would be impossible.

We have seen that 2-degenerate graphs G with $\Delta(G) \geq 4$ are class 1, but that 2-degenerate graphs with one vertex of degree 2 and all others of degree 3 are class 2. In the next section we will give evidence that characterizing the class of 2-degenerate graphs G with $\Delta(G) \leq 3$ is probably not easy even when G is planar.

4. Planar Graphs

We begin with the following definition.

Definition A graph G is called overfull if $|E(G)|/\lfloor \frac{1}{2}|V(G)| \rfloor > \Delta(G)$.

It is clear that an overfull graph G must be class 2, since no color class can contain more than $\lfloor \frac{1}{2}|V(G)| \rfloor$ edges. We saw in the previous section that a 2-degenerate graph with degree sequence $3^{2k}2$ (i.e., $2k$ vertices of degree 3 and one of degree 2) is class 2. It is easily checked that a graph with this degree sequence is overfull.

On the other hand, there exist class 2 planar graphs which are not themselves overfull. To form such a graph, let H denote the graph K_4 with one edge subdivided. Form G by taking two copies of H and then joining the degree two vertex in each by an edge. Then G is planar, class 2 but not overfull. Of course G contains a subgraph H with $\Delta(H) = \Delta(G)$ which is overfull. This suggests our next definition.

Definition. A subgraph H of G is called $\Delta(G)$*-overfull* if $\Delta(H) = \Delta(G)$ and H is overfull.

It is apparent that any graph G containing a $\Delta(G)$-overfull subgraph is certainly class 2. On the other hand, there are graphs G with no $\Delta(G)$-overfull subgraph which are nonetheless class 2 (e.g., the Petersen graph.) However every example of a class 2 graph G with no $\Delta(G)$-overfull subgraph of which we are aware is nonplanar. This leads us to make the following conjecture, though we must admit that the only evidence for the conjecture is that neither we nor anyone to whom we have mentioned the question knows of a counterexample.

Conjecture 1. *A planar graph G is class 1 if and only if G does not contain a $\Delta(G)$-overfull subgraph.*

It is interesting to note that there are class 2 graphs G with crossing number one which do not contain a $\Delta(G)$-overfull subgraph (e.g., the Petersen graph with a vertex deleted).

Conjecture 1 would have a number of important corollaries. Before giving these however, we require the following fact.

Theorem 6. *The degree sequence of any overfull planar graph has one of the following forms* $2^{2k+1}, 3^{2k}2, 4^{2k}2, 4^{2k+1}3^2, 4^{2k+1}, 5^{2k}2, 5^{2k+1}43, 5^{2k}4^3$ *or* $5^{2k}4$.

Proof: It is easy to verify that each degree sequence above belongs to an overfull graph.

To show that there are no other degree sequences of overfull planar graphs, we observe first that an overfull graph must have odd order. Consider the possible maximum degree of an overfull planar graph. First, since $|E(G)| \leq 3(|V(G)| - 2)$ for any planar graph G, it is readily seen that if a planar graph G satisfies $\Delta(G) \geq 6$, it cannot be overfull.

Suppose next that G is a planar graph with $\Delta(G) = 5$. If G has $2k$ vertices of degree 5 and $2t + 1$ other vertices, then we find

$$\frac{|E|}{\lfloor \frac{1}{2}|V| \rfloor} \leq \frac{5k + 4t + 2}{k + t} \leq 5,$$

unless $t = 0$ or 1. Thus G has either one or three vertices of degree < 5. It is easy to verify that G must have one of the degree sequences $5^{2k}2$, $5^{2k}4^3$ or $5^{2k}4$. If G has $2k + 1$ vertices of degree 5 and $2t$ other vertices, then again (1) holds unless $t = 0$ or 1. It is readily checked that G must have a degree sequence of the form $5^{2k+1}43$.

If $\Delta(G) \leq 4$, analogous arguments show that the degree sequence of G is one of those listed in the theorem. ∎

We now turn to some important implications of Conjecture 1.

Implication 1 of Conjecture 1. (Four Color Theorem—Edge Coloring Version) *Every cubic planar block is class 1.*

Proof: Let G be a cubic planar block. If G were not class 1, then by Conjecture 1 G contains a 3-overfull subgraph H which, by Theorem 6, has a degree sequence of the form $3^{2k}2$. The edge belonging to $E(G) - E(H)$ incident to the degree 2 vertex of H would be a cut edge in G, contradicting the fact that G was a block. ∎

Since Conjecture 1 implies the Four Color Theorem, it almost certainly will be difficult to prove. On the other hand, one might hope to use the Four Color Theorem to give a proof of Conjecture 1.

Vizing [16] and later Melnikov [14] proved that any planar graph G with $\Delta(G) \geq 8$ is class 1, and Vizing conjectured that this was true for all planar G with $\Delta(G) \geq 6$. This conjecture would be an easy corollary of Conjecture 1 and Theorem 6.

Implication 2 of Conjecture 1. (Vizing's Planar Graph Conjecture [16]) *If G is a planar graph with $\Delta(G) \geq 6$, then G is class* 1.

Proof: By Theorem 6, there are no 6-overfull planar graphs. So by Conjecture 1, G is class 1. ∎

Another conjecture which has received a good deal of attention is the Critical Graph Conjecture. It was first stated by Jacobsen [9] and, in a slightly different form, by Beineke and Wilson [1]. Jacobson called a connected graph *critical* if it is class 2 and the removal of any edge lowers the edge chromatic number. The Critical Graph Conjecture states that every critical graph is off odd order. This conjecture turned out to be false; the first counterexample was given by Goldberg [5]. A brief history of the rise and fall of the conjecture is given by Chetwynd and Wilson in [2]. However, all of the counterexamples to the conjecture which have appeared in the literature are nonplanar. Thus it is interesting to ask if there are any planar counterexamples. An implication of Conjecture 1 is that there are not.

Implication 3 of Conjecture 1. *Every critical planar graph has odd order.*

Proof: Suppose G is a critical planar graph of even order. Since G is class 2, Conjecture 1 implies that G contains a $\Delta(G)$-overfull subgraph H. By Theorem 6, H has odd order, and hence H is a proper subgraph of G. So if e is any edge incident to a vertex in $V(G) - V(H)$, then $G-e$ still contains H. Hence $\chi'(G-e)$ is still $\Delta(G) + 1$, contrary to the assumption that G is critical. ∎

The algorithmic difficulty of determining the class of an arbitrary graph was open for some time. Finally Holyer [7] showed that it is NP-hard to determine the edge-chromatic number even of cubic graphs. However Holyer's work left open the algorithmic difficulty of determining the class of an arbitrary planar graph. We now show that if Conjecture 1 is true, then there exists a polynomial algorithm for this problem.

Implication 4 of Conjecture 1. *There exists a polynomial algorithm to determine the class of any planar graph.*

Proof: By Conjecture 1, the class of any planar graph G depends on the existence of a $\Delta(G)$-overfull subgraph in G. By Theorem 6, there are only a finite number of possible types of degree sequences for $\Delta(G)$-overfull subgraphs of G. In the sequel we will assume $\Delta(G) = 5$, and indicate how one might determine if there is a 5-overfull subgraph with degree sequence $5^{2k}4^3$ in polynomial time; a similar algorithm will easily be seen to exist for each of the other types of degree sequences of overfull planar graphs. Thus the desired polynomial algorithm will exist.

To determine if G contains a subgraph with degree sequence $5^{2k}4^3$, we first select a set $\{x_1, x_2, x_3\}$ of vertices with $d_G(x_i) \geq 4$ for $i = 1, 2, 3$ as candidates for the degree 4 vertices in the subgraph H. Clearly the only vertices which could then be in the subgraph are those in the set $S = \{x_1, x_2, x_3\} \cup \{v_1 \mid \deg v = 5\}$. For $i = 1, 2, 3$ select (if possible) a set N_i of four neighbors of x_i in S. Once the choice of N_1, N_2, N_3 is made, the remainder of the construction is forced: Remove from S any neighbors of x_1, x_2 or x_3 which do not belong to $\bigcup_{i=1}^{3} N_i \cup \{x_1, x_2, x_3\}$. Clearly each vertex in $S_1 = \bigcup_1^3 N_i - \{x_1, x_2, x_3\}$ belongs to S. Assuming S_j is contained in S, check whether $S_{j+1} \equiv S_j \cup N_G(S_j) - \{x_1, x_2, x_3\}$ is contained in S for $j = 1, 2, \ldots$. One of two things must happen: Either $S_{j+1} \subseteq S_j$ for some $j \geq 1$, in which case the subgraph induced by $S_j \cup \{x_1, x_2, x_3\}$ will have degree sequence $5^{2k}4^3$, or else $S_{j+1} \not\subseteq S$ for some $j \geq 1$, in which case the choice of N_1, N_2, N_3 was no good. We proceed to another choice of N_1, N_2, N_3 or, once these choices are exhausted, to another choice of $\{x_1, x_2, x_3\}$. Thus we can either find a subgraph H with degree sequence $5^{2k}4^3$, or establish that none exists, in polynomial time. ∎

Our final implication of Conjecture 1 states that if a planar graph has sufficiently large edge-connectivity, then its class depends only on whether G itself is overfull.

Implication 5 of Conjecture 1. *Let G be a $(\Delta(G) - 1)$-edge-connected planar graph. Then G is class 1 if and only if G itself is not overfull.*

Proof: Suppose G is a $(\Delta(G) - 1)$-edge-connected planar graph which is class 2 but is not itself overfull. Since G is class 2, Conjecture 1 implies that G contains a *nonspanning* $\Delta(G)$-overfull subgraph H, which must have one of the degree sequences given in Theorem 6. But it is easily checked that there are at most $\Delta(G)$-2 edges possible between $V(H)$ and $V(G) - V(H)$, contradicting the fact that G is $(\Delta(G) - 1)$-edge-connected. ∎

We note that Implication 5 would immediately imply the following result, which seems an important conjecture in its own right, though apparently weaker than Conjecture 1.

Conjecture 2. *Let G be a planar block with $\Delta(G) = 3$. If G contains $s \geq 2$ vertices of degree 2, then G is class 1.*

Theorem 7. *Conjecture 1 implies Conjecture 2.*

Proof: Since G is 2-edge-connected and $\Delta(G) = 3$ it follows by Implication 5 that G is class 1 unless G is itself a 3-overfull graph. This means, by Theorem 6, that the degree sequence of G has the form $3^{2k}2$. This contradicts the fact that G contains $s \geq 2$ vertices of degree 2. ∎

Remarkably, however, even Conjecture 2 with $s = 2$ is strong enough to imply the Four Color Theorem! The following theorems will be useful for establishing this.

Halin's Theorem [6]. *Every cubic block contains an edge e such that $G - e$ is a block.*

Izbicki's Theorem [8]. *Let G be a class 1 graph in which each vertex has degree 1 or $\Delta(G)$. Given any $\Delta(G)$-edge-coloring of G, let f_i denote the number of terminal edges of color i, $1 \leq i \leq \Delta(G)$. Then all the f_i's have the same parity.*

Theorem 8. *Conjecture 2 with $s = 2$ implies the Four Color Theorem (i.e., that every cubic planar block has class 1).*

Proof: Let G be a cubic planar block. By Halin's Theorem, G contains an edge $e = (x, y)$ such that $G - e$ is a block. Conjecture 2 with $s = 2$ assures us $G - e$ is 3-edge-colorable. In any such 3-edge-coloring, Izbicki's Theorem tells us that the same color is missing at both x and y. This color is available for e to complete a 3-edge-coloring of G. ∎

We observe that exactly the same argument yields the following theorem of Jaeger [10]: Every cubic block with crossing number one is class 1. (Since the Petersen graph is a class 2 cubic block with crossing number two, Jaeger's result is in some sense best possible.)

Another implication of Conjecture 2 with $s = 2$ is that if we begin with a cubic, planar block G (which must be class 1 by the Four-Color-Theorem), and subdivide any pair of edges in G, the resulting planar block is class 1. It is interesting to note that there exist class 1 cubic nonplanar blocks with crossing number 2 for which this is false. We note that the cubic block with crossing number 2 and two edges subdivided shown in Figure 2 has class 2, although the underlying cubic block is easily seen to be class 1. To see that the graph G in Fig. 2 is class 2, note that in any 3-edge-coloring of G, e_1, e_2, e_3 must be colored differently, say with 1,2,3 respectively. Then e_4 must be colored 1 or 3; by symmetry assume 3. Then (e_5, e_6) must be colored (1,2) or (2,1). In either case, the rest of the coloring is forced until we reach an edge with no available color.

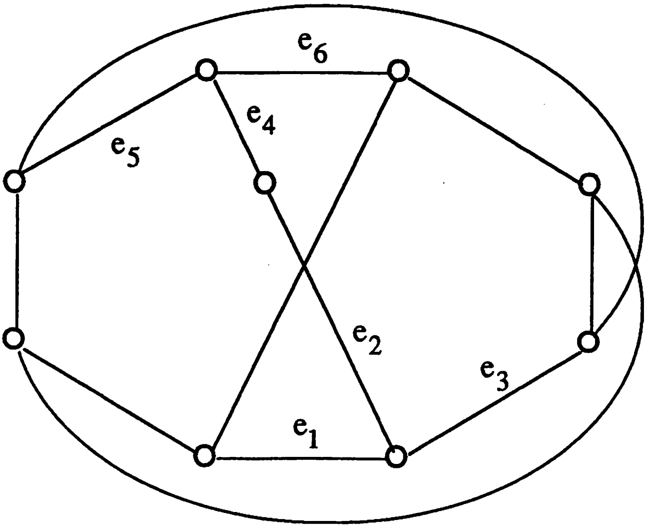

Figure 2

Finally, on the evidence that a simple proof of the Four Color Theorem is unlikely, it appears that Conjecture 2, if true, will be very difficult to prove unless the Four Color Theorem is used in the proof.

We conclude with one final conjecture which, although a special case of Conjecture 2, nonetheless suffices to prove Conjecture 2.

Conjecture 2'. *Let G be a planar bock with $\Delta(G) \geq 3$ containing $s \geq 2$ vertices of degree 2. If for any degree 2 vertex v, the neighbors u_1, u_2 of v are nonadjacent of degree 3, then G is class 1.*

Theorem 8. *Conjecture 2' implies Conjecture 2.*

Proof: Suppose Conjecture 2' is true but Conjecture 2 is false. Let G be a planar block with $\Delta(G) = 3$ and $s \geq 2$ vertices of degree 2 which is a smallest counterexample to Conjecture 2. By Conjecture 2', it must be the case that for some degree 2 vertex v in G, either u_1, u_2 are adjacent or at least one of u_1, u_2 have degree 2.

Case 1. u_1, u_2 are adjacent.

Contract $\{v, u_1, u_2\}$ to a single vertex of degree 2. The resulting graph H is class 1 by the minimality assumption on G. We can easily extend a 3-edge-coloring of H to a 3-edge-coloring of G.

Case 2. u_1 has degree 2.

If $s \geq 3$, contract v and u_1; by the minimality of G, the resulting graph H can be 3-edge-colored and then so can G. If $s = 2$, then v and u_1 are the only vertices of degree 2 in G. Let x be the other vertex adjacent to u_1. If x and u_1 are adjacent, then $H = G - v - u_1$ has two vertices of degree 2, and by the minimality of G can be 3-edge-colored. If x, u_2 are not adjacent, contract $\{v, u_1, u_2\}$ to obtain a cubic planar block H, which can be 3-edge-colored by the Four Color Theorem. Either way, the 3-edge-coloring of H can easily be extended to G.

In either case, G is 3-edge-colorable. This contradiction then completes the proof. ∎

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