On Tricovers of Pairs by Quintuples: $v \equiv 1 \pmod 4$

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Abstract. A tricover of pairs by quintuples on a v-element set V is a family of 5-element subsets of V, called blocks, with the property that every pair of distinct elements of V occurs in at least three blocks. If no other such tricover has fewer blocks, the tricover is said to be minimum, and the number of blocks in a minimum tricover is the tricovering number $C_3(v,5,2)$, or simply $C_3(v)$. It is well known that $C_3(v) \geq \lceil v\lceil 3(v-1)/4\rceil/5\rceil = B_3(v)$, where $\lceil x\rceil$ is the smallest integer that is at least x. It is shown here that if $v \equiv 1 \pmod 4$, then $C_3(v) = B_3(v) + 1$ for $v \equiv 9$ or $17 \pmod{20}$, and $C_3(v) = B_3(v)$ otherwise.

1. Introduction.

Let V be a finite set of cardinality v. A (k,t)-cover of index λ is a family of k-element subsets of V, called blocks, with the property that every t-element subset of V occurs in at least λ of the blocks. The covering number $C_\lambda(v,k,t)$ is defined to be the number of blocks in a minimum (as opposed to minimal) (k,t)-cover of index λ on V. Many authors have been involved in determining the covering numbers known to date (see bibliography).

The object of this paper is to determine $C_3(v,5,2)$ for all $v \equiv 1 \pmod 4$. The corresponding covers are called tricovers of pairs by quintuples.

For $v > k > t > 0$ let

$$B_\lambda(v,k,t) = \left\lceil \frac{v}{k}\left\lceil \frac{v-1}{k-1}\cdots\left\lceil \frac{v-t+1}{k-t+1}\lambda\right\rceil\cdots\right\rceil\right\rceil,$$

where $\lceil x\rceil$ is the smallest integer that is at least x. Schönheim [26] observed that the quantity $B_\lambda(v,k,t)$ is a lower bound for $C_\lambda(v,k,t)$.

For simplicity let $C_\lambda(v,5,2)$ be denoted by $C_\lambda(v)$ and $B_\lambda(v,5,2)$ be denoted by $B_\lambda(v)$.

Hanani [9] has shown that if $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 1 \pmod{k}$, then $C_\lambda(v,k,2) \geq B_\lambda(v,k,2) + 1$. Thus, if $v \equiv 9$ or $17 \pmod{20}$, then $C_3(v) \geq B_3(v) + 1$.

A balanced incomplete block design BIBD(v,k,λ) is a pair (V,B), where V is a v-element set and B is a family of k-element subsets of V, called blocks, such that every pair of distinct elements of V occurs in exactly λ blocks. Hanani [9] has shown that there exists a BIBD($v, 5, \lambda$) for all integers $v \geq 5$ which satisfy the relations

$$\lambda(v-1) \equiv 0 \pmod 4$$

and

$$\lambda v(v-1) \equiv 0 \pmod{20}$$

with the exception of $v = 15$ and $\lambda = 2$.

Since the relation $C_\lambda(v) = B_\lambda(v)$ holds for balanced incomplete block designs, it is immediate that $C_3(v) = B_3(v)$ for $v \equiv 1$ or $5 \pmod{20}$.

We combine these various results into the following theorem.

Theorem 1. *If* $v \equiv 1$ *or* $5 \pmod{20}$, *then* $C_3(v) = B_3(v)$. *If* $v \equiv 9$ *or* $17 \pmod{20}$, *then* $C_3(v) \geq B_3(v) + 1$. *If* $v \equiv 13 \pmod{20}$ *then* $C_3(v) \geq B_3(v)$.

In the next four sections we will show that we always have equality in Theorem 1. We do this by constructing appropriate covering designs.

We require several other types of combinatorial configurations. We need some of these types only for $\lambda = 1$ so we will drop the λ from our notation for these types. A pairwise balanced design PBD[$\{5, w^\star\}; v$] is a pair (V, B), where V is a v-element set and B is a family of subsets of V called blocks, such that there is a special block W of size w, all the other blocks have size 5, and every pair of distinct elements of V occurs in exactly 1 block. The blocks that are not W are called ordinary blocks. They all have size 5. A resolvable balanced incomplete block design RBIBD($v, k, 1$) is a BIBD($v, k, 1$) in which the blocks can be divided into disjoint classes such that each element of V occurs in exactly one of the blocks of each class. These are called resolution classes.

A group divisible design GDD[$a^\alpha b^\beta \dots d^\delta; K; \lambda$] is a triple (V, G, B) where

- (i) V is a finite set,
- (ii) G is a family of disjoint subsets of V called groups, whose union is V,
- (iii) exactly α of the groups have size a, exactly β of the groups have size $b, \dots,$ exactly δ of the groups have size d, and there are no other groups,
- (iv) B is a family of subsets of V, called blocks, and the size of every block is an element of the set K,
- (v) each block contains at most one element from any given group,
- (vi) every pair of distinct elements of V belongs to exactly one of the groups or to exactly λ of the blocks.

A transversal design $TD_\lambda(k; v)$ is a $GDD[v^k; \{k\}; \lambda]$. For $\lambda = 1$ such a design is equivalent to a set of $k-2$ mutually orthogonal Latin squares, and is, of course, denoted by $TD(k; v)$.

An incomplete transversal design $TD(k; v) - TD(k; w)$ is a quadruple (X, Y, G, B), such that X is a kv-element set; G is a family of disjoint v-element subsets, $G_1, G_2, \ldots, G_k$ of X, called groups, whose union is X; Y is a subset of X, called the hole, satisfying $|Y \cap G_i| = w$ for all i; B is a family of k-element subsets of X, called blocks, such that each block intersects each group in exactly one point, no block contains more than one element of Y, and every pair of elements of distinct groups, not both in Y, belongs to exactly one block.

For the existence of transversal designs, our authority is Brouwer [3] unless another reference is given. Similarly, for the existence of resolvable balanced incomplete block designs and balanced incomplete block designs, see Mathon and Rosa [14].

2. Some constructions.

For any integer $n \geq 5$, we set $\Delta_\lambda(n) = C_\lambda(n) - B_\lambda(n)$.

The following lemmas are useful for our constructions.

Lemma 1. *If a $PBD[\{5, w^\star\}; v]$ exists, then $\Delta_\lambda(v) \leq \Delta_\lambda(w)$ for all λ.*

Proof: Let W be the special block of size w in the PBD, and let V be the set of elements not in this block. Thus, the cardinality of V is $v-w$. Let $F(\alpha)$ denote the number of ordinary blocks that contain α. For $\alpha \in V$ we have $F(\alpha) = (v-1)/4$ and for $\alpha \in W$ we have $F(\alpha) = (v-w)/4$. This gives us $v \equiv w \equiv 1 \pmod 4$. Moreover, if F is the total number of ordinary blocks, then we have

$$5F = \sum_\alpha F(\alpha) = (v-w)\frac{v-1}{4} + w\frac{v-w}{4} = \frac{v(v-1)}{4} - \frac{w(w-1)}{4}.$$

This gives us

$$\lambda F = \frac{v(v-1)\lambda}{20} - \frac{w(w-1)\lambda}{20} = \left\lceil \frac{v(v-1)\lambda}{20} \right\rceil - \left\lceil \frac{w(w-1)\lambda}{20} \right\rceil = B_\lambda(v) - B_\lambda(w).$$

If we replicate the ordinary blocks in our PBD λ times, and adjoin the $C_\lambda(w)$ blocks of a minimum $(5,2)$-cover of W of index λ, then we get a covering of index λ of a v-element set with $\lambda F + C_\lambda(w)$ blocks of size 5. Therefore,

$$C_\lambda(v) \leq \lambda F + C_\lambda(w) = B_\lambda(v) - B_\lambda(w) + C_\lambda(w) = B_\lambda(v) + \Delta_\lambda(w).$$

It follows that $\Delta_\lambda(v) \leq \Delta_\lambda(w)$. ∎

Lemma 2. *There exists a PBD*$[\{5,(4u+1)^\star\}; 16u+5]$, *and* $\Delta_\lambda(16u+5) \leq \Delta_\lambda(4u+1)$ *for all positive integers* u *and* λ.

Proof: Hanani, Ray-Chaudhuri, and Wilson [10] have shown that there exists an RBIBD$(12u+4,4,1)$, say D. Such a design has $4u+1$ resolution classes. Let $x_1, x_2, \ldots, x_{4u+1}$ be a set of points not occurring in D. By adjoining x_i to each block of the ith resolution class of D, for $1 \leq i \leq 4u+1$, and adding a new block $\{x_1, x_2, \ldots, x_{4u+1}\}$ we obtain a PBD$[\{5,(4u+1)^\star\}; 16u+5]$, and Lemma 1 yields $\Delta_\lambda(16u+5) \leq \Delta_\lambda(4u+1)$. ∎

Lemma 3. *Let* m *and* t *be integers such that* $0 \leq t \leq m$, $m \equiv 0$ *or* $1 \pmod 5$, *and either there exists a TD*$(6;m)$ *or* $m = 10$. *Then* $\Delta_\lambda(20m+4t+1) \leq \Delta_\lambda(4t+1)$ *for all* λ.

Proof: We can construct a GDD$[4^5;\{5\};1]$ by removing one element from a BIBD$(21,5,1)$, and a GDD$[4^6;\{5\};1]$ by removing one element from a BIBD$(25,5,1)$.

If there exists a TD$(6;m)$ we start with it and remove $m-t$ elements from one of the groups, giving us a GDD$[m^5t^1;\{5,6\};1]$. We inflate each of the points of this GDD by a factor of 4, replace each block of size 5 by the blocks of a GDD$[4^5;\{5\};1]$ on the corresponding points, and each block of size 6 by the blocks of a GDD$[4^6;\{5\};1]$ on the corresponding points. This gives us a GDD$[(4m)^5(4t)^1;\{5\};1]$. Take one extra point X at infinity and adjoin it to each of the groups of this last GDD, replace each of the resulting sets of size $4m+1$ by the blocks of a BIBD$(4m+1,5,1)$. We now have a PBD$[\{5,(4t+1)^\star\}; 20m+4t+1]$. Applying Lemma 1 we get our result.

On the other hand, if $m = 10$ we start with a TD$(6;10)-$ TD$(6;2)$ which has been constructed by Brouwer [4]. We remove $m-t$ elements from one of the groups G_0 in such a way that the two points x, y, of G_0, that are also in the hole are either both removed or neither removed. Inflate each point by a factor of 4 as before, replace the blocks of sizes 5 and 6 as before with the blocks of the designs GDD$[4^5;\{5\};1]$ and GDD$[4^6;\{5\};1]$, respectively. If x and y have both been removed, add the blocks of a GDD$[8^5;\{5\};1]$ on the 40 points obtained from the hole. If x and y have not been removed add the blocks of a GDD$[8^6;\{5\};1]$ on the 48 points obtained from the hole. Such a GDD has been constructed by Brouwer [2]. Then we complete the construction as before. ∎

It is worth noting that there exists a TD$(6;m)$ or all $m \equiv 0$ or $1 \pmod 5$, exception of $m = 6$ and possibly $m \in \{10,26,30\}$, as is shown in [3] and [29].

Lemma 4. *Suppose that there exists an RBIBD*$(m,5,1)$, *and let* t *be an integer satisfying* $0 \leq t < (m-1)/4$. *Then* $\Delta_\lambda(4m+4t+1) \leq \Delta_\lambda(4t+1)$ *for all* λ.

Proof: Taking the blocks of one of the resolution classes of the RBIBD as groups we obtain a GDD$[5^{m/5};\{5\};1]$ with at least t resolution classes of blocks. We adjoin a new group consistng of t new points $x_1, \ldots x_t$ to our design, and adjoin x_i

to the blocks of the ith resolution class. This gives us GDD$[5^{m/5}t^1; \{5,6\}; 1]$. As in the proof of Lemma 3, we inflate each point by a factor of 4, replace the block of sizes 5 and 6 by the blocks of the designs GDD$[4^5; \{5\}; 1]$ and GDD$[4^6; \{5\}; 1]$, respectively. This gives us a GDD$[20^{m/5}(4t)^1; \{5\}; 1]$. We now adjoin a point X at infinity to each group, replace each of the resulting sets of size 21 by a BIBD$(21, 5, 1)$ to obtain a PBD$[\{5, (4t+1)^*\}; 4m + 4t + 1]$ and our result follows from Lemma 1. ∎

Lemma 5. *Let u and w be odd positive integers, and let b be an integer satisfying $0 \leq b \leq w$. Suppose that there exists a PBD$[\{5, w^*\}; u]$ and an incomplete transversal design TD$(5; u - w + b) -$ TD$(5; b)$. Then $\Delta_\lambda(5u - 4w + 4b) \leq \Delta_\lambda(4b + w)$ for all λ.*

Proof: Let $G_0, G_1, \ldots, G_4$ be the groups of the incomplete transversal design. Let Y be the hole of this design. Let Z be a set of $(w - b)$ points that is disjoint from the set of points of our design. Let $F_i = G_i \cup Z$ and let $H_i = (Y \cap G_i) \cup Z$. For each i take the blocks of the PBD$[\{5, w^*\}; u]$ on the points of F_i in such a way that the block of size w is H_i. The blocks of our incomplete transversal design combined with the ordinary blocks of these five PBDs and the block $Y \cup Z$ of size $4b + w$ give us a PBD$[\{5, (4b+w)^*\}; 5u - 4w + 4b]$. Our result now follows from Lemma 1. ∎

In order to apply Lemma 5 we will need some incomplete transversal designs. Most of the ones we will need follow from the following lemma, which is due essentially to Wilson. It is stated and proved in the form that we need it in Brouwer and van Rees [5].

Lemma 6. *Let $m > 1$ and suppose that a TD$(k+1; t)$, a TD$(k; m)$), and a TD$(k; m+1)$ all exist; and that $0 \leq s \leq t$. Then a TD$(k; mt + s) -$ TD$(k; s)$ exists.*

3. Tricovers for $v \equiv 17 \pmod{20}$.

This is the easiest of our cases. Here we have the following result.

Theorem 2. *Let $v = 20m + 17$. Then $C_3(v) = B_3(v) + 1$.*

Proof: By Theorem 1 we have $C_3(v) \geq B_3(v) + 1$. To complete the proof we must construct a tricover on v points with $B_3(v) + 1$ quintuples.

First consider the case $m > 0$. Let D be a BIBD$(20m + 21, 5, 1)$ constructed on the set $\{1, 2, \ldots, 20m + 15, y_1, y_2, \ldots, y_6\}$. We can renumber the points so that D contains the blocks

$$\begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5, \\ 1 & 2 & 3 & y_4 & y_6, \\ 4 & 5 & 6 & y_5 & y_6. \end{array}$$

Delete these three blocks, and replace the symbols y_1, y_2, y_3 by the symbol y. Similarly, replace the symbols y_4, y_5, y_6 by the symbol z. To the resulting set of blocks, adjoin the three blocks

$$\begin{array}{ccccc} 1 & 2 & 4 & 5 & z, \\ 1 & 3 & 4 & 6 & z, \\ 2 & 3 & 5 & 6 & z. \end{array}$$

We now adjoin the blocks of a BIBD$(20m+15, 5, 2)$ on the set $\{1, 2, \ldots, 20m+15\}$, which exists since $v \neq 15$. It is easily verified that this collection of blocks forms a tricover of $20m + 17$ points with $B_3(20m + 17) + 1$ blocks.

Now consider the case $v = 17$. Hanani [9] has shown that there exists a $TD_\lambda(7; n)$ for all $\lambda \geq 2$ and all positive integers n. Setting $\lambda = n = 3$ and removing two groups, we obtain a GDD$[3^5; \{5\}; 3]$. By adjoining two new points X, Y to each group, replacing the resulting blocks by three copies of themselves, and using the blocks of our GDD we obtain a tricover on 17 points with $B_3(17) + 1 = 42$ blocks. ∎

4. Tricovers for $v \equiv 9 \pmod{20}$.

Here $v = 9, 29, 49$ need to be treated individually.

Lemma 7. *For $v \in \{9, 29, 49\}$, there exists a tricover with $B_3(v) + 1$ blocks.*

Proof: For $v = 9$ let the point set be the integers modulo 9. We have $B_3(9) = 11$. Our 12 blocks are

$$\begin{array}{cccccc} 3i & 3i+2 & 3i+4 & 3i+6 & 3i+8, & i = 0, 1, 2 \\ 0 & 1 & 2 & 3 & 6 & \bmod 9. \end{array}$$

For $v = 29$, let the point set be a point at infinity X and the 28 ordered pairs (i, j), where i is an integer modulo 7 and j is an integer modulo 4. The blocks

are

$(2i,0)$	$(2i+1,0)$	$(2i+2,0)$	$(2i+3,0)$	$(2i+4,0)$	$i=0,1,2,3$
X	$(0,0)$	$(4,1)$	$(6,2)$	$(2,3)$	mod $(7,-)$
X	$(0,0)$	$(5,1)$	$(3,2)$	$(4,3)$	mod $(7,-)$
X	$(0,0)$	$(6,1)$	$(4,2)$	$(3,3)$	mod $(7,-)$
$(0,0)$	$(6,0)$	$(0,1)$	$(0,2)$	$(0,3)$	mod $(7,-)$
$(0,0)$	$(2,0)$	$(0,1)$	$(0,2)$	$(0,3)$	mod $(7,-)$
$(0,0)$	$(4,0)$	$(0,1)$	$(0,3)$	$(1,3)$	mod $(7,-)$
$(0,0)$	$(3,0)$	$(0,2)$	$(1,2)$	$(5,3)$	mod $(7,-)$
$(0,0)$	$(2,0)$	$(1,2)$	$(3,3)$	$(6,3)$	mod $(7,-)$
$(0,0)$	$(1,1)$	$(2,1)$	$(2,2)$	$(4,2)$	mod $(7,-)$
$(0,0)$	$(1,1)$	$(2,1)$	$(3,2)$	$(6,2)$	mod $(7,-)$
$(0,0)$	$(2,1)$	$(4,1)$	$(5,2)$	$(6,3)$	mod $(7,-)$
$(0,0)$	$(3,1)$	$(5,1)$	$(2,2)$	$(6,3)$	mod $(7,-)$
$(0,0)$	$(3,1)$	$(6,1)$	$(2,2)$	$(5,3)$	mod $(7,-)$
$(0,0)$	$(4,1)$	$(6,1)$	$(3,2)$	$(2,3)$	mod $(7,-)$
$(0,1)$	$(1,1)$	$(4,1)$	$(3,3)$	$(5,3)$	mod $(7,-)$
$(0,2)$	$(1,2)$	$(2,2)$	$(4,2)$	$(2,3)$	mod $(7,-)$
$(0,2)$	$(2,3)$	$(3,3)$	$(4,3)$	$(6,3)$	mod $(7,-)$.

For $v = 49$ we proceed as follows: Let W be a 40-element set, and let X be a nine element set disjoint from W, say $X = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$. Let $C_1, C_2, \dots, C_{13}$ be the resolution classes of an RBIBD$(40,4,1)$ on W.

Adjoin a_1 to the blocks in C_1, C_2, C_3. Adjoin a_2 to the blocks in C_1, C_2, C_3. Adjoin a_3 to the blocks in C_4, C_5, C_6. Adjoin a_4 to the blocks in C_4, C_5, C_6. Adjoin a_5 to the blocks in C_7, C_8, C_9. Adjoin a_6 to the blocks in C_7, C_8, C_9. Adjoin a_7 to the blocks in C_{10}, C_{11}, C_{12}. Adjoin a_8 to the blocks in C_{10}, C_{11}, C_{12}.

Now adjoin a_9 to the blocks in C_{13} and take each of the resulting quintuples twice. Finally, take the 82 quintuples of a BIBD$(41,5,1)$ on the 41 element set obtained by adjoining a_9 to W. This gives us the 342 quintuples which cover all pairs exactly three times, except for the pairs of elements of X, which are not covered at all. Now $C_3(9) = 12$. Therefore, there is a tricovering of our 49-element set with $B_3(49) + 1 = 354$ quintuples. ∎

Lemma 8. *Suppose that* $v = 80m + 69$. *Then* $C_3(v) = B_3(v) + 1$.

Proof: Set $u = 5m+4$ in Lemma 2 and use Theorem 2 to get $\Delta_3(v) \le \Delta_3(20m+17) = 1$. ∎

Lemma 9. *Suppose that* $v = 100m+9$ *or* $100m+89$. *Then* $C_3(v) = B_3(v) + 1$.

Proof: For $v = 9$ our result follows from Lemma 7. For the remaining cases, note that if $u = 20m + 21$ or $u = 20m + 25$ and $m \ge 0$, then there exists a

BIBD($u, 5, 1$). Let B be a block of this design. Taking B as the special block we obtain a PBD[$\{5, 5^*\}; u$]. By [3] and [28] there exist three mutually orthogonal Latin squares for all orders $n > 10$, so that there exists a TD($5; u-4$) – TD($5; 1$) for all of the above u. Applying Lemma 5 with $w = 5$ and $b = 1$ and using the result for $v = 9$ establishes the lemma. ∎

Theorem 3. *Let* $v = 20m + 9$. *Then* $C_3(v) = B_3(v) + 1$.

Proof: By Theorem 1 we have $C_3(v) \geq B_3(v) + 1$. To complete the proof we must show that $\Delta_3(v) \leq 1$. In view of Lemma 9, we need consider only v congruent to 29, 49, 69 modulo 100. By Lemma 7, Lemma 8, and Lemma 9, we have $C_3(u) = B_3(u) + 1$ for $u \in \{9, 29, 49, 69, 89\}$. Applying Lemma 3 we obtain the required result for all v except 129, 149, 169, 269. Wilson has constructed a PBD[$\{5, 29^*\}; 129$]. See Lamken, Mills, and Wilson [13]. We apply Lemma 1 to obtain the result for $v = 129$. Furthermore, $v = 149$ is covered by Lemma 8. For $v = 269$, note that there exists a RBIBD($65, 5, 1$). Applying Lemma 4 with $t = 2$ yields the result.

For $v = 169$, we will apply Lemma 5 with $u = 37$, $w = 9$, and $b = 5$. For this, a PBD[$\{5, 9^*\}; 37$] and a TD($5; 33$) – TD($5; 5$) are needed. If we set $u = 2$ in Lemma 2, we obtain this PBD. Applying Lemma 6 with $k = 5$, $s = 5$, $m = 4$, $t = 7$, gives us this incomplete transversal design. ∎

5. Tricovers for $v \equiv 13 \pmod{20}$.

We first treat the cases with $v < 100$.

Here is a $(13, 5, 2)$ tricover with $B_3(13) = 24$ blocks.

$X0178$	$Y0289$	$Z0459$	$XYZ08$	01234
$X0357$	$Y0369$	$Z0467$	$XYZ19$	01256
$X1369$	$Y1458$	$Z1279$	$XYZ27$	34789
$X2459$	$Y1467$	$Z1358$	$XYZ34$	56789
$X2468$	$Y2357$	$Z2368$	$XYZ56$	

Next we come to $v = 33$.

Here is a $(33, 5, 2)$ cover with 54 blocks. The points are the 33 ordered pairs (i, j), where i is an integer modulo 3 and j is an integer modulo 11. The blocks

are

(0,0)	(1,0)	(0,1)	(0,2)	(0,3)	mod (3,–)
(0,0)	(1,1)	(0,4)	(1,4)	(0,5)	mod (3,–)
(0,0)	(1,2)	(0,6)	(0,9)	(0,10)	mod (3,–)
(0,0)	(1,3)	(1,5)	(2,5)	(0,7)	mod (3,–)
(0,0)	(2,4)	(1,6)	(2,6)	(0,8)	mod (3,–)
(0,0)	(2,7)	(1,8)	(1,9)	(2,9)	mod (3,–)
(0,0)	(1,7)	(2,8)	(1,10)	(2,10)	mod (3,–)
(0,1)	(1,1)	(2,2)	(1,7)	(1,8)	mod (3,–)
(0,1)	(1,3)	(2,3)	(2,6)	(2,8)	mod (3,–)
(0,1)	(1,4)	(2,7)	(0,9)	(1,10)	mod (3,–)
(0,1)	(1,5)	(0,6)	(1,9)	(2,9)	mod (3,–)
(0,1)	(0,5)	(1,6)	(0,10)	(2,10)	mod (3,–)
(0,2)	(1,2)	(1,5)	(1,6)	(1,7)	mod (3,–)
(0,2)	(1,3)	(0,4)	(0,9)	(1,9)	mod (3,–)
(0,2)	(2,3)	(2,4)	(0,10)	(1,10)	mod (3,–)
(0,2)	(1,4)	(2,5)	(0,8)	(1,8)	mod (3,–)
(0,3)	(1,4)	(2,6)	(0,7)	(1,7)	mod (3,–)
(0,3)	(2,5)	(2,8)	(1,9)	(0,10)	mod (3,–).

In this covering the pairs of the form $(i,9),(i+1,9)$ and those of the form $(i,10),(i+1,10)$ are covered three times, while all other pairs are covered exactly once. This shows that $C_1(33) \le 54$. Since $C_1(33) \ge B_1(33)+1 = 54$, we have $C_1(33) = B_1(33)+1 = 54$, which is a new result.

Here is a $(33,5,2)$ bipacking with 105 blocks. The points are the ordered pairs (i,j), where i is an integer modulo 3 and j is an integer modulo 11. The blocks

are

(0,0)	(1,0)	(0,1)	(1,1)	(0,2)	mod (3,–)
(0,0)	(1,0)	(2,1)	(1,2)	(0,10)	mod (3,–)
(0,0)	(1,2)	(2,2)	(0,3)	(0,4)	mod (3,–)
(0,0)	(0,3)	(1,3)	(0,4)	(2,4)	mod (3,–)
(0,0)	(1,3)	(2,3)	(0,5)	(1,5)	mod (3,–)
(0,0)	(2,3)	(0,6)	(1,6)	(0,7)	mod (3,–)
(0,0)	(1,4)	(0,6)	(2,6)	(1,7)	mod (3,–)
(0,0)	(1,4)	(2,7)	(0,8)	(1,8)	mod (3,–)
(0,0)	(2,4)	(0,8)	(0,9)	(1,9)	mod (3,–)
(0,0)	(0,5)	(1,5)	(2,7)	(1,9)	mod (3,–)
(0,0)	(2,5)	(2,6)	(2,8)	(0,10)	mod (3,–)
(0,0)	(2,5)	(1,8)	(2,8)	(2,10)	mod (3,–)
(0,0)	(1,6)	(0,9)	(2,9)	(1,10)	mod (3,–)
(0,0)	(0,7)	(1,7)	(2,9)	(1,10)	mod (3,–)
(0,1)	(1,1)	(1,2)	(1,7)	(2,8)	mod (3,–)
(0,1)	(1,2)	(0,5)	(0,6)	(2,9)	mod (3,–)
(0,1)	(0,3)	(1,4)	(0,9)	(0,10)	mod (3,–)
(0,1)	(0,3)	(1,5)	(2,7)	(2,9)	mod (3,–)
(0,1)	(1,3)	(1,5)	(0,8)	(2,10)	mod (3,–)
(0,1)	(1,3)	(2,6)	(0,7)	(0,10)	mod (3,–)
(0,1)	(2,3)	(1,6)	(2,8)	(1,9)	mod (3,–)
(0,1)	(2,3)	(2,6)	(0,8)	(0,9)	mod (3,–)
(0,1)	(0,4)	(1,4)	(1,9)	(1,10)	mod (3,–)
(0,1)	(2,4)	(0,5)	(1,6)	(1,8)	mod (3,–)
(0,1)	(0,4)	(2,5)	(0,6)	(2,10)	mod (3,–)
(0,1)	(2,4)	(2,5)	(1,7)	(2,7)	mod (3,–)
(0,2)	(1,2)	(1,5)	(0,6)	(0,9)	mod (3,–)
(0,2)	(0,3)	(0,6)	(2,8)	(2,10)	mod (3,–)
(0,2)	(0,3)	(0,7)	(0,8)	(1,9)	mod (3,–)
(0,2)	(1,3)	(2,7)	(2,8)	(1,10)	mod (3,–)
(0,2)	(2,3)	(2,7)	(2,9)	(0,10)	mod (3,–)
(0,2)	(1,4)	(2,5)	(1,6)	(1,10)	mod (3,–)
(0,2)	(2,4)	(1,5)	(1,7)	(0,10)	mod (3,–)
(0,2)	(0,4)	(0,5)	(1,8)	(0,9)	mod (3,–)
(0,2)	(0,4)	(1,6)	(1,7)	(0,8)	mod (3,–).

In this bipacking the pairs of the form $(i, 10), (i+1, 10)$ are not covered at all, while all other pairs are covered exactly twice. This shows that the bipacking number $P_2(33) \geq 105$. Since $P_2(33) \leq [33[64/4]/5] = 105$ we get $P_2(33) = 105$.

If we combine the (33, 5, 2) cover and the (33, 5, 2) bipacking we get a (33, 5, 2) tricover with $B_3(33) = 159$ blocks.

Setting $u = 3$ in Lemma 2, we obtain $\Delta_3(53) \leq \Delta_3(13) = 0$.

To handle the case $v = 73$ we construct a PBD[$\{5, 13^*\}; 73$]. The points of the special block of size 13 are the integers $0, 1, 2, 3, 4$ and A, B, C, D, E, F, G, H. The other points are the 60 ordered pairs (i, j), where i is an integer modulo 15 and j is an integer modulo 4. The quintuples are

$$h(h+5g,0)\ (h+5g+1,0)\ (h+5g+3,0)\ (h+5g+4,2)\ \mathrm{mod}(-,4)$$
$$h(h+5g+2,0)\ (h+5g+2,1)\ (h+5g+2,2)\ (h+5g+2,3),$$

where $h = 0, 1, 2, 3, 4$ and $g = 0, 1, 2$; and

A	$(0,0)$	$(3,1)$	$(9,2)$	$(5,3)$	mod $(15,-)$
B	$(0,0)$	$(4,1)$	$(5,2)$	$(12,3)$	mod $(15,-)$
C	$(0,0)$	$(5,1)$	$(2,2)$	$(11,3)$	mod $(15,-)$
D	$(0,0)$	$(7,1)$	$(10,2)$	$(14,3)$	mod $(15,-)$
E	$(0,0)$	$(9,1)$	$(13,2)$	$(3,3)$	mod $(15,-)$
F	$(0,0)$	$(11,1)$	$(6,2)$	$(9,3)$	mod $(15,-)$
G	$(0,0)$	$(12,1)$	$(8,2)$	$(7,3)$	mod $(15,-)$
H	$(0,0)$	$(14,1)$	$(7,2)$	$(4,3)$	mod $(15,-)$
$(0,0)$	$(4,0)$	$(9,0)$	$(2,1)$	$(10,1)$	mod $(15,-)$
$(0,1)$	$(4,1)$	$(10,1)$	$(2,2)$	$(9,2)$	mod $(15,-)$
$(0,2)$	$(4,2)$	$(9,2)$	$(2,3)$	$(10,3)$	mod $(15,-)$
$(0,0)$	$(7,0)$	$(2,3)$	$(8,3)$	$(13,3)$	mod $(15,-)$

Using Lemma 1 we obtain $\Delta_3(73) \leq \Delta_3(13) = 0$.

Lamken, Mills, and Wilson [13] have constructed a PBD[$\{5, 13^*\}; 93$]. Using Lemma 1 again we obtain $\Delta_3(93) \leq \Delta_3(13) = 0$.

Theorem 4. *Let* $v = 20m + 13$. *Then* $C_3(v) = B_3(v)$.

Proof: By Theorem 1 we have $C_3(v) \geq B_3(v)$. To complete the proof we must show that $\Delta_3(v) \leq 0$.

We have already shown that $\Delta_3(v) = 0$ for $v = 13, 33, 53, 73$, and 93. Using these results we apply Lemma 3 to obtain the desired result for all v except 133, 153, 173, 193, 273, 293, 393, 613.

For $v = 133$ we use Lemma 2 with $u = 8$.

Fr $v = 153$ we set $u = 2$ in Lemma 2 to obtain a PBD[$\{5, 9^*\}; 37$]. Since there exists a TD(5; 29) there exists a TD(5; 29) – TD(5; 1). Using Lemma 5 with $u = 37$, $w = 9$, $b = 1$, we obtain $\Delta_3(153) \leq \Delta_3(13) = 0$.

For $v = 173$, we have just seen that there exists a PBD[$\{5, 9^*\}; 37$]. Setting $k = 5$, $s = 6$, $m = 4$, $t = 7$, in Lemma 6 we obtain a TD(5; 34) – TD(5; 6). We now use Lemma 5 with $u = 37$, $w = 9$, $b = 6$.

For $v = 193$, we take one block of a BIBD(41, 5, 1) as a special block to obtain a PBD[$\{5, 5^*\}; 41$]. We set $k = 5$, $s = 2$, $t = 9$, $m = 4$, in Lemma 6 and get a TD(5; 38) – TD(5; 2). We now apply Lemma 5 with $u = 41$, $w = 5$, $b = 2$.

It is known that there exists an RBIBD$(u, 5, 1)$ for $u = 65, 85, 125$. For $v = 273$ and 293 we apply Lemma 4 with $m = 65$ and $t = 3$ and 8, respectively. For $v = 393$ we apply Lemma 4 with $m = 85$ and $t = 13$. For $v = 613$ we apply Lemma 4 with $m = 125$ and $t = 28$, and use the fact that we already know that $\Delta_3(113) = 0$. ∎

Theorems 1, 2, 3, and 4, together yield the following.

Theorem 5. *If* $v \equiv 1 \pmod 4$, *then* $C_3(v) = B_3(v) + 1$ *for* $v \equiv 9$ *or* 17 (mod 2)0, *and* $C_3(v) = B_3(v)$ *otherwise.*

References

1. A.M. Assaf and L.P. Singh, *Packing designs with block size 5 and index 2: the case v odd.* (Preprint).
2. A.E. Brouwer, *Mutually orthogonal Latin squares*, Math. Cent. Report **ZN81** (August 1978).
3. A.E. Brouwer, *The number of mutually orthogonal Latin squares — a table up to order 10,000*, Math. Cent. Report **ZW123** (June 1979).
4. A.E. Brouwer, *Four MOLS of order 10 with a hole of order 2*, J. Statist. Planning and Inference **10** (1984), 203–205.
5. A.E. Brouwer and G.H.J van Rees, *More mutually orthogonal Latin squares*, Discrete Math. **39** (1982), 263–281.
6. M.K. Fort, Jr. and G.A. Hedlund, *Minimal covering of pairs by triples*, Pacific J. Math. **8** (1958), 709–719.
7. B. Gardner, *Results on coverings of pairs with special reference to coverings by quintuples*, Congr. Numer. **5** (1971), 169–178.
8. G. Haggard, *On the function* $N(3, 2, \lambda, v)$, Congr. Numer. **6** (1972), 243–250.
9. H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11** (1975), 255–369.
10. H. Hanani, D.K. Ray-Chaudhuri, and R.M. Wilson, *On resolvable designs*, Discrete Math. **3** (1972), 343–357.
11. A.Hartman, W.H. Mills, and R.C. Mullin, *Covering triples by quadruples: An asymptotic soluton*, J. Combin Theory Ser. A **41** (1986), 117–138.
12. E.R. Lamken, W.H. Mills, R.C. Mullin and S.A. Vanstone, *Coverings of pairs by quintuples*, J. Combin. Theory Ser. A **44** (1987), 49–68.
13. E.R. Lamken, W.H. Mills, and R.M. Wilson, *Four pairwise balanced designs*, Designs, Codes and Cryptography **1** (1991), 63–68.
14. R. Mathon and A. Rosa, *Tables of parameters of BIBD's with* $r \leq 41$ *including existence, enumeration and resolvability results: An update*, Ars Combin. **30** (1990), 65–96.
15. W.H. Mills, *On the covering of pairs by quadruples I*, J. Combin. Theory Ser. A **13** (1972), 55-78.

16. W.H. Mills, *On the covering of pairs by quadruples II*, J. Combin. Theory Ser. A **15** (1973), 138–166.
17. W.H. Mills, *Covering problems*, Congr. Numer. **8** (1973), 23–52.
18. W.H. Mills, *On the covering of triples by quadruples*, Congr. Numer. **10** (1974), 563–581.
19. W.H. Mills, *A covering of pairs by quintuples*, Ars Combin. **18** (1984), 21–31.
20. W.H. Mills and R.C. Mullin, *Covering pairs by quintuples: The case v congruent to* $3 \pmod 4$, J. Combin. Theory Ser. A **49** (1988), 308–322.
21. R. C.Mullin, *On covering pairs by quintuples: The cases $v \equiv 3$ or 11 modulo 20*, J. Combin. Math. Combin. Comput. **2** (1987), 133–146.
22. R. C.Mullin, *On the determination of the covering numbers* $C(2,5,v)$, J. Combin. Math. Combin. Comput. **4** (1988), 123–132.
23. R. C.Mullin and J.D. Horton, *Bicovers of pairs by quintuples: v even*, Ars Combin. **26** (1988), 197–228.
24. R.C.Mullin, J.D. Horton, and W.H. Mills, *On bicovers of pairs by quintuples: v odd, $v \not\equiv 3$ (mod 10)*, Ars Combinatoria **31** (1991), 3–19.
25. R.C.Mullin, E. Nemeth, R.G. Stanton, and W.D. Wallis, *Isomorphism properties of some small covers*, Utilitas Math. **29** (1986), 269–283.
26. J.Schönheim, *On coverings*, Pacific J. Math. **14** (1964), 1405–1411.
27. R.G. Stanton and M.J. Rogers, *Packings and coverings by triples*, Ars Combin. **13** (1982), 61–69.
28. D.T. Todorov, *Three mutually orthogonal Latin squares of order 14*, Ars Combin. **20** (1985), 45–47.
29. D.T. Todorov, *Four mutually orthogonal Latin squares of order 20*, Ars Combin. **27** (1989), 63–65.
30. R. M. Wilson, *Concerning the number of mutually orthogonal Latin squares*, Discrete Math. **9** (1974), 181–198.
31. R. M. Wilson, *Contructions and uses of pairwise balanced designs*, Combinatorics ,Part I; Math. Centre Tracts **55** (1975), 18–41, Amsterdam.