On Zero-Sum Turan Numbers: Stars and Cycles

Yair Caro

Department of Mathematics
School of Education
University of Haifa - Oranim
Tivon 36-910, ISRAEL

Abstract. Let Z_k be the cyclic group of order k. Let H be a graph. A function $c: E(H) \to Z_k$ is called Z_k-coloring of the edge set $E(H)$ of H. A subgraph $G \subset H$ is called zero-sum (with respect to a Z_k-coloring) if $\sum_{e \in E(G)} c(e) \equiv o \pmod{k}$. Define the zero-sum Turan numbers as follows. $T(n, G, Z_k)$ is the maximum number of edges in a Z_k-colored graph on n vertices, not containing a zero-sum copy of G. Extending a result of [BCR], we prove:

THEOREM. *Let $m \geq k \geq 2$ be integers, $k \mid m$. Suppose $n > 2(m-1)(k-1)$ then*

$$T(n, K_{1,m}, Z_k) = \begin{cases} \frac{(m+k-2)\cdot n}{2} - 1, & n-1 \equiv m \equiv k \equiv o \pmod 2 \\ \lfloor \frac{(m+k-2)\cdot n}{2} \rfloor, & \textit{otherwise.} \end{cases}$$

1. Introduction.

Bialostocki and Dierker [BD1, BD2] raised the following variant of the Ramsey numbers.

Let Z_k be the cyclic group of order k. Let H be a graph. A function $c: E(H) \to Z_k$ is called a Z_k-coloring of the edges of H. A subgraph $G \subset H$ is called zero-sum (with respect to a Z_k-coloring), if $\sum_{e \in E(G)} c(e) \equiv o \pmod{k}$.

Define $R(G, Z_k)$ to be the minimal integer t, such that for every Z_k-coloring of the edges of K_t, there exists a zero-sum copy of G. (One must assume $k \mid e(G)$, $e(G) = |E(G)|$).

Bialostocki and Dierker [BD1,BD2] proved among many results that

$$R(K_{1,n}, Z_n) = \begin{cases} 2n & n \equiv 1 \pmod 2 \\ 2n-1 & n \equiv o \pmod 2. \end{cases}$$

Later this result has been greatly extended to yield [CAR], for $k \mid n$

$$R(K_{1,n}, Z_k) = \begin{cases} n+k-1 & n \equiv k \equiv o \pmod 2 \\ n+k & \text{otherwise.} \end{cases}$$

In [BCR] the related Turan numbers were introduced. $T(n, G, Z_k)$ is the maximum number of edges in a Z_k-colored graph on n vertices, not containing a zero-sum copy of G.

Among the many results of [BCR] the following result is the starting point of this paper.

$$T(n, K_{1,m}, Z_m) = \begin{cases} (m-1)n - 1 & n-1 \equiv m \equiv o \pmod 2 \\ (m-1)n & \text{otherwise.} \end{cases}$$

The main concern of this paper is the investigation of $T(n, K_{1,m}, Z_k)$ for $k \mid m$. We shall make use of the following theorem of Erdös-Ginzburg-Ziv.

Theorem A. [EGZ] *Let* $\{a_1, a_2, \ldots, a_{(t+1)k-1}\}$ *be a sequence of integers. There exists a subset* $I \subset \{1,2,\ldots,(t+1)k-1\}$, $|I| = t\cdot k$ *such that* $\sum_{i\in I} a_i \equiv o \pmod{k}$.

The next result dealt with the extremal cases in the EGZ-theorem.

Theorem B. [CAR] *Let* $A = \{a_1, a_2, \ldots, a_{(t+1)k-2}\}$ *be a sequence of integers.*

Suppose there exists no subset $I \subset \{1,2,\ldots,(t+1)k-2\}$, $|I| = t\cdot k$ *such that* $\sum_{i\in I} a_i \equiv o \pmod{k}$, *then*

(1) *the members of* A *belong to exactly two residue classes of* Z_k, *and further, each residue class contains* $-1 \pmod{k}$ *members from* A.
(2) *If* k *is even, then the residue classes are of distinct parity.*

Two further results that we shall need later are:

Theorem C. [HAR] *Let* K_n *be the complete graph on* n *vertices.*

(1) *if* $n \equiv 1 \pmod{2}$, *then* K_n *is the edge disjoint union of* $\frac{n-1}{2}$ *hamiltonian cycles* C_n.
(2) *if* $n \equiv o \pmod{2}$, *then* K_n *is the edges disjoint union of* $\frac{n-2}{2}$ *hamiltonian cycles* C_n *and one perfect matching* M.

Theorem D. [SSP] [BOL] [YAP] *Let* G *and* H *be graphs on* n *vertices, with maximum degree* $\Delta(G)$ *and* $\Delta(H)$. *Suppose* $2\Delta(G)\Delta(H) < n$, *then there is a packing of* G *and* H *(that is, one can place edge disjoint copies of* G *and* H *within* n *vertices).*

Lastly, our notation is standard and we refer to the books of Harary and Bollobas ([HAR], [BOL]).

2. Results.

We start with the exact determination of $T(n, K_{1,m}, Z_2)$.

Theorem 1. *If* $2 \mid m$ *then*

$$T(n, K_{1,m}, Z_2) = \begin{cases} \binom{n}{2} & n \leq m \\ \frac{m\cdot n}{2} & n \equiv o \pmod{2},\ n > m \\ \frac{m\cdot n}{2} - 1 & n \equiv 1 \pmod{2},\ n > m. \end{cases}$$

Proof: The case $n \leq m$ is trivial so we assume $n > m$. Let G be an extremal graph that realizes $T(n, K_{1,m}, Z_2)$.

If $\Delta(G) \geq m+1$, then by Theorem A there exists a zero-sum copy of $K_{1,m}$ (take in Theorem A, $k = 2$, $t = \frac{m}{2}$), hence, $e(G) \leq \frac{m\cdot n}{2}$.

Suppose $n \equiv 1 \pmod{2}$. If there exists a vertex $v \in V(G)$ such that $\deg v < m$ then clearly $e(G) \leq \frac{m\cdot n}{2} - 1$, otherwise G is an m-regular graph. As G contains no zero-sum copy of $K_{1,m}$, the number of edges, incident with a vertex v, that are colored by 1 must be odd. The graph induced by the edges colored 1 has $n \equiv 1 \pmod 2$ vertices and all its vertices are of odd degree which is clearly impossible. Hence, we have shown that

$$T(n, K_{1,m}, Z_2) \leq \begin{cases} \frac{m \cdot n}{2} & n \text{ even} \\ \frac{m \cdot n}{2} - 1 & n \text{ odd.} \end{cases}$$

To prove the lower bound we consider two cases.

Case 1: $m \equiv 2 \pmod 4$.

Consider a connected m-regular graph G on n vertices. Since $n > m$ such a graph G exists by Theorem C, in fact, take $\frac{m}{2}$ hamiltonian cycles C_n.

As m is even G is eulerian. Color its edges alternately by 0 and 1, along the eulerian trail.

As $m \equiv 2 \pmod 4$ we shall have in each vertex v of G, $\frac{m}{2}$ edges colored 1 and $\frac{m}{2}$ edges colored 0, except perhaps the last (= first) vertex on the trail which is dependent on the parity of n. So if $n \equiv o \pmod 2$ the construction is exact, and if $n \equiv 1 \pmod 2$ then we shall delete the last edge of the trail and G is Z_2-colored without zero-sum copy of $K_{1,m}$. Hence, we proved

$$T(nkK_{1,m}, Z_2) \geq \begin{cases} \frac{m \cdot n}{2} & n \text{ even} \\ \frac{m \cdot n}{2} - 1 & n \text{ odd} \end{cases}$$

completing the proof of the theorem for $m \equiv 2 \pmod 4$.

Case 2: $m \equiv o \pmod 4$.

Let G be the union of $\frac{m}{2}$ hamiltonian cycles C_n, (again G exists by Theorem C).

Color one cycle by the color 1. The remaining edges form a connected $m-2$ regular graph, $m - 2 \equiv 2 \pmod 4$.

Apply the eulerian coloring, as in Case 1, to obtain a Z_2-colored graph G in which in every vertex there are $\frac{m}{2} + 1$ edges colored 1, and $\frac{m}{2} + 1$ edges colored 0, except perhaps the two vertices forming the last edge on the eulerian trail in which case we delete this last edge if $n \equiv 1 \pmod 2$.

One can see that G contains no sero-sum copy of $K_{1,m}$, and again

$$T(n, K_{1,m}, Z_2) \geq \begin{cases} \frac{m \cdot n}{2} & n \text{ even} \\ \frac{m \cdot n}{2} - 1 & n \text{ odd} \end{cases}$$

completing the proof. ∎

We are now ready for the main result, namely,

Theorem 2. *Let $k, m, n \geq 2$ be positive integers, such that $k \mid m$, and $n > 2(m-1)(k-1)$. Then*

$$T(n, K_{1,m}, Z_k) = \begin{cases} \frac{(m+k-2)n}{2} - 1 & n - 1 \equiv k \equiv m \equiv o \pmod 2 \\ \lfloor \frac{(m+k-2) \cdot n}{2} \rfloor & \text{otherwise.} \end{cases}$$

Proof: Let us first show that

$$\left\lfloor \frac{(m-1)n}{2} \right\rfloor + \left\lfloor \frac{(k-1)n}{2} \right\rfloor \leq T(n, K_{1,m}, \mathbf{Z}_k) \leq \left\lfloor \frac{(m+k-2)n}{2} \right\rfloor.$$

Suppose G realizes $T(n, K_{1,m}, \mathbf{Z}_k)$. If $\Delta(G) \geq m + k - 1$ then by Theorem A, there exists a zero-sum copy of $K_{1,m}$ (take in Theorem A, $t = \frac{m}{k}$). Hence, $e(G) \leq \lfloor \frac{m+k-2)n}{2} \rfloor$.

On the other hand, let $H(t, n)$ be the extremal Turan graph on n vertices for $K_{1,t}$, then $\Delta(H(m, n)) = m-1, \Delta(H(k, n)) = k-1$ (see, for example, [SIM]) and by Theorem D there is a packing of $H(m, n)$ and $H(k, n)$ into a graph G on n vertices.

Color the edges that belong to $H(m, n)$ by color 1.

Color the edges that belong to $H(k, n)$ by color 0.

Clearly, there exists no zero-sum $\pmod{k}$ copy of $K_{1,m}$. Hence, $T(n, K_{1,m}, \mathbf{Z}_k) \geq T(n, K_{1,m}) + T(n, K_{1,k}) = \lfloor \frac{(m-1)n}{2} \rfloor + \lfloor \frac{(k-1)n}{2} \rfloor$, (for $T(n, K_{1,m})$ see, for example, [SIM]).

Consider now three cases.

Case 1: $n \equiv o \pmod{2}$

Here, $\lfloor \frac{(m-1)n}{2} \rfloor + \lfloor \frac{(k-1)n}{2} \rfloor = \lfloor \frac{(m+k-2)n}{2} \rfloor$ and this case is solved.

Case 2: $n \equiv 1 \pmod{2},\ m \equiv o \pmod{2},\ k \equiv 1 \pmod{2}$.

$\lfloor \frac{(m-1)n}{2} \rfloor + \lfloor \frac{(k-1)n}{2} \rfloor = \frac{(m-1)n}{2} - \frac{1}{2} + \frac{(k-1)n}{2} = \frac{(m+k-2)n}{2} - \frac{1}{2} = \lfloor \frac{(m+k-2)n}{2} \rfloor$, and this case is also solved.

As $k \mid m$, the only case that remains is

Case 3: $n \equiv 1 \pmod{2},\ m \equiv k \equiv o \pmod{2}$.

In this case $\frac{(m+k-2)n}{2} - 1 \leq T(n, K_{1,m}, \mathbf{Z}_k) \leq \frac{(m+k-2)n}{2}$.

We shall improve the upper bound by 1.

Let G be a graph that realizes $T(n, K_{1,m}, \mathbf{Z}_k)$ in Case 3.

If for some vertex v, $\deg v \leq m + k - 3$, we are done as $2e(G) = \sum_v \deg v \leq n(m + k - 2) - 1$ (recall that $\Delta(G) \leq m + k - 2$). Hence, G must be a $m + k - 2$ regular graph.

Apply now Theorem B. The only possibility to avoid a zero-sum copy of $K_{1,m}$ is when in each vertex there are $-1 \pmod{k}$ edges colored α, and $-1 \pmod{k}$ edges colored β, for some $\alpha, \beta \in \mathbf{Z}_k$ and also α, β are of distinct parity.

Asume w.l.o.g that α is of odd parity. Consider the subgraph $H \subset G$ on $n \equiv 1 \pmod{2}$ vertices induced by the edges colored α. In each vertex v of H there are $-1 \pmod{k}$ edges, hence, $\deg_H v \equiv 1 \pmod{2}$ for all $v \in V(H)$ which is impossible as is well known. Hence, we proved that if $n - 1 \equiv m \equiv k \equiv o \pmod{2}$ then

$$T(n, K_{1,m}, \mathbf{Z}_k) \leq \frac{(m+k-2)n}{2} - 1$$

completing the proof of the theorem. ∎

Remark: For the sake of completeness it is of interest to close the gap for the cases in which $m < n \leq 2(m-1)(k-1)$. Our next result concerns cycles. Let F_t be the family of all cycles of at least t vertices that is, $F_t = \{C_n, n \geq t\}$. We pose the following problem.

Determine $T(n, F_t, Z_k)$, where $T(n, F_t, Z_k)$ is the maximal number of edges in a Z_k-colored graph G on n vertices, not containing a zero-sum $\pmod{k}$ copy of any cycle $C_n, n \geq t$.

Theorem 3. $T(n, F_3, Z_2) = \lfloor \frac{3(n-1)}{2} \rfloor$.

Proof: Let G be an extremal graph that realizes $T(n, F_3, Z_2)$.

If there exist in G two vertices x and y and three vertex disjoint paths which connect them, then clearly there are two paths, the sum of their edges is $0 \pmod{2}$. These two paths constitute a zero-sum cycle. Hence, for any pair of vertices x, y the connectivity $\kappa(x, y) \leq 2$.

For such graphs it is well known (see, for example, [**BOL**, p. 30]) that $e(G) \leq \lfloor \frac{3(n-1)}{2} \rfloor$, and the bound attained by graphs in which every block is K_3 except perhaps one block, which is either a single edge K_2 or a cycle C_4.

In such an extremal graph G, color all the edges belonging to a block of type K_3 by 1, a block of type K_2 color 1, and in a block of type C_4 color three edges by 0 and the remaining edge by 1. Clearly, no cycle in G is zero-sum $\pmod{2}$. ∎

One may ask what happens if instead of F_3 we take F_t in Theorem 3. We cannot give an exact solution to this problem but we can show that a linear upper bound exists.

Theorem 4. *For every $t \geq 3$ there exists a positive constant $c(t)$ s.t*

$$T(n, F_t, Z_2) \leq c(t) \cdot n.$$

Proof: Recall the definition of topological complete graph TK_p (see, for example, [**BOL**, p. 368]). Mader [**MAD**] was the first to show that the Turan numbers of TK_p are bounded above by a linear function, namely, $T(n, TK_p) \leq c_1(p)n$, $c_1(p) > 0$ depend on p only.

Let G be an extremal graph that realizes $T(n, F_t, Z_2)$.

Suppose G contains a copy of a topological complete graph $H = TK_{3(\lceil \frac{t}{2} \rceil - 1)+2}$. Then in H there are two vertices x, y and three vertex disjoint paths of length at least $\lceil \frac{t}{2} \rceil$ connecting x and y (this is because such x and y exist in $K_{3(\lceil \frac{t}{2} \rceil - 1)+2}$). Now two of the paths add to $0 \pmod{2}$ as in Theorem 3. The resulting cycle is of length at least $2\lceil \frac{t}{2} \rceil \geq t$ and is zero-sum $\pmod{2}$ contradicting the choice of G.

Hence, $T(n, F_t, Z_2) \leq T(n, TK_{3(\lceil \frac{t}{2} \rceil - 1)+2}) \leq c(t)n$ by Mader's theorem. ∎

We choose to close with the following problem, due also to Bialostocki.

Problem: Let t and k be given integers, $t \geq 3$, $k \geq 2$.

Does there exist a constant $c(t, k)$ such that

$$T(n, F_t, Z_k) \leq c(t, k) \cdot n.$$

Acknowledgement.

I am indebted to the referee for many helpful comments.

References

[BCR]. A. Bialostocki, Y. Caro, Y. Roditty, *On zero-sum Turan numbers*, Ars Combinatoria **29a** (1990), 117–127.

[BD1]. A. Bialostocki, P. Dierker, *On the Erdös-Ginzburg-Ziv theorem and the Pamsey numbers for stars and matchings.* (to appear).

[BD2]. A. Bialostocki, P. Dierker, *Zero-sum Ramsey theorems*, Congressus Numerantium **70** (1990), 119–130.

[BOL]. B. Bollobas, "Extremal Graph Theory", Academic Press, 1978.

[CAR]. Y. Caro, *On zero sum Ramsey numbers – stars.* (to appear).

[EGZ]. P. Erdös, A. Ginzburg, A. Ziv, *Theorem in additive number theory*, Bull. Research Council **1OF** (1961), 41–43, Israel.

[HAR]. F. Harary, "Graph Theory", Addison Wesley, 1972.

[MAD]. W. Mader, *Homomorphiesätze für Graphen*, Math. Ann. **178** (1968), 154–168.

[SIM]. M. Simonovits, *Extremal graph theory*, in "Selected Topics in Graph Theory – 2", L. Beineke, R. Wilson, editors, Academic Press, 1983.

[SSP]. N. Sauer, J. Spencer, *Edge disjoint placement of graphs*, J. Combin. Theory **Ser B 25** (1978), 295–302.

[YAP]. H.P. Yap, *Some topics in graph theory*, Lecture Note Series 108, London Mathematical Society.