## ON THE EXISTENCE OF A DOUBLE EXTENSION OF $PG(3,2)$

Alphonse Baartmans
Department of Mathematics
Michigan Technological University
Houghton, MI U.S.A. 49931-1295

Joseph Yucas
Department of Mathematics
Southern Illinois University–Carbondale
U.S.A. 62901-4408

**Abstract.** In this paper we give a necessary condition for the Steiner system $S(3,4,16)$ obtained from a one point extension of the points and lines of $PG(3,2)$ to be further extendable to a Steiner system $S(4,5,17)$.

### 1. Introduction.

It is not yet known whether a Steiner system $S(4,5,17)$ exists. One possible way of constructing such a Steiner system would be by twice extending the Steiner system $S(2,3,15)$ formed by the points and lines of $PG(3,2)$. A one point extension of this Steiner system $S(2,3,15)$ is easily obtained by adding a new point, say $\infty_1$ to the point set of $PG(3,2)$ and taking as blocks all sets of the form $\{\ell \cup \{\infty_1\} \mid \ell \text{ is a line in } PG(3,2)\}$ and all sets of the form $\{\pi \backslash \ell \mid \pi$ is a plane in $PG(3,2), \ell$ is a line in $PG(3,2)$, and $\ell \subset \pi\}$. We will denote this one point extension of $PG(3,2)$ by $PG_1(3,2)$. It is a Steiner system $S(3,4,16)$. The main purpose of this paper is to study the existence of a one point extension of $PG_1(3,2)$.

Section 2 is preliminary in nature and lists several propositions on $PG(3,2)$ needed in Section 3. The first few are certainly well known in the folklore. In Section 3 we show that no one point extension of $PG_1(3,2)$ with a certain uniformity property (stated before Lemma 3.5) exists, see Theorem 3.12. The problem in general, however, remains unsettled.

### 2. Preliminaries.

By an *oval* of $PG(3,2)$ we will mean a collection of four points in a plane of $PG(3,2)$ no three of which are colinear.

**Proposition 2.1.** *Let* $O = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *be an oval of* $PG(3,2)$.

(i) *(i) There is a unique plane* $\pi$ *of* $PG(3,2)$ *containing* $O$ *and* $\pi \backslash O$ *is a line in* $\pi$.

(ii) $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ *(vector addition in* $PG(3,2)$*).*

Proof: $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a collection of four points in a plane $\pi$ of $PG(3,2)$ no three of which are colinear. Since two planes of $PG(3,2)$ intersect in three points we see that $\pi$ is the unique plane containing $O$. Further, since $\alpha_1, \alpha_2$, and $\alpha_3$ are not colinear, $\pi$ is spanned by $\alpha_1$, $\alpha_2$, and $\alpha_3$. Consequently, $\alpha_4 \in \{\alpha_1 + \alpha_2,$

$\alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ since $\alpha_4 \in \pi$. This implies $\alpha_4 = \alpha_1 + \alpha_2 + \alpha_3$ for otherwise we contradict the fact that no three of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are colinear. Finally, notice that $\pi \backslash O = \{\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$ is a line in $\pi$. ∎

An *ovoid* of $PG(3,2)$ will be a collection of 4 non-coplanar point of $PG(3,2)$.

**Proposition 2.2.** *Let $O = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be an ovoid of $PG(3,2)$. There is a unique plane $\pi$ of $PG(3,2)$ with $O \subset \overline{\pi}$, the complement of $\pi$ in $PG(3,2)$.*

Proof: Let $p_i$, $0 \leq i \leq 3$, be the number of planes of $PG(3,2)$ intersecting $O$ in $i$ points. We must show $p_0 = 1$. Since $PG(3,2)$ has 15 planes it suffices to show that $p_1 + p_2 + p_3 = 14$. Since $\alpha_1, \alpha_2, \alpha_3$, and $\alpha_4$ are non-coplanar, there are exactly 4 planes intersecting $O$ in 3 points. They are the spans of $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$, $\{\alpha_1, \alpha_3, \alpha_4\}$, and $\{\alpha_2, \alpha_3, \alpha_4\}$. Hence, $p_3 = 4$. Every pair of points $\alpha_i, \alpha_j$, $1 \leq i, j \leq 4$, occurs in two of the above planes. Since every pair of points occurs in three planes, every pair $\alpha_i, \alpha_j$ must occur in one additional plane. Consequently, there are $\binom{4}{2} = 6$ planes that contain exactly two points of $O$, that is, $p_2 = 6$. Finally, every point $\alpha_i$, $1 \leq i \leq 4$, occurs three times in the planes intersecting $O$ in three points and three times in the planes intersecting $O$ in two points. Since every point occurs in seven planes there is a unique plane which intersects $O$ at $\alpha_i$. Consequently, $p_1 = 4$ and $p_1 + p_2 + p_3 = 4 + 6 + 4 = 14$. ∎

A *triangle* of $PG(3,2)$ will be a set of three linearly independent points of $PG(3,2)$.

**Proposition 2.3.** *Let $\pi$ be a plane in $PG(3,2)$. A four point set in $\overline{\pi}$ is either an oval or an ovoid of $PG(3,2)$. Moreover, $\overline{\pi}$ contains 14 ovals and 56 ovoids.*

Proof: Suppose $O = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \overline{\pi}$. Since the sum of any three of these $\alpha_i$ is in $\overline{\pi}$ we see that no three of the $\alpha_i$ are colinear. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$ then $O$ is an oval. If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 0$ then $\alpha_4$ is not in the span of $\alpha_1, \alpha_2$, and $\alpha_3$ hence, $\alpha_1, \alpha_2, \alpha_3$, and $\alpha_4$ are non-coplanar and $O$ is an ovoid. Now, the number of triangles in $\overline{\pi}$ is $\binom{8}{3} = 56$ and so there are $56/4 = 14$ sets of the form $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ in $\overline{\pi}$. These are the ovals in $\overline{\pi}$. Since there are $\binom{8}{4} = 70$ four point subsets of $\overline{\pi}$ there must be $70 - 14 = 56$ ovoids in $\overline{\pi}$. ∎

**Corollary 2.4.** *Let $\pi_1$ and $\pi_2$ be planes of $PG(3,2)$. $\overline{\pi}_1 \cap \overline{\pi}_2$ is an oval of $PG(3,2)$.*

Proof: Since $|\pi_1 \cap \pi_2| = 3$, $|\overline{\pi}_1 \cap \overline{\pi}_2| = 4$. By Proposition 2.3, $\overline{\pi}_1 \cap \overline{\pi}_2$ is an oval or an ovoid. But $\overline{\pi}_1 \cap \overline{\pi}_2$ is not an ovoid by Proposition 2.2. ∎

**Proposition 2.5.** *Let $\pi$ be a plane of $PG(3,2)$.*

(i) *The ovals in $\overline{\pi}$ form a $(v, b, r, k, \lambda_2) = (8, 14, 7, 4, 3)$ 3-design with $\lambda_3 = 1$.*

(ii) *The ovoids in $\overline{\pi}$ form an $(8, 56, 28, 4, 12)$ 3-design with $\lambda_3 = 4$.*

Proof: If $\alpha_1, \alpha_2, \alpha_3$ are three points in $\overline{\pi}$ then they determine a unique oval in $\overline{\pi}$, namely, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. Hence, $\lambda_3 = 1$ and (i) follows. For (ii), notice that the collection of all 4-subsets of $\overline{\pi}$ forms a 3-design with $\lambda_3 = 5$. Since a 4 element subset of $\overline{\pi}$ is either an oval or an ovoid and the collection of ovals in $\overline{\pi}$ forms a 3-design with $\lambda_3 = 1$, (ii) must follow. ∎

**Proposition 2.6.** *Let $\pi$ be a plane of $PG(3,2)$ and let $T = \{\alpha_1, \alpha_2, \alpha_3\}$ be a triangle in $\pi$. There are exactly two $(7,3,1)$ designs of triangles of $\pi$ containing the block $T$. They are*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| (i) | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | (ii) | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
| | $\alpha_1$ | $\alpha_1 + \alpha_2$ | $\alpha_2 + \alpha_3$ | | $\alpha_1$ | $\alpha_1 + \alpha_2$ | $\alpha_1 + \alpha_2 + \alpha_3$ |
| | $\alpha_1$ | $\alpha_1 + \alpha_3$ | $\alpha_1 + \alpha_2 + \alpha_3$ | | $\alpha_1$ | $\alpha_1 + \alpha_3$ | $\alpha_2 + \alpha_3$ |
| | $\alpha_2$ | $\alpha_1 + \alpha_2$ | $\alpha_1 + \alpha_2 + \alpha_3$ | | $\alpha_2$ | $\alpha_1 + \alpha_2$ | $\alpha_1 + \alpha_3$ |
| | $\alpha_2$ | $\alpha_1 + \alpha_3$ | $\alpha_2 + \alpha_3$ | | $\alpha_2$ | $\alpha_2 + \alpha_3$ | $\alpha_1 + \alpha_2 + \alpha_3$ |
| | $\alpha_3$ | $\alpha_1 + \alpha_2$ | $\alpha_1 + \alpha_3$ | | $\alpha_3$ | $\alpha_1 + \alpha_2$ | $\alpha_2 + \alpha_3$ |
| | $\alpha_3$ | $\alpha_2 + \alpha_3$ | $\alpha_1 + \alpha_2 + \alpha_3$ | | $\alpha_3$ | $\alpha_1 + \alpha_3$ | $\alpha_1 + \alpha_2 + \alpha_3$ |

Proof: If $\{\alpha_1, \alpha_2, \alpha_3\}$ is a block in a $(7,3,1)$ design then $\alpha_1$ must appear in two more blocks. Also, the pair $\{\alpha_1, \alpha_1 + \alpha_2\}$ must appear once in some block, thus, $\{\alpha_1, \alpha_1 + \alpha_2, \beta\}$ and $\{\alpha_1, \gamma, \delta\}$ must be blocks in the design for some $\beta, \gamma, \delta \in \pi$. Clearly, $\beta, \gamma, \delta \notin \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$ so $\beta, \gamma, \delta \in \{\alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. If $\beta = \alpha_1 + \alpha_3$ then $\gamma, \delta \in \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. But $\{\alpha_1, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not a triangle, hence, $\beta \in \{\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$.

**Case 1.** $\beta = \alpha_2 + \alpha_3$.

Here the three blocks containing $\alpha_1$ must be $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ and $\{\alpha_1, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. There must be two other blocks containing $\alpha_2$ and one of them must contain the pair $\alpha_2, \alpha_1 + \alpha_2$. Hence, $\{\alpha_2, \alpha_1 + \alpha_2, \iota\}$ and $\{\alpha_2, \xi, \eta\}$ are blocks in the design for some $\iota, \xi, \eta \in \pi$. Clearly, $\iota, \xi, \eta \notin \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$ and so $\iota, \xi, \eta \in \{\alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. Notice that $\{\alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not a triangle so $\iota \in \{\alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$. If $\iota = \alpha_1 + \alpha_3$ then $\{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3\}$ and $\{\alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ are the other two sets containing $\alpha_2$ besides $\{\alpha_1, \alpha_2, \alpha_3\}$. The other two sets containing $\alpha_3$ must be of the form $\{\alpha_3, \alpha_1 + \alpha_2, \rho\}$ and $\{\alpha_3, \sigma, \tau\}$ for some $\rho, \sigma, \tau \in \pi$. Notice that $\rho \neq \alpha_2 + \alpha_3$ and $\rho \neq \alpha_1 + \alpha_3$ for the pairs $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2, \alpha_1 + \alpha_3$ already occur in previous blocks. Hence, $\rho = \alpha_1 + \alpha_2 + \alpha_3$, but $\{\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ is not a triangle. Consequently, $\iota \neq \alpha_1 + \alpha_3$ and so $\iota = \alpha_1 + \alpha_2 + \alpha_3$. The blocks containing $\alpha_2$ are $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$ and $\{\alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$. The remaining two blocks containing $\alpha_3$ are then forced to be $\{\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3\}$ and $\{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ and we have the $(7,3,1)$ design in (i).

**Case 2.** $\beta = \alpha_1 + \alpha_2 + \alpha_3$.

Here a similar argument works forcing the $(7,3,1)$ design containing $\{\alpha_1, \alpha_2, \alpha_3\}$ to be the design in (ii). ∎

**Proposition 2.7.** *Let $\pi$ be a plane of $PG(3,2)$. There are exactly eight $(7,3,1)$ designs of triangles of $\pi$ and each triangle in $\pi$ occurs as a block in exactly two of them.*

Proof: Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a triangle in $\pi$. Applying Proposition 2.6 to the triangles $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_2 + \alpha_3\}$ and $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$, respectively, and writing $1, 2, 3, 12, 13$, etc., for $\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3\}$, etc., we obtain the following eight $(7,3,1)$ designs of triangles:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1. | (i) | 1 | 2 | 3 | (ii) | 1 | 2 | 3 |
| | | 1 | 12 | 23 | | 1 | 12 | 123 |
| | | 1 | 13 | 123 | | 1 | 13 | 23 |
| | | 2 | 12 | 123 | | 2 | 12 | 13 |
| | | 2 | 13 | 23 | | 2 | 23 | 123 |
| | | 3 | 12 | 13 | | 3 | 12 | 23 |
| | | 3 | 23 | 123 | | 3 | 13 | 123 |
| 2. | (i) | 1 | 2 | 13 | (ii) | 1 | 2 | 13 |
| | | 1 | 12 | 123 | | 1 | 12 | 23 |
| | | 1 | 3 | 23 | | 1 | 3 | 123 |
| | | 2 | 12 | 23 | | 2 | 12 | 3 |
| | | 2 | 3 | 123 | | 2 | 123 | 23 |
| | | 13 | 12 | 3 | | 13 | 12 | 123 |
| | | 13 | 123 | 23 | | 13 | 3 | 23 |
| 3. | (i) | 1 | 2 | 23 | (ii) | 1 | 2 | 23 |
| | | 1 | 12 | 3 | | 1 | 12 | 13 |
| | | 1 | 123 | 13 | | 1 | 123 | 3 |
| | | 2 | 12 | 13 | | 2 | 12 | 123 |
| | | 2 | 123 | 3 | | 2 | 3 | 13 |
| | | 23 | 12 | 123 | | 23 | 12 | 3 |
| | | 23 | 3 | 13 | | 23 | 123 | 13 |
| 4. | (i) | 1 | 2 | 123 | (ii) | 1 | 2 | 123 |
| | | 1 | 12 | 13 | | 1 | 12 | 3 |
| | | 1 | 23 | 3 | | 1 | 23 | 13 |
| | | 2 | 12 | 3 | | 2 | 12 | 23 |
| | | 2 | 23 | 13 | | 2 | 13 | 3 |
| | | 123 | 12 | 23 | | 123 | 12 | 13 |
| | | 123 | 13 | 3 | | 123 | 23 | 3 |

Notice that each of the 28 triangles in $\pi$ occurs as a block in exactly two of these eight designs. Further, by Proposition 2.6, a triangle occurs in exactly two

$(7,3,1)$ designs of triangles, hence, the above eight are the only $(7,3,1)$ designs of triangles of $\pi$. ∎

**3. The existence condition.**

$PG_1(3,2)$ has as its point set the fifteen points of $PG(3,2)$ together with the new point $\infty_1$. A set of points is a block in this extension if it is a line of $PG(3,2)$ together with $\infty_1$ or if it is an oval in $PG(3,2)$. This one point extension has sixteen points in blocks of size four with any three points occurring exactly once, that is, this one point extension is a Steiner system $S(3,4,16)$.

Assuming that a one point extension of $PG_1(3,2)$ exists, it would be an $S(4,5,17)$ with an additional point, say $\infty_2$, added to the point set of $PG_1(3,2)$. The blocks containing $\infty_2$ would be all lines of $PG(3,2)$ together with $\infty_1$ and $\infty_2$ and all complements of lines in planes of $PG(3,2)$ together with $\infty_2$. Now, in this one point extension of $PG_1(3,2)$ every point must occur 140 times. Notice that $\infty_1$ is already accounted for in 35 blocks since there are 35 lines in $PG(3,2)$. Consequently, there must be 105 additional blocks in this one point extension of $PG_1(3,2)$ which contain $\infty_1$. Let $A_1$ be the collection of these 105 blocks containing $\infty_1$ and set $A = \{S \subset PG(3,2) \mid S \cup \{\infty_1\} \in A_1\}$.

**Proposition 3.1.** *If $S \subseteq A$ then $S$ is an ovoid and, hence, is a four element linearly independent subset of $PG(3,2)$.*

Proof: No three of the four points of $S$ are colinear for then we would have a four point set (a line with $\infty_1$) occurring in two blocks of our one point extension of $PG_1(3,2)$. If no three of the four points of $S$ are colinear but $S$ is contained in some plane $\pi$ then $S$ must be the complement of some line in $\pi$. But then $S \cup \{\infty_1\}$ and $S \cup \{\infty_2\}$ are two blocks of our one point extension of $PG_1(3,2)$, again a contradiction. ∎

**Proposition 3.2.** *The elements of $A$ form the blocks of a $(15,105,28,4,6)$ design.*

Proof: Recall that our one point extension of $PG_1(3,2)$ is a 4-design with $\lambda_4 = 1$. Three points occur in $\binom{17-3}{4-3} / \binom{5-3}{4-3} = 7$ blocks and two point occur in $\binom{17-2}{4-2} / \binom{5-2}{4-2} = 35$ blocks of this one point extension of $PG_1(3,2)$. Let $C_1$ be the collection of all blocks of this one point extension of $PG_1(3,2)$ containing $\infty_1$ and let $C = \{B \backslash \{\infty_1\} \mid B \in C_1\}$. Then $C$ forms the blocks of a 3-design with parameters $(16,140,35,4,7)$ and $\lambda_3 = 1$. Let $L = \{\ell \cup \{\infty_2\} \mid \ell$ is a line of $PG(3,2)\}$. Then $C = A \cup L$ and $A \cap L = \emptyset$ so we can count the occurrences of a point, or of two points of $PG(3,2)$, in the blocks of $A$ by counting the occurrences in $L$ and $C$. A point of $PG(3,2)$ occurs on seven lines and, hence, occurs in seven blocks of $L$. Consequently, a point of $PG(3,2)$ must occur in $35-7 = 28$ blocks of $A$. Similarly, two points determine a unique line in $PG(3,2)$. Since they must occur in seven blocks of $C$, the two points of $PG(3,2)$ must occur in six blocks of $A$. The result now follows. ∎

Notice that if $T$ is a triangle in $PG(3,2)$ then the four point set $T \cup \{\infty_1\}$ must occur exactly once as a subset of some block of our one point extension of $PG_1(3,2)$. Since $T$ is not a line, this block containing $T \cup \{\infty_1\}$ must be in $A$. Hence, given a triangle of $PG(3,2)$ there is a unique block in $A$ containing it.

Let $\pi$ be a plane of $PG(3,2)$ and let $S \in A$. Notice that $|S \cap \pi| \leq 3$. Let $A_i$ be the collection of sets $S \in A$ satisfying $|S \cap \pi| = i$, $i = 0,1,2,3$.

**Proposition 3.3..** *$|A_0| = 7$, $|A_1| = 28$, $|A_2| = 42$ and $|A_3| = 28$.*

Proof: There are 28 triangles in $\pi$. As remarked above, each triangle occurs exactly once as a subset of a block of $A$, hence, $|A_3| = 28$. Now, a pair of points of $\pi$ occurs in four triangles of $\pi$ so every pair of points of $\pi$ occurs in four sets of $A_3$. Since every pair of points occurs six times we see that every pair of points of $\pi$ occurs in two sets of $A_2$. There are 21 distinct pairs of points of $\pi$, hence, $|A_2| = 42$. Every point of $\pi$ occurs in 12 triangles of $\pi$, hence, every point of $\pi$ occurs in 12 sets of $A_3$. Similarly, every point of $\pi$ occurs in $6 \cdot 2 = 12$ sets of $A_2$. Since every point must occur 28 times, every point of $\pi$ occurs in four sets of $A_1$. Consequently, $|A_1| = 7 \cdot 4 = 28$. The remaining sets of $A$ must be in $A_0$. There are $150 - 28 - 42 - 28 = 7$ of them. ∎

Let $B_\pi$ be the sets of $A$ whose elements belong to $\overline{\pi}$.

**Proposition 3.4.** *$\{B_\pi \mid \pi$ a plane of $PG(3,2)\}$ is a partition of $A$ into fifteen families each of size seven.*

Proof: By Proposition 3.1 each set in $A$ is an ovoid and, hence, is in $\overline{\pi}$ for some unique plane $\pi$ by Proposition 2.2. By Proposition 3.3, $|B_\pi| = 7$ for each plane $\pi$. ∎

We now make a basic assumption that if $\pi$ is a plane and $\alpha, \beta$ are points of $PG(3,2)$ then $\{\alpha, \beta\}$ does not occur as a subset of a unique set $S \in B_\pi$.

**Lemma 3.5.** *Let $\pi$ be a plane of $PG(3,2)$ and suppose $\alpha, \beta$ are points of $PG(3,2)$. If $\{\alpha, \beta\} \subseteq S \in B_\pi$ then $\{\alpha, \beta\}$ occurs exactly twice as a subset of a set in $B_\pi$.*

Proof: Suppose $S = \{\alpha, \beta, \delta_1, \delta_2\} \in B_\pi$. Since any two sets in $B_\pi$ can intersect in at most two points the pair $\{\alpha, \beta\}$ can occur at most three times as a subset of a set in $B_\pi$. Notice that the pairs $\{\alpha, \delta_1\}$, $\{\alpha, \delta_2\}$, $\{\beta, \delta_1\}$, $\{\beta, \delta_2\}$, and $\{\delta_1, \delta_2\}$, must occur again by our basic assumption and none of these pairs can occur again together or again with $\alpha$ or $\beta$. Consequently, the pair $\{\alpha, \beta\}$ cannot occur three times for then $|B_\pi| \geq 8$. ∎

**Proposition 3.6.** *For each plane $\pi$ of $PG(3,2)$ the sets in $B_\pi$ are the blocks of a $(7,7,4,4,2)$ design.*

Proof: We first show that if $\alpha, \beta \in \overline{\pi}$ and if $\alpha$ is in a set of $B_\pi$ and if $\beta$ is in a set of $B_\pi$ then $\{\alpha, \beta\}$ is a subset of a set of $B_\pi$. Suppose the contrary, that $\alpha$ occurs, $\beta$ occurs but $\{\alpha, \beta\}$ does not. Let $S = \{\alpha, \delta_1, \delta_2, \delta_3\} \in B_\pi$. Since $\{\alpha, \delta_1\}$, $\{\alpha, \delta_2\}$, and $\{\alpha, \delta_3\}$ must occur again we see that there are three additional blocks which contain $\alpha$. Similarly there are four blocks containing $\beta$. If $\alpha$ and $\beta$ do not occur together then $|B_\pi| \geq 8$, a contradiction. Now let $P_\pi$ be the point set corresponding to the blocks in $B_\pi$. Suppose $|P_\pi| = m$. We show $m = 7$. Let $X = \{(\{\alpha, \beta\}, B) \mid \alpha, \beta \in B,\ B \in B_\pi\}$. $|X| = \binom{m}{2} \cdot 2 = \binom{4}{2} \cdot 7$, hence, $m^2 - m - 42 = 0$ and $m = 7$. ∎

The unique point of $PG(3,2)$ in $\overline{\pi}$ which is not used in $B_\pi$ will be called the *translation point of* $\pi$ and denoted $\gamma_\pi$.

Recall that an arbitrary point $\alpha$ of $PG(3,2)$ must occur 28 times in the blocks of $A$. Since $\alpha$ occurs four times in the blocks of $B_\pi$ for those planes $\pi$ for which $\alpha \in \overline{\pi}$ and $\alpha$ is not a translation point for $\pi$ and since $\alpha$ is in eiqht complemented planes we see that $\alpha$ must be the translation point of some plane. Hence, a point of $PG(3,2)$ is a translation point of one and only one plane.

The following Corollary follows immediately from Proposition 3.6.

**Corollary 3.7.** *For each plane* $\pi$ *of* $PG(3,2)$, *the sets* $\gamma_\pi + B$, $B \in B_\pi$, *are the blocks of a* $(7,7,4,4,2)$ *design in* $\pi$.

**Corollary 3.8.** *Let* $\pi$ *be a plane of* $PG(3,2)$. $\{\pi \backslash (\gamma_\pi + B) \mid B \in B_\pi\}$ *is a collection of triangles of* $\pi$ *which are the blocks of a* $(7,7,3,3,1)$ *design in* $\pi$.

Proof: First notice that $\gamma_\pi + B$, $B \in B_\pi$, is a set of four points in $\pi$ and, hence, are linearly dependent. Since $B$ is not an oval, $\gamma_\pi + B$ must contain a line and thus $\pi \backslash (\gamma_\pi + B)$ is a triangle. Let $D = \{\gamma_\pi + B \mid B \in B_\pi\}$ and let $\alpha \in \pi$. $\alpha$ occurs in four of the seven blocks of $D$, hence, $\alpha$ occurs in three of their complements in $\pi$. Also a pair $\alpha, \beta \in \pi$ occurs in two blocks of $D$ with $\alpha$ occurring four times and $\beta$ occurring four times. Hence, there is one block of $D$ in which neither $\alpha$ nor $\beta$ occurs. That is, the pair $\alpha, \beta$ occurs in one complement of a block in $D$. ∎

**Proposition 3.9.** *Let* $\pi$ *be a plane of* $PG(3,2)$ *with translation point* $\gamma_\pi$. *Suppose* $\pi'$ *is another plane of* $PG(3,2)$ *with* $\gamma_\pi \in \overline{\pi}'$. *There is a block* $\{\gamma_\pi, \alpha_1, \alpha_2, \alpha_3\}$ *in* $B_{\pi'}$, *such that* $\pi$ *is spanned by* $\alpha_1$, $\alpha_2$ *and* $\alpha_3$.

Proof: $\pi \cap \pi'$ is an oval. Say $\pi \cap \pi' = \{\gamma_\pi, \delta_1, \delta_2, \delta_3\}$ with $\gamma_\pi + \delta_1 + \delta_2 + \delta_3 = 0$. The triangles $\{\gamma_\pi, \delta_1, \delta_2\}$, $\{\gamma_\pi, \delta_1, \delta_3\}$ and $\{\gamma_\pi, \delta_2, \delta_3\}$ must occur as subsets of some sets of $A$. Since $\gamma_\pi$ is the translation point for $\pi$ and since no other complemented plane contains three of $\gamma_\pi, \delta_1, \delta_2$, and $\delta_3$, there must be three sets of $A$ say $\{\gamma_\pi, \delta_1, \delta_2, \alpha_1\}$, $\{\gamma_\pi, \delta_1, \delta_3, \alpha_2\}$ and $\{\gamma_\pi, \delta_2, \delta_3, \alpha_3\}$ in $B_{\pi'}$. Notice that $\alpha_1, \alpha_2, \alpha_3$ are distinct for otherwise a triangle would be appearing more than once. $\alpha_1, \alpha_2$, and $\alpha_3$ are points in $\overline{\pi}'$, and are not in $\{\gamma_\pi, \delta_1, \delta_2, \delta_3\}$ so $\alpha_1, \alpha_2, \alpha_3 \in \pi$. Now the pairs $\{\gamma_\pi, \alpha_1\}$, $\{\gamma_\pi, \alpha_2\}$, and $\{\gamma_\pi, \alpha_3\}$ must appear again as subsets of sets in $B_{\pi'}$, and since $\gamma_\pi$ already appears three times $\{\gamma_\pi, \alpha_1, \alpha_2, \alpha_3\}$ must be in $B_{\pi'}$. ∎

Notice that in the above proof, the four blocks in $B_{\pi'}$ containing $\gamma_\pi$ are $\{\gamma_\pi\delta_1, \delta_2, \alpha_1\}$, $\{\gamma_\pi\delta_1, \delta_3, \alpha_2\}$, $\{\gamma_\pi\delta_2, \delta_3, \alpha_3\}$ and $\{\gamma_\pi\alpha_1, \alpha_2, \alpha_3\}$. Now, $\pi$ intersects $\{\delta_1, \delta_2, \alpha_1\}$, $\{\delta_1, \delta_3, \alpha_2\}$ and $\{\delta_2, \delta_3, \alpha_3\}$ each in exactly one point.

**Example 3.10.** The above remark enables us to find the plane for which an arbitrary point $\gamma$ is the translation point. For example, suppose

| | | | | |
|---|---|---|---|---|
| 1. | $\gamma + \delta_2$ | $\delta_1 + \delta_2$ | $\delta_1 + \delta_3$ | $\delta_1 + \delta_2 + \delta_3$ |
| 2. | $\gamma$ | $\delta_1 + \delta_2$ | $\delta_1 + \delta_2 + \delta_3$ | $\delta_1$ |
| 3. | $\gamma$ | $\delta_1$ | $\delta_1 + \delta_3$ | $\gamma + \delta_2$ |
| 4. | $\gamma + \delta_2 + \delta_3$ | $\delta_1$ | $\delta_1 + \delta_2$ | $\delta_1 + \delta_3$ |
| 5. | $\delta_1$ | $\gamma + \delta_2$ | $\delta_1 + \delta_2 + \delta_3$ | $\gamma + \delta_2 + \delta_3$ |
| 6. | $\gamma + \delta_2 + \delta_3$ | $\gamma_1 + \delta_3$ | $\delta_1 + \delta_2 + \delta_3$ | $\gamma$ |
| 7. | $\gamma + \delta_2 + \delta_3$ | $\gamma$ | $\gamma + \delta_2$ | $\delta_1 + \delta_2$ |

are the seven blocks of $B_\pi$ for some plane $\pi$ and consider the point $\gamma$. $\gamma$ appears in the four blocks $B_1 = \{\gamma, \delta_1 + \delta_2, \delta_1 + \delta_2 + \delta_3, \delta_1\}$, $B_2 = \{\gamma, \delta_1, \delta_1 + \delta_3, \gamma + \delta_2\}$, $B_3 = \{\gamma, \gamma + \delta_2 + \delta_3, \delta_1 + \delta_3, \delta_1 + \delta_3, \delta_1 + \delta_2 + \delta_3\}$, and $B_4 = \{\gamma, \gamma + \delta_2 + \delta_3, \gamma + \delta_2, \delta_1 + \delta_2\}$. Let $\pi_i$ be the plane spanned by $X_i = B_i \backslash \{\gamma\}$. Notice that $X_2 \cap \pi_1 = \{\delta_1, \delta_1 + \delta_3\}$, $X_4 \cap \pi_2 = \{\gamma + \delta_2 + \delta_3, \gamma + \delta_2\}$, and $X_3 \cap \pi_4 = \{\gamma + \delta_2 + \delta_3, \delta_1 + \delta_2 + \delta_3\}$ but $|X_i \cap \pi_3| = 1$ for $i = 1,2,4$. By the remark above, $\gamma$ must be the translation point for $\pi_3$. ∎

**Proposition 3.11.** *A does not exist.*

Proof: Let $\pi$ be a plane of $PG(3,2)$. Recall that $B_\pi$ is a $(7,7,4,4,2)$ design which arises from a $(7,7,3,3,1)$ design of triangles of $\pi$. Fix a triangle in this design, say $\{\alpha_1, \alpha_2, \alpha_3\}$ and let $\alpha_4 = \gamma_\pi$, the translation point of $\pi$. By Proposition 2.6 we have two cases to consider:

**Case 1.** Suppose the $(7,7,3,3,1)$ design of triangles of $\pi$ is derived using construction (i) of Proposition 2.6. This design of triangles is then

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 1 | 12 | 23 |
| 1 | 13 | 123 |
| 2 | 12 | 123 |
| 2 | 13 | 23 |
| 3 | 12 | 13 |
| 3 | 23 | 123 |

where we are again writing $1,2,3$ for $\alpha_1, \alpha_2, \alpha_3$, and $12, 13$, etc., for $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_3$, etc. Recall then that $B_\pi$ is constructed from this $(7,7,3,3,1)$ design by first complementing in $\pi$ and then translating by $\alpha_4$. It is

| | | | |
|---|---|---|---|
| 124 | 134 | 234 | 1234 |
| 24 | 34 | 134 | 1234 |
| 24 | 34 | 124 | 234 |
| 14 | 34 | 134 | 234 |
| 14 | 34 | 124 | 1234 |
| 14 | 24 | 234 | 1234 |
| 14 | 24 | 124 | 134. |

Let $\pi'$ be the plane whose translation point is $\alpha_1 + \alpha_4$. $\alpha_1 + \alpha_4$ appears four times in the blocks of $B_\pi$ and, thus, by Proposition 3.9, $\pi'$ must be the span of one of

| | | | |
|---|---|---|---|
| *i)* | 34 | 134 | 234 |
| *ii)* | 34 | 124 | 1234 |
| *iii)* | 24 | 234 | 1234 |
| *iv)* | 24 | 124 | 134 |

By the same procedure as in Example 3.10 we see that $\pi' = \{34, 124, 1234, 123, 12, 3, 4\}$ and thus $\overline{\pi} \cap \overline{\pi'} = \{14, 24, 134, 234\}$. Now, by Proposition 2.7 there are eight possibilities for $B_{\pi'}$. Using the triangles $\{3, 4, 12\}$, $\{3, 4, 123\}$, $\{3, 4, 124\}$ and $\{3, 4, 1234\}$ and constructions (i) and (ii) of Proposition 2.6, as in Proposition 2.7, we can list these eight possibilities as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1. | 13 | 234 | 2 | 23 | 2. | 13 | 234 | 2 | 23. |
| | 1 | 24 | 234 | 23 | | 1 | 24 | 234 | 2 |
| | 1 | 24 | 13 | 2 | | 1 | 24 | 13 | 23 |
| | 134 | 24 | 234 | 2 | | 134 | 24 | 2 | 23 |
| | 134 | 24 | 13 | 23 | | 134 | 24 | 13 | 234 |
| | 134 | 1 | 2 | 23 | | 134 | 1 | 234 | 23 |
| | 134 | 1 | 13 | 234 | | 134 | 1 | 13 | 2 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3. | 24 | 13 | 2 | 23 | 4. | 24 | 13 | 2 | 23 |
| | 1 | 24 | 234 | 2 | | 1 | 24 | 234 | 23 |
| | 1 | 13 | 234 | 23 | | 1 | 13 | 234 | 2 |
| | 134 | 24 | 234 | 23 | | 134 | 234 | 2 | 23 |
| | 134 | 13 | 234 | 2 | | 134 | 24 | 13 | 234 |
| | 134 | 1 | 2 | 23 | | 134 | 1 | 24 | 2 |
| | 134 | 1 | 24 | 13 | | 134 | 1 | 13 | 23 |

5.

| | | | |
|---|---|---|---|
| 24 | 13 | 234 | 23 |
| 1 | 234 | 2 | 23 |
| 1 | 24 | 13 | 2 |
| 134 | 24 | 2 | 23 |
| 134 | 13 | 234 | 2 |
| 134 | 1 | 24 | 234 |
| 134 | 1 | 13 | 23 |

6.

| | | | |
|---|---|---|---|
| 24 | 13 | 234 | 23 |
| 1 | 24 | 2 | 23 |
| 1 | 13 | 234 | 2 |
| 134 | 24 | 234 | 2 |
| 134 | 13 | 2 | 23 |
| 134 | 1 | 234 | 23 |
| 134 | 1 | 24 | 13 |

7.

| | | | |
|---|---|---|---|
| 24 | 13 | 234 | 2 |
| 1 | 24 | 13 | 23 |
| 1 | 13 | 234 | 23 |
| 134 | 234 | 2 | 23 |
| 134 | 24 | 13 | 23 |
| 134 | 1 | 24 | 234 |
| 134 | 1 | 13 | 2 |

8.

| | | | |
|---|---|---|---|
| 24 | 13 | 234 | 2 |
| 1 | 234 | 2 | 23 |
| 1 | 24 | 13 | 23 |
| 134 | 24 | 234 | 23 |
| 134 | 13 | 2 | 23 |
| 134 | 1 | 24 | 2 |
| 134 | 1 | 13 | 234 |

Now, as in Example 3.10 we can list the planes corresponding to the translation points $\alpha_2 + \alpha_4$, $\alpha_1 + \alpha_3 + \alpha_4$ and $\alpha_2 + \alpha_3 + \alpha_4$ (the remaining points in $\overline{\pi} \cap \overline{\pi}'$). Using $B_\pi$ and the eight possibilities for $B_{\overline{\pi}'}$, we obtain:

| (7,7,4,4,2) design | Translation point | Plane corresponding to the translation point | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $B_\pi$ | 24 | 14 | 234 | 1234 | 123 | 23 | 1 | 4 |
| $B_{\pi'}(1)$ | 24 | 1 | 234 | 23 | 1234 | 123 | 4 | 14 |
| $B_{\pi'}(2)$ | 24 | 134 | 2 | 23 | 1234 | 124 | 3 | 14 |
| $B_{\pi'}(3)$ | 24 | 134 | 1 | 13 | 34 | 4 | 3 | 14 |
| $B_{\pi'}(4)$ | 24 | 1 | 234 | 23 | 1234 | 123 | 4 | 14 |
| $B_{\pi'}(5)$ | 24 | 134 | 2 | 23 | 1234 | 124 | 3 | 14 |
| $B_{\pi'}(6)$ | 24 | 134 | 1 | 13 | 34 | 4 | 3 | 14 |
| $B_{\pi'}(7)$ | 24 | 13 | 234 | 2 | 124 | 123 | 34 | 14 |
| $B_{\pi'}(8)$ | 24 | 13 | 234 | 2 | 124 | 123 | 34 | 14 |
| $B_\pi$ | 134 | 14 | 24 | 124 | 12 | 2 | 1 | 4 |
| $B_{\pi'}(1)$ | 134 | 24 | 13 | 23 | 1234 | 34 | 12 | 14 |
| $B_{\pi'}(4)$ | 134 | 1 | 24 | 2 | 124 | 12 | 4 | 14 |
| $B_\pi$ | 234 | 14 | 34 | 134 | 13 | 3 | 1 | 4 |
| $B_{\pi'}(4)$ | 234 | 134 | 2 | 23 | 1234 | 124 | 3 | 14 |

Notice that using $B_\pi$ and then $B_{\pi'}(2)$ to find the plane with translation point $\alpha_2 + \alpha_4$ we obtain two different planes, a contradiction. Hence, $B_{\pi'} \neq B_{\pi'}(2)$. Similarly, we can eliminate possibilities 3, 5, 6, 7 and 8 using the point $\alpha_2 + \alpha_4$. Finally, using $\alpha_1 + \alpha_3 + \alpha_4$ we can eliminate possibility 1 and using $\alpha_2 + \alpha_3 + \alpha_4$ we can eliminate possibility 4. Consequently, $A$ cannot exist.

Case 2. Suppose this $(7,7,3,3,1)$ design of triangles of $\pi$ is derived using construction (ii) of Proposition 2.6. Here the argument is similar to that as in Case 1 and, thus, we just list the results of our computation. $B_\pi$ is

| | | | |
|---|---|---|---|
| 124 | 134 | 234 | 1234 |
| 24 | 34 | 134 | 234 |
| 24 | 34 | 124 | 1234 |
| 14 | 34 | 234 | 1234 |
| 14 | 34 | 124 | 134 |
| 14 | 24 | 134 | 1234 |
| 14 | 24 | 124 | 234 |

$\pi' = \{24, 134, 1234, 123, 13, 2, 4\}$ and $\overline{\pi} \cap \overline{\pi}' = \{14, 34, 124, 234\}$. Using the four triangles $\{2,4,13\}$, $\{2,4,134\}$, $\{2,4,1234\}$ and $\{2,4,123\}$ to construct the eight possibilities for $B_{\pi'}$, and then checking planes corresponding to translation points in $\overline{\pi} \cap \overline{\pi}'$ we obtain:

| $(7,7,4,4,2)$ design | Translation point | Plane corresponding to the translation point | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $B_\pi$ | 34 | 14 | 234 | 1234 | 123 | 23 | 1 | 4 |
| $B_{\pi'}(1)$ | 34 | 1 | 234 | 23 | 1234 | 123 | 4 | 14 |
| $B_{\pi'}(2)$ | 34 | 124 | 3 | 23 | 1234 | 134 | 2 | 14 |
| $B_{\pi'}(3)$ | 34 | 124 | 3 | 23 | 1234 | 134 | 2 | 14 |
| $B_{\pi'}(4)$ | 34 | 124 | 1 | 12 | 24 | 4 | 2 | 14 |
| $B_{\pi'}(5)$ | 34 | 12 | 234 | 3 | 134 | 123 | 24 | 14 |
| $B_{\pi'}(6)$ | 34 | 12 | 234 | 3 | 134 | 123 | 24 | 14 |
| $B_{\pi'}(7)$ | 34 | 124 | 1 | 12 | 24 | 4 | 2 | 14 |
| $B_{\pi'}(8)$ | 34 | 1 | 234 | 23 | 1234 | 123 | 4 | 14 |
| $B_\pi$ | 124 | 14 | 34 | 134 | 13 | 3 | 1 | 4 |
| $B_{\pi'}(1)$ | 124 | 34 | 12 | 23 | 1234 | 24 | 13 | 14 |
| $B_{\pi'}(8)$ | 124 | 1 | 34 | 3 | 134 | 13 | 4 | 14 |
| $B_\pi$ | 234 | 14 | 24 | 124 | 12 | 2 | 1 | 4 |
| $B_{\pi'}(8)$ | 234 | 124 | 3 | 23 | 1234 | 134 | 2 | 14 |

∎

In summary, we have

**Theorem 3.12.** *If a one point extension of $PG_1(3,2)$ exists then there is a plane $\pi$ and points $\alpha, \beta$ in $PG(3,2)$ such that $\{\alpha, \beta\}$ occurs as a subset of a unique block of $B_\pi$.*

## References

1. T. Beth, D. Jungnickel, and H. Lenz, "Design Theory", Cambridge University Press, Zurich, 1986.
2. J.W.P. Hirschfield, "Finite Projective Spaces of Three Dimensions", Clarendon Press, Oxford, 1985.