# On the Existence of Balanced Graphs with Given Edge-Toughness and Edge-connectivity

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## ABSTRACT

*The edge-toughness $\tau_1(G)$ of a graph $G$ is defined as*

$$\tau_1(G) = \min\left\{\frac{|X|}{\omega(G-X)-1} \;\middle|\; X \text{ is an edge-cutset of } G\right\},$$

*where $\omega(G-X)$ denotes the number of components of $G-X$. Call a graph $G$ balanced if $\tau_1(G) = |E(G)|/(\omega(G-E(G))-1)$. It is known that for any graph $G$ with edge-connectivity $\lambda(G)$, $\lambda(G)/2 < \tau_1(G) \leq \lambda(G)$. In this paper we prove that for any integer $r$, $r \geq 2$ and any rational number $s$ with $r/2 < s \leq r$, there always exists a balanced graph $G$ such that $\lambda(G) = r$ and $\tau_1(G) = s$.*

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## 1. Introduction.

Let $G$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $\omega(G)$ the number of (connected) components of $G$. A subset $S$ of $V(G)$ is called a *vertex-cutset* of $G$ if $\omega(G-S)>1$. The *vertex-toughness* of a graph G, denoted by $\tau(G)$, is defined as:

$$\tau(G)=\min\left\{\frac{|S|}{\omega(G-S)} \;\middle|\; S \quad \text{is a vertex-cutset of} \quad G\right\},$$

with the convention that $\min\emptyset=+\infty$. The notion of $\tau(G)$, which was first introduced by Chvátal [2], has received much attention recently (see, for instance [3,4]).

A subset $X$ of $E(G)$ is called an *edge-cutset* of $G$ if $\omega(G-X)>1$. Chvátal [2] also defined the "edge-toughness" of $G$, considered as the dual concept of $\tau(G)$, as

$$\min\left\{\frac{|X|}{\omega(G-X)} \;\middle|\; X \quad \text{is an edge-cutset of} \quad G\right\}.$$

This parameter is, however, not of much interest as he showed that it is exactly one half of $\lambda(G)$, the *edge-connectivity* of $G$.

Tutte [8] and Nash-Williams [5] obtained independently the following result:

**Theorem A.** *A connected graph $G$ has $s$ edge-disjoint spanning trees if and only if $|X|\geq s\big(\omega(G-X)-1\big)$ for all $X\subseteq E(G)$.*

This suggests that one may alter the above definition of edge-toughness by replacing $\omega(G-X)$ with $\omega(G-X)-1$. Thus, as introduced in [6], the *edge-toughness* of $G$, denoted by $\tau_1(G)$, is defined as:

$$\tau_1(G)=\min\left\{\frac{|X|}{\omega(G-X)-1} \;\middle|\; X \quad \text{is an edge-cutset of} \quad G\right\}.$$

We note that the edge-toughness of a graph will remain fixed if any of its vertices is blown up to a highly connected graph. In order to avoid such triviality, we define below a graph called balanced graph, whose edge-toughness may change by simply blowing any of its vertices to a highly connected graph. Also, the edge-toughness of a balanced graph can be easily determined. (Note that the edge-toughness of a graph is not easily calculated and it is not clear whether it can be computed in polynomial time.)

A graph is said to be *balanced* if $\tau_1(G) = |E(G)|/\big(\omega(G - E(G)) - 1\big)$. Note that a graph $G$ is balanced if and only if $\tau_1(G) = |E(G)|/(|V(G)| - 1)$.

Following an argument given by Chvátal [2], the following result was shown in [6].

**Theorem B.** *Let $G$ be a connected graph of order $p$, $p \geq 2$. Then*

$$\frac{\lambda(G)}{2} < \frac{p\lambda(G)}{2(p-1)} \leq \tau_1(G) \leq \lambda(G).$$

By definition and Theorem A, it follows that a graph $G$ has $s$ edge-disjoint spanning trees if and only if $\tau_1(G) \geq s$. In connection with Theorems A and B, the following result was also proved in [6].

**Theorem C.** *For any two positive integers $r$ and $s$ with $r/2 < s \leq r$, there exists a balanced graph $G$ such that $\lambda(G) = r$, $\tau_1(G) = s$ and $G$ can be factored into exactly $s$ spanning trees.*

The following problem arises naturally. "Given an integer $r$, $r \geq 2$, and a *rational* number $s$ with $r/2 < s \leq r$, does there always exist a balanced graph $G$ such that $\lambda(G) = r$ and $\tau_1(G) = s$?" The objective of this paper is to give an affirmative answer to this question. (Note that without imposing the condition of balance, such a graph exists trivially.)

Let $A$ and $B$ be any two subsets of $V(G)$. Denote by $e_G(A, B)$ the number of edges of $G$ joining a vertex of $A$ to a vertex of $B$. For other terminology and notation not explained here, we refer to [1].

## 2. Terminology and basic results.

For a real $x$, we shall denote by $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) the largest (resp. least) integer less than (resp. greater than) or equal to $x$. As usual, let $\kappa(G)$ and $\delta(G)$ denote respectively the *vertex-connectivity* and the *minimum degree* of $G$. We begin with the following two known results on vertex- and edge-connectivity of a graph.

**Theorem D** (Whitney [9]). *For any graph $G$ of order $p$ and size $q$,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \lfloor \frac{2q}{p} \rfloor.$$

The following necessary and sufficient condition for a graph to be balanced will be used to prove our main result.

**Theorem E** (Peng and Tay [7]). *Let $G$ be a nontrivial graph of order $p$ and size $q$. Then $G$ is balanced if and only if, for every subgraph $H$ of $G$,*

$$|E(H)| \leq \frac{q}{p-1}(|V(H)| - 1).$$

For regular graphs, we have:

**Lemma 1.** *Let $G$ be a $k$-regular graph of order $p$ and size $q$. If $\lambda(G) = k$, then $G$ is balanced.*

**Proof.** By Theorem B, $\tau_1(G) \geq pk/\big(2(p-1)\big) = q/(p-1)$. □

Given any two graphs $G_1$ and $G_2$, the *join* of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and whose edge set is

$$E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \quad \text{and} \quad v \in V(G_2)\}.$$

We shall write $G_1 + v$ for $G_1 + G_2$ if $V(G_2) = \{v\}$.

The following lemma and its corollary will be needed to prove our main result.

**Lemma 2.** *Let $G$ be a nontrivial graph of order $p$ and size $q$. If $G$ is balanced, then $G + v$ is also balanced.*

**Proof.** Let $H = G + v$. Suppose that $H$ is not balanced. Since $(q+p)/p = |E(H)|/(|V(H)|-1)$, there exists a subgraph $F$ of $H$ such that $|E(F)|/(|V(F)|-1) > (q+p)/p$ by Theorem E. Note that $v \in F$. Otherwise, $F$ is a subgraph of $G$ and $|E(F)|/(|V(F)|-1) > (q+p)/p > q/(p-1)$, which contradicts the assumption that $\tau_1(G) = q/(p-1)$. We now write $F = F^* + v$ for some subgraph $F^*$ of $G$. Observe that

$$\frac{|E(F)|}{|V(F)|-1} = \frac{|E(F^*)|+|V(F^*)|}{|V(F^*)|} > \frac{q+p}{p}$$

or

$$\frac{|E(F^*)|}{|V(F^*)|} > \frac{q}{p}.$$

Since $|E(F^*)| < q$ and both $|V(F^*)|$ and $p$ are greater than one, we have:

$$\frac{|E(F^*)|}{|V(F^*)|-1} > \frac{q}{p-1},$$

which is a contradiction. □

**Corollary.** *Let $G$ be a nontrivial balanced graph. Then for any complete graph $K_n$, $K_n + G$ is also balanced.*

To end this section, we include the following simple result in arithmetic which will be found useful in obtaining some inequalities.

**Lemma 3.** *Let $a, b, x$ and $y$ be positive integers. Then*

$$\frac{a+x}{b+y} \leq \frac{a}{b} \Longleftrightarrow \frac{a}{b} \geq \frac{x}{y}.$$

# 3. Circulant wheels.

In this section, we shall introduce a class of graphs called *circulant wheels*, which will play an important role in the proof of our main result.

For any two integers $a$ and $d$ with $a \geq d \geq 2$, we denote by $S(a,d)$ the following set of integers:

$$S(a,d) = \left\{ i \;\middle|\; \left\lceil \frac{id}{a} \right\rceil < \left\lceil \frac{(i+1)d}{a} \right\rceil,\ i = 0, 1, \ldots, a-1 \right\}.$$

Thus, $S(15,9) = \{0,1,3,5,6,8,10,11,13\}$ and $S(8,5) = \{0,1,3,4,6\}$.

Note that (i) $0 \in S(a,d)$,
(ii) $S(a,a) = \{0,1,\ldots,a-1\}$,
and (iii) $a-1 \in S(a,d)$ if and only if $a = d$.

Now, assume that $a \geq 3$ and $a > d$, and let $b$ be an *even* integer such that $b \leq d$. Denote by $W(a,d,b)$ the graph, called *circulant wheel*, obtained in the following ways:

(i) Draw an $a$-gon and label its vertices by the integers $0,1,\ldots,a-1$.

(ii) Join two vertices $i$ and $j$ of the $a$-gon by an edge if and only if $i-j \equiv h(mod\ a)$ where $h \in \{2,3,\ldots,b/2\}$.

(iii) Add a new vertex $v$ adjacent to each vertex in $S(a,d)$.

The circulant wheel $W(8,5,4)$ is shown in Figure 1. Our aim in the remainder of this section is to determine the edge-connectivity and edge-toughness of the graph $W(a,d,b)$.

Two distinct integers $x$ and $y$ are *consecutive* in the *residue class modulo a* if $x, y \in \{0,1,\ldots,a-1\}$ and $|x-y| \in \{1, a-1\}$. Thus $7,8,0,1,2$ are five consecutive integers in the residue class modulo 9.

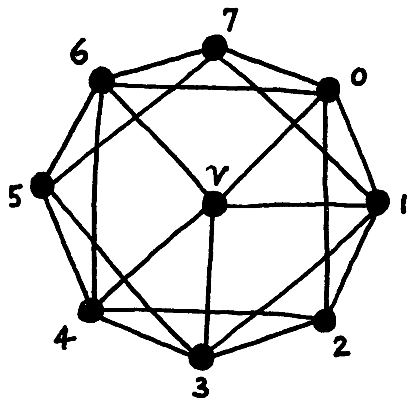


**Figure 1. The graph $W(8,5,4)$**

Two useful properties of the set $S(a,d)$ are given below.

**Lemma 4.** (1) $|S(a,d)| = d$.

(2) *For any set $T$ of $n$ consecutive integers in the residue class modulo $a$ $(n \le a)$,*

$$|T \cap S(a,d)| < \frac{nd}{a} + 1.$$

**Proof.** (1) First we note that $\lceil id/a \rceil < \lceil (i+1)d/a \rceil$ if and only if there exists an integer $t$ such that $id/a \le t < (i+1)d/a$. Since $d \le a$, the number of such an integer $t$ in the interval is at most one. Thus there is a one to one correspondence between the integers $t$ and the elements of $S(a,d)$. Since the number of the integers $t$ satisfying $0 \times d/a \le t < (a-1+1)d/a$ is $d$, we conclude that $|S(a,d)| = d$.

(2) Let $T = \{h + i \mid i = 1, 2, \ldots, n\}$. If $0 \notin T$, then by an argument similar to that given in (1) above, the number of elements of $T$ in $S(a,d)$ is equal to the number of integers $t$ satisfying $(h+1)d/a \le t < (h+n+1)d/a$, which is less than $nd/a + 1$. If $0 \in T$, then $h + r = 0$ for some $r \in \{1, 2, \ldots, n\}$. Let $T_1 = \{h+1, h+2, \ldots, h+r-1\}$ and $T_2 = \{h+r, h+r+1, \ldots, h+n\}$. Note that $T_1$ may be empty. Again, by note (iii), we have

$$|T_1 \cap S(a,d)| \le (r-1)\frac{d}{a}$$

and

$$|T_2 \cap S(a,d)| \le \left\lceil (n-r+1)\frac{d}{a} \right\rceil < (n-r+1)\frac{d}{a} + 1.$$

Thus

$$|T \cap S(a,d)| = |T_1 \cap S(a,d)| + |T_2 \cap S(a,d)| < \frac{nd}{a} + 1,$$

as required. □

We now have:

**Theorem 1.** *The graph $W(a,d,b)$ is balanced with edge-connectivity $b$.*

**Proof.** For convenience, we write $G$ for $W(a,d,b)$. We first prove that $\lambda(G) = b$. It is clear from the construction that $G - v$ is a $b$-regular graph. Thus $\lambda(G - v) \le b$ by Theorem D. On the other hand, in order to disconnect $G - v$, it is necessary to remove two suitable disjoint subsets with $b/2$ consecutive vertices each from the $a$-gon. Thus $\lambda(G - v) \ge \kappa(G - v) \ge b$ by Theorem D, and we have $\lambda(G - v) = b$. Note that $deg(v) = d \ge b$ by Lemma 4(1). Thus $\lambda(G) \ge b$. On the other hand, as $a > d$, there is at least one vertex on the $a$-gon of degree $b$. Thus $\lambda(G) \le \delta(G) = b$ by Theorem D, and we conclude that $\lambda(G) = b$.

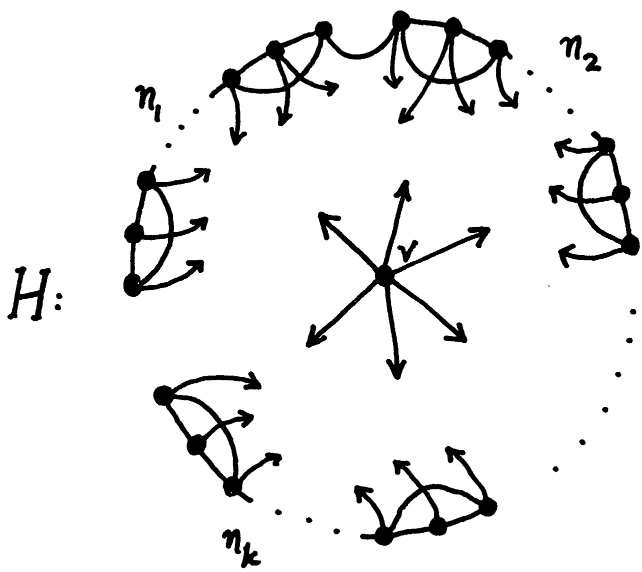


**Figure 2.**

To prove that $G$ is balanced, we first note that $p = |V(G)| = a + 1$ and $q = |E(G)| = ab/2 + d$. Now let $H$ be any subgraph of $G$ with $p'$ vertices and $q'$ edges. Since $G - v$ is $b$-regular and $\lambda(G - v) = b$, we have by Lemma 1, $G - v$ is a balanced graph. Thus if $v \notin V(H)$, then by Theorem E,

$$\frac{q'}{p' - 1} \le \frac{|E(G - v)|}{|V(G - v)| - 1} = \frac{ab}{2(a - 1)}.$$

Since $ab/2(a-1) \le b \le d$, we have by Lemma 3,

$$\frac{q'}{p'-1} \le \frac{ab}{2(a-1)} \le \frac{ab/2+d}{a-1+1} = \frac{q}{p-1}.$$

Suppose now that $v \in V(H)$. If $H-v$ contains the $a$-gon, it is obvious that $q'/(p'-1) \le q/(p-1)$. Otherwise, let $H-v$ be the union of $k$ ($\ge 1$) paths of order $n_1, n_2, \dots, n_k$ respectively (see Figure 2). Then $p' = \sum_{i=1}^{k} n_i + 1$ and by Lemma 4(2),

$$q' \le \sum_{i=1}^{k}(n_i-1) + \sum_{i=1}^{k} n_i \frac{(b-2)}{2} + \sum_{i=1}^{k}(n_i \frac{d}{a}+1)$$
$$= \frac{b}{2}\sum_{i=1}^{k} n_i + \frac{d}{a}\sum_{i=1}^{k} n_i.$$

Thus

$$\frac{q'}{p'-1} \le \frac{b}{2} + \frac{d}{a} = \frac{q}{p-1}.$$

By Theorem E, $G$ is a balanced graph. □

# 4. Main results

We shall prove in this section the following main result of this paper.

**Theorem 2.** *For any integer $r$, $r \ge 2$, and for any rational number $s$ satisfying $r/2 < s \le r$, there exists a balanced graph $G$ such that $\lambda(G) = r$ and $\tau_1(G) = s$.*

The case when $s$ is an integer is contained in Theorem C. It remains to prove the result in Theorem 2 for noninteger $s$. To get to this, we first consider in what follows two special cases.

**Theorem 3.** *Let $t$ be an even positive integer, and let $s$ be any rational number satisfying $t/2 < s < t/2+1$. Then there exist infinitely many balanced graphs $G$ of different orders and sizes such that $\lambda(G) = t$ and $\tau_1(G) = s$.*

**Proof.** Let $s = m/n$ where $m, n$ are integers and let $\alpha$ be any natural number satisfying $\alpha \ge 2t/(2m-tn)$. Denote

$d = \alpha(m - tn/2)$. Note that $\alpha n > d \geq t \geq 2$ because $\alpha \geq 2t/(2m - tn)$ and $s < t/2 + 1$. Therefore we can construct the circulant wheel $G = W(\alpha n, d, t)$ and by Theorem 1, $G$ is a balanced graph with $\lambda(G) = t$ and $\tau_1(G) = s$. Note that if $\alpha_1 \neq \alpha_2$, then $G_1 = W(\alpha_1 n, \alpha_1(m - tn/2), t)$ and $G_2 = W(\alpha_2 n, \alpha_2(m-tn/2), t)$ are balanced graphs of different orders and sizes, but $\lambda(G_1) = \lambda(G_2) = t$ and $\tau_1(G_1) = \tau_1(G_2) = s$. □

**Theorem 4.** *Let $r$ be an odd integer, $r \geq 3$, and let $s$ be any rational number satisfying $r/2 < s < (r+1)/2$. Then there exist infinitely many balanced graphs $G$ of different orders and sizes such that $\lambda(G) = r$ and $\tau_1(G) = s$.*

**Proof.** Let us write $s = m/n$ where $m$ and $n$ are integers, and let $\alpha$ be any natural number such that $\alpha n$ is *even* and $\alpha \geq 2r/(2m-rn)$. Denote $d = \alpha(m-rn/2)$. Since $\alpha \geq 2r/(2m-rn)$ and $s < (r+1)/2$, we have $\alpha n > d \geq r$. Thus we can construct the circulant wheel $W(\alpha n, d, r-1)$.

Let $G$ be a graph obtained from $W(\alpha n, d, r-1)$ by joining $\alpha n/2$ pairs of vertices which are diametrically opposite in the $\alpha n$-gon of $W(\alpha n, d, r-1)$. We claim that $G$ is a balanced graph with $\lambda(G) = r$ and $\tau_1(G) = s$.

We first show that $\lambda(G) = r$. It is clear from the construction that $G - v$ is an $r$-regular graph. Thus $\lambda(G - v) \leq r$ by Theorem D. We now show that $\lambda(G - v) \geq r$. Note that this follows from Theorem D if we can show that at least $r$ vertices must be removed to disconnect the graph $G - v$. We observe that in order to disconnect this graph, it is necessary to remove two suitable disjoint subsets, each with $(r-1)/2$ consecutive vertices, along the $\alpha n$-gon to break the circumferential connection, and at least one more vertex to break the diameteric connection. Thus at least $r$ vertices must be removed to disconnect $G - v$. Hence $\lambda(G - v) \geq r$ and therefore $\lambda(G - v) = r$. Since $\alpha n > deg(v) \geq r$, $\delta(G) = r$. We thus have the desired result.

Next we show that $G$ is a balanced graph with $\tau_1(G) = s$. Note that $|V(G)| = p = \alpha n + 1$ and $|E(G)| = q = \alpha nr/2 + d = \alpha m$. Let $H$ be any subgraph of $G$ with $p'$ vertices and $q'$ edges. If $v \notin V(H)$, then by Theorem E, Lemmas 1 and 3 together with the inequality $\alpha nr/2(\alpha n - 1) \leq r \leq d$, we have

$$\frac{q'}{p'-1} \leq \frac{|E(G-v)|}{|V(G-v)|-1} = \frac{\alpha nr/2}{\alpha n - 1} \leq \frac{\alpha nr/2 + d}{\alpha n - 1 + 1} = \frac{q}{p-1}.$$

Suppose now that $v \in V(H)$. If $H-v$ contains the $\alpha n$-gon, it is clear that $q'/(p'-1) \leq q/(p-1)$. Otherwise, we let $H-v$ be the union of $k$ $(\geq 1)$ paths of order $n_1, n_2, \ldots, n_k$ respectively (see Figure 3). Then $p' = \sum_{i=1}^{k} n_i + 1$, and by Lemma 4(2),

$$q' \leq \sum_{i=1}^{k}(n_i - 1) + \sum_{i=1}^{k}(\frac{r-2}{2})n_i + \sum_{i=1}^{k}(n_i \frac{d}{\alpha n} + 1)$$
$$= \frac{r}{2}\sum_{i=1}^{k} n_i + \frac{d}{\alpha n}\sum_{i=1}^{k} n_i.$$

Thus

$$\frac{q'}{p'-1} \leq \frac{r}{2} + \frac{d}{\alpha n} = \frac{q}{p-1}.$$

By Theorem E, $G$ is a balanced graph with $\tau_1(G) = s$ as required. Note that different values of $\alpha$ give rise to balanced graphs $G$ of different orders and sizes with $\lambda(G) = r$ and $\tau_1(G) = s$. The proof is now complete. □

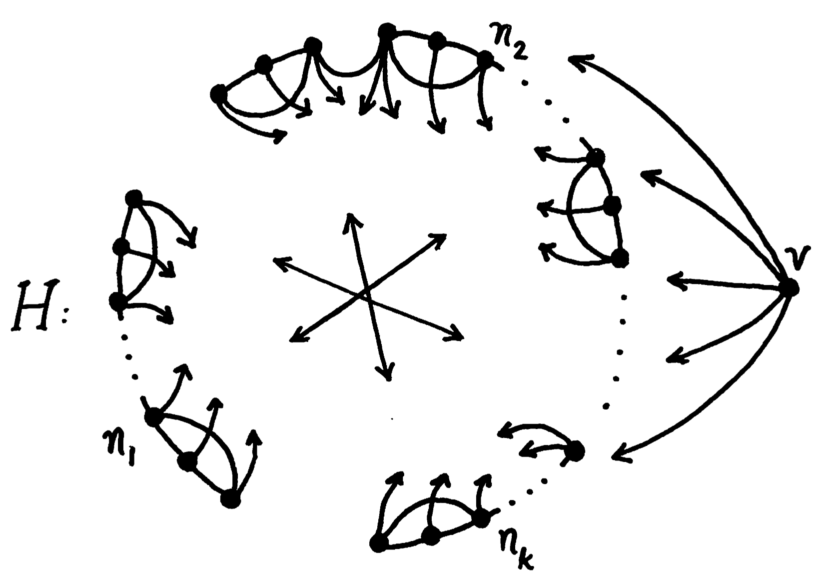


**Figure 3.**

**Theorem 5.** *Let $r$ be an integer, $r \geq 2$, and let $s$ be any noninteger rational number satisfying $r/2 < s < r$. Then there exist infinitely many balanced graphs $G$ of different orders and sizes such that $\lambda(G) = r$ and $\tau_1(G) = s$.*

**Proof.** We need only to consider the case where there exists an integer $\beta \geq r/2$ such that $\beta < s < \beta + 1$; for otherwise, $r$ is odd and $r/2 < s < (r+1)/2$, and in this case it is equivalent to Theorem 4.

The case $\beta = r/2$ has been proved in Theorem 3. We may now assume that $r \geq 3$ and note that $r > \beta$. Let us write $s = m/n$ where $m, n$ are integers, and let $\alpha$ be any natural number satisfying

$$\alpha > \max\left\{\frac{2\beta - 1}{n}, \frac{r-1}{2(m - n\beta)}, \frac{(2\beta - r)(r+1)}{2(\beta n + n - m)}\right\}.$$

With this value of $\alpha$, we let

$$p^* = \alpha n + r - 2\beta$$

and

$$d^* = \alpha(m - n\beta) + \frac{(2\beta - r)(r-1)}{2}.$$

Note that $p^* \geq 3$ since $\alpha > (2\beta - 1)/n$ and $r \geq 3$.

**Claim.** $p^* > d^* \geq 2(r - \beta)$.

We first prove the left inequality.

Observe that

$$\begin{aligned} p^* > d^* &\iff \alpha n + r - 2\beta > \alpha(m - n\beta) + \frac{(2\beta - r)(r-1)}{2} \\ &\iff \alpha(\beta n + n - m) > 2\beta - r + \frac{(2\beta - r)(r-1)}{2} \\ &\iff \alpha > \frac{(2\beta - r)(r+1)}{2(\beta n + n - m)}. \end{aligned}$$

The last inequality holds by our choice of $\alpha$. Thus we have $p^* > d^*$.

Now we show the right inequality. Observe that

$$\begin{aligned} d^* \geq 2(r-\beta) &\iff \alpha(m-n\beta) + \frac{(2\beta-r)(r-1)}{2} \geq 2(r-\beta) \\ &\iff \alpha(m-n\beta) \geq \frac{3r-2\beta-2\beta r+r^2}{2} \\ &\iff \alpha \geq -\left[\frac{(2\beta-r)(r-1)-4(r-\beta)}{2(m-n\beta)}\right]. \end{aligned}$$

Note that in the last inequality above, if $\beta = (r+1)/2$, then $\alpha \geq (r-1)/2(m-n\beta)$; and if $\beta \geq (r+2)/2$, then the quotient in the square bracket is always positive. Thus by the choice of $\alpha$, we have $d^* \geq 2(r-\beta)$ as desired.

The above claim enables us to construct the circulant wheel $W(p^*, d^*, 2(r-\beta))$. Define the join

$$G = K_{2\beta-r} + W(p^*, d^*, 2(r-\beta)).$$

We shall show that the graph $G$ is balanced with $\lambda(G) = r$ and $\tau_1(G) = s$. We first prove that $G$ is balanced with $\tau_1(G) = s$.

If $2\beta = r$, then $\tau_1(G) = \tau_1(W(p^*, d^*, 2(r-\beta))) = 2(r-\beta)/2 + d^*/p^* = \alpha m/\alpha n = s$. If $2\beta > r$, then by the Corollary to Lemma 2, $\tau_1(G) = |E(G)|/(|V(G)|-1)$. But $|V(G)| = (2\beta - r) + (p^* + 1) = \alpha n + 1$ and $|E(G)| = (2\beta-r)(2\beta-r-1)/2 + (2\beta-r)(p^*+1) + 2(r-\beta)p^*/2 + d^* = \alpha m$. Thus $G$ is a balanced graph with $\tau_1(G) = s$, as required.

To prove that $\lambda(G) = r$, we proceed as follows. Since $\alpha > (2\beta-1)/n$, we have $p^* \geq r$. Let $S$ be any non-empty proper subset of $V(G)$. Write $G^*$ for $W(p^*, d^*, 2(r-\beta))$. If $S \cap V(K_{2\beta-r}) = V(K_{2\beta-r})$, then

$$e_G(S, V(G)-S) \geq \begin{cases} p^* + 1 > r & \text{if } S \cap V(G^*) = \emptyset, \\ 2\beta - r + \lambda(G^*) = r & \text{otherwise.} \end{cases}$$

If $S \cap V(K_{2\beta-r}) \neq V(K_{2\beta-r})$ and $S \cap V(K_{2\beta-r}) \neq \emptyset$, then we let $u \in S \cap V(K_{2\beta-r})$ and $v \in V(K_{2\beta-r}) - S$. Thus

$$\begin{aligned} e_G(S, V(G)-S) &\geq e_G(v, S \cap V(G^*)) + e_G(u, V(G^*) - S) \\ &= p^* + 1 > r. \end{aligned}$$

If $S \cap V(K_{2\beta-r}) = \emptyset$, then

$$e_G(S, V(G) - S) \geq \begin{cases} p^* + 1 > r & \text{if } S = V(G^*), \\ 2\beta - r + \lambda(G^*) = r & \text{otherwise.} \end{cases}$$

Since $e_G(S, V(G) - S) \geq r$ for any non-empty proper subset $S$ of $V(G)$, we have $\lambda(G) \geq r$. On the other hand, by Theorem D, $\lambda(G) \leq \delta(G) = r$ because $p^* > d^*$ implies the existence of a vertex in $G^*$ of degree $2(r - \beta)$. Hence $\lambda(G) = r$, as desired.

Observe that different values of $\alpha$ give rise to balanced graphs $G$ of different orders and sizes with $\lambda(G) = r$ and $\tau_1(G) = s$. The proof is now complete. □

Finally, we note that our main result (i.e., Theorem 2) now follows from Theorems 5 and C.

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# References

[1] M. Behzad, G. Chartrand, and L. Lesniak-Foster, *Graphs and Digraphs*, Wadsworth, Belmont, Calif. (1979).

[2] V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Mathematics* 5(1973) 215–228.

[3] H. Enomoto, Toughness and the existence of $k$-factors II, *Graphs and Combinatorics* 2(1986) 37–42.

[4] H. Enomoto, B. Jackson, D. Katerinis and A. Saito, Toughness and the existence of $k$-factors, *J. Graph Theory* 9(1985) 87–95.

[5] C. St. J. A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, *J. London Maths. Soc.* 36(1961) 445–450.

[6] Y. H. Peng, C. C. Chen and K. M. Koh, On the edge-toughness of a graph (I), *SEA Bull. Math.* Vol. 12 No. 2 (1988) 109–122.

[7] Y. H. Peng and T. S. Tay, On the edge-toughness of a graph (II), (submitted).

[8] W. T. Tutte, On the problem of decomposing a graph into $n$ connected factors, *J. London Maths. Soc.* 36(1961) 221–230.

[9] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* 54 (1932) 150–168.