ON THE WELL COVEREDNESS OF PRODUCTS OF GRAPHS

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1. Introduction

In this paper we discuss finite undirected simple graphs. For any undefined terms see [2] and [10]. For any graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G respectively. For $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G and, more generally, $N_G(S) = \bigcup_{v \in S} N_G(v)$ for $S \subseteq V(G)$. A set of mutually nonadjacent vertices of a graph is said to be independent. The maximum size of an independent set in a graph G is called the independence number of the graph and is denoted by $\alpha(G)$. A graph is called well covered if every maximal independent set is a maximum independent set (equivalently every independent set is contained in a maximum independent set). For example, the graphs in Figure 1 are well covered. A graph G is said to be very well covered if G is a well covered graph without isolated vertices and $\alpha(G) = |V(G)|/2$.

The concept of a well covered graph was introduced by Plummer [11] in 1970. Until now, however, only a few classes of well covered graphs have been studied. For example, Staples [14] studied properties of the W_n classes of graphs, where a graph G belongs to class W_n if $|V(G)| \geq n$ and every n disjoint independent sets in G are contained in n disjoint maximum independent sets. The W_n classes form a descending chain $W_1 \supseteq W_2 \supseteq W_3 \supseteq \ldots$ and W_1 is the class of well covered graphs. Staples [13] and later Favaron [4] gave a characterization of very well covered graphs. These graphs include bipartite well covered graphs which were also characterized by Ravindra [12]. Berge [1], among other things, presents relationships between the class of well covered graphs and some other classes of graphs. Finbow and Hartnell [5] characterized well covered graphs of girth at least 8. Recently Finbow, Hartnell, and Nowakowski in [8] and [9] have completely described well covered graphs of girth at least 5 and well covered graphs containing neither a cycle C_4 nor a cycle C_5 as a subgraph. The cubic, planar, 3-connected graphs which are well covered have been

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characterized in [3] by Campbell and Plummer (see Proposition 5 below). Other subclasses of the well covered graphs were studied in [6], [7], and [15].

The concept of the well coveredness of a graph has attracted our attention in relation to products of graphs. In this paper we study the (very) well coveredness of graphs formed from other graphs by various operations. Conditions for the corona, the join, the disjunction, the conjunction, the lexicographic product, and the cartesian product of graphs to be (very) well covered are established based upon the factors. Many questions relating to the well coveredness of product graphs are still open, and we present a few of them.

Before proceeding we state a few necessary results. As in [2], a vertex x of a graph G is called a critical vertex of G if $\alpha(G - x) \neq \alpha(G)$, or equivalently, if every maximum independent set of G contains x.

Proposition 1 [2]. *If a graph G has no critical vertices, then $\alpha(G) \leq |V(G)|/2$.* □

Proposition 2. *If G is a well covered graph without isolated vertices, then G has no critical vertices.*

Proof. Let x be any vertex of G. It is enough to show that there exists a maximum independent set in G that does not contain x. Let y be any neighbour of x and let I be any maximal independent set that contains y. Certainly, $x \notin I$. In addition, since G is well covered and I is a maximal independent set in G, I is a maximum independent set in G. This implies the result. □

Corollary 1. *If G is a well covered graph without isolated vertices, then $\alpha(G) \leq |V(G)|/2$.*

Proof. Immediate from Propositions 1 and 2. □

Proposition 3 [4]. *A graph G without isolated vertices is very well covered if and only if G has a perfect matching M and for every edge $vu \in M$,*

(1) vu does not belong to a triangle and

(2) every vertex of $N_G(v)$ is adjacent to every vertex of $N_G(u)$. □

Proposition 4 [12]. *A bipartite graph G without isolated vertices is well covered if and only if G has a perfect matching M and for every $vu \in M$, the subgraph induced by $N_G(v) \cup N_G(u)$ is a complete bipartite graph.* □

An immediate consequence of Propositions 3 and 4 is

Corollary 2. *A bipartite graph without isolated vertices is very well covered if and only if it is well covered.* □

Proposition 5 [3]. *There are exactly four cubic, planar, 3-connected, well covered graphs and they are shown in Figure 1.* □

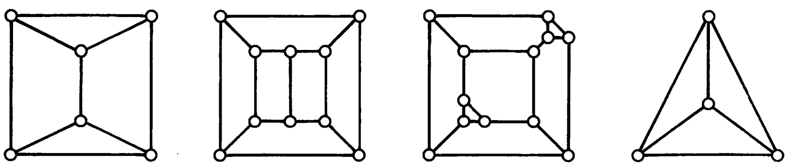

Figure 1.

2. The corona of graphs

For a graph G and a family $\mathcal{H} = \{H_v : v \in V(G)\}$ of graphs indexed by the vertices of G, the corona $G \circ \mathcal{H}$ of G and $\mathcal{H}$ is the disjoint union of G and H_v, $v \in V(G)$, with additional edges joining each vertex v of G to all vertices of H_v. If all the graphs of the family $\mathcal{H}$ are isomorphic to one and the same graph H then we shall write $G \circ H$ instead of $G \circ \mathcal{H}$.

The following results specify when the corona $G \circ \mathcal{H}$ is a (very) well covered graph.

Theorem 1. *Let G be a graph, and let $\mathcal{H} = \{H_v : v \in V(G)\}$ be a family of nonempty graphs indexed by the vertices of G. Then the corona $G \circ \mathcal{H}$ is a well covered graph if and only if $\mathcal{H}$ consists of complete graphs.*

Proof. Assume that $G \circ \mathcal{H}$ is a well covered graph. For every vertex $v \in V(G)$, let I_v be any maximum independent set in H_v. It is easy to see that $I = \bigcup_{v \in V(G)} I_v$ is a maximal (and thus, maximum) independent set in $G \circ \mathcal{H}$. We claim that H_v is a complete graph for every $v \in V(G)$. Suppose to the contrary that H_{v_0} is not a complete graph for some $v_0 \in V(G)$. Then $|I_{v_0}| > 1$ and by removing I_{v_0} from I and replacing it by $\{v_0\}$, we form the set I' which is also a maximal independent set in $G \circ \mathcal{H}$ but which is smaller than I, a contradiction. This implies that the above condition is necessary for the corona $G \circ \mathcal{H}$ to be well covered.

We now assume that each graph of the family $\mathcal{H}$ is complete. Let I be a maximal independent set in $G \circ \mathcal{H}$. It follows from the definition of $G \circ \mathcal{H}$

and the choice of I that either $v \in I$ or $|I \cap V(H_v)| = 1$ for every $v \in V(G)$; for if there were a vertex v_0 in G such that $v_0 \notin I$ and $I \cap V(H_{v_0}) = \emptyset$, then, for any $x \in V(H_{v_0})$, the set $I \cup \{x\}$ would be a larger independent set in $G \circ \mathcal{H}$ which is impossible. This implies that each maximal independent set in $G \circ \mathcal{H}$ has exactly $|V(G)|$ elements. Hence $G \circ \mathcal{H}$ is well covered. □

Corollary 3. *For any graph G and a positive integer n, the corona $G \circ K_n$ is a well covered graph.* □

The above theorem and its proof immediately yield the next corollary.

Corollary 4. *For a graph G and a family $\mathcal{H}$ of nonempty graphs indexed by the vertices of G, the corona $G \circ \mathcal{H}$ is very well covered if and only if $G \circ \mathcal{H} = G \circ K_1$.* □

3. The lexicographic product

In this section we study maximal independent sets of a lexicographic graph. Then we establish necessary and sufficient conditions for the (very) well coveredness of lexicographic products of graphs. For a graph G and a family $\mathcal{H} = \{H_v : v \in V(G)\}$ of nonempty graphs indexed by the vertices of G, the lexicographic product $G[\mathcal{H}]$ of G and $\mathcal{H}$ is the graph having vertex set $V(G[\mathcal{H}]) = \bigcup_{v \in V(G)} \{(v, u) : u \in V(H_v)\} = \bigcup_{v \in V(G)} \{v\} \times V(H_v)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G[\mathcal{H}]$ are adjacent whenever either $[v_1 u_1 \in E(G)]$ or $[v_1 = u_1$ and $v_2 u_2 \in E(H_{v_1})]$. If all the graphs of the family $\mathcal{H}$ are isomorphic to one and the same graph H then we shall write $G[H]$ instead of $G[\mathcal{H}]$. For a subset S of $V(G[\mathcal{H}])$, we denote $\pi_G(S) = \{x \in V(G) : \exists_{y \in V(H_x)} (x, y) \in S\}$ and $\pi_{H_x}(S) = \{y \in V(H_x) : (x, y) \in S\}$ for every $x \in \pi_G(S)$.

The join $G_1 + G_2$ of two graphs G_1 and G_2 is defined as the disjoint union of G_1 and G_2 with additional edges joining each vertex of G_1 with each vertex of G_2. It is obvious that the join $G_1 + G_2$ is isomorphic to the lexicographic product $K_2[\{G_1, G_2\}]$.

We begin by describing maximal independent sets in the lexicographic product of graphs.

Proposition 6. *Let G be a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ a family of nonempty graphs indexed by the vertices of G. A subset S of $V(G[\mathcal{H}])$ is a maximal independent set in $G[\mathcal{H}]$ if and only if $\pi_G(S)$ is a maximal independent set in G, and for every $v \in \pi_G(S)$, the set $\pi_{H_v}(S)$ is a maximal independent set in the graph H_v.*

Proof. Assume that the set $S \subseteq V(G[\mathcal{H}])$ is a maximal independent set in $G[\mathcal{H}]$. It is obvious from the definition of the lexicographic product that the set $\pi_G(S)$ is independent in G, and for every $v \in \pi_G(S)$, the set $\pi_{H_v}(S)$ is independent in H_v. We claim that $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v for $v \in \pi_G(S)$. First suppose to the contrary that $\pi_G(S)$ is not a maximal independent set in G. Then there is $v_0 \in V(G) - \pi_G(S)$ such that the set $\pi_G(S) \cup \{v_0\}$ is independent in G. Hence, for every $x \in V(H_{v_0})$, the set $S \cup \{(v_0, x)\}$ would be a greater independent set in $G[\mathcal{H}]$, a contradiction. Similarly, the set $\pi_{H_v}(S)$ (for $v \in \pi_G(S)$) is a maximal independent set in H_v, as otherwise there is $x \in V(H_v) - \pi_{H_v}(S)$ such that $\pi_{H_v}(S) \cup \{x\}$ is independent in H_v and then $S \cup \{(v, x)\}$ would be a greater independent set in $G[\mathcal{H}]$, which is impossible.

On the other hand, if $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v, $v \in \pi_G(S)$, then S is a maximal independent set in $G[\mathcal{H}]$; for if not, then there is a vertex $(v_0, x_0) \in V(G[\mathcal{H}]) - S$ such that $S \cup \{(v_0, x_0)\}$ is independent in $G[\mathcal{H}]$ and then $\pi_G(S \cup \{(v_0, x_0)\}) = \pi_G(S) \cup \{v_0\}$ or $\pi_{H_{v_0}}(S \cup \{(v_0, x_0)\}) = \pi_{H_{v_0}}(S) \cup \{x_0\}$ is a greater independent set in G or in H_{v_0}, respectively, which is impossible. This completes the proof. □

We are now ready to show conditions for the lexicographic product of graphs to be well covered.

Theorem 2. *Let G be a graph and $\mathcal{H} = \{H_v : v \in V(G)\}$ a family of nonempty graphs indexed by the vertices of G. Then the lexicographic product $G[\mathcal{H}]$ is a well covered graph if and only if G and $\mathcal{H}$ satisfy the following two conditions:*

(1) each graph H_v of the family $\mathcal{H}$ is well covered,

(2) $\sum_{v \in S_G} \alpha(H_v) = \sum_{u \in S'_G} \alpha(H_u)$ for every two maximal independent sets S_G and S'_G of G.

Proof. We begin by assuming that $G[\mathcal{H}]$ is a well covered graph. First we claim that every graph H_v from $\mathcal{H}$ is well covered. For if not, let H_{v_0} be a counterexample. Then H_{v_0} has two maximal independent sets of different cardinality, say I_{v_0} and I'_{v_0}. Let $S_G \subseteq V(G) - \{v_0\}$ be such that $S_G \cup \{v_0\}$ is a maximal independent set in G. For every $v \in S_G$, let I_v be any maximal independent set in H_v. Since $|I_{v_0}| \neq |I'_{v_0}|$, Proposition 6 implies that $\bigcup_{v \in S_G} \{(v, x) : x \in I_v\} \cup \{(v_0, y) : y \in I_{v_0}\}$ and $\bigcup_{v \in S_G} \{(v, x) : x \in I_v\} \cup \{(v_0, t) : t \in I'_{v_0}\}$ are maximal independent sets of different cardinality in $G[\mathcal{H}]$, which contradicts our assumption. Hence, each graph of the family $\mathcal{H}$ is well covered if the graph $G[\mathcal{H}]$ is well covered.

Let S_G and S'_G be two maximal independent sets in G. We now claim that $\sum_{v\in S_G}\alpha(H_v)=\sum_{v\in S'_G}\alpha(H_v)$. To prove this, let J_v be any maximum independent set in H_v for $v\in S_G\cup S'_G$. Proposition 6 and the assumption on $G[\mathcal{H}]$ imply that $S=\bigcup_{v\in S_G}\{(v,x):x\in J_v\}$ and $S'=\bigcup_{v\in S'_G}\{(v,x):x\in J_v\}$ are maximum independent sets in $G[\mathcal{H}]$. Hence $|S|=|S'|$ and then from the observation $|\{(v,x):x\in J_v\}|=|J_v|=\alpha(H_v)$ (for $v\in S_G\cup S'_G$) we have $\sum_{v\in S_G}\alpha(H_v)=|S|=|S'|=\sum_{v\in S'_G}\alpha(H_v)$, and our assertion follows.

For the converse, assume G and $\mathcal{H}$ satisfy the conditions (1) and (2). We shall prove that $G[\mathcal{H}]$ is a well covered graph. To this end, assume that S is a maximal independent set in $G[\mathcal{H}]$. Then, by Proposition 6, $\pi_G(S)$ is a maximal independent set in G and $\pi_{H_v}(S)$ is a maximal independent set in H_v for every $v\in\pi_G(S)$. Since $S=\bigcup_{v\in\pi_G(S)}\{(v,x):x\in\pi_{H_v}(S)\}$ and $|\pi_{H_v}(S)|=\alpha(H_v)$ (by (1)), $|S|=\sum_{v\in\pi_G(S)}|\{(v,x):x\in\pi_{H_v}(S)\}|=\sum_{v\in\pi_G(S)}|\pi_{H_v}(S)|=\sum_{v\in\pi_G(S)}\alpha(H_v)$. Consequently, by (2), every two maximal independent sets in $G[\mathcal{H}]$ have the same cardinality and therefore $G[\mathcal{H}]$ is a well covered graph. □

Corollary 5. *The lexicographic product $G[H]$ of two nonempty graphs G and H is a well covered graph if and only if G and H are well covered graphs; if graphs G and H are nonempty and one of them is without isolated vertices, then the lexicographic product $G[H]$ is very well covered if and only if exactly one of G and H is very well covered and the second is totally disconnected, i.e., without edges.*

Proof. The first part of the assertion easily follows from Theorem 2. Thus we shall only prove the second part. Let $a=|V(G)|$ and $b=|V(H)|$.

We first assume that $G[H]$ is very well covered. Then G and H are well covered (by the first part of the corollary), and $\alpha(G[H])=|V(G[H])|/2=ab/2$. Moreover, it follows from Proposition 6 that $\alpha(G[H])=\alpha(G)\alpha(H)$. Since G or H is without isolated vertices, Corollary 1 implies that $\alpha(G)\leq a/2$ or $\alpha(H)\leq b/2$. Therefore $ab/2=\alpha(G)\alpha(H)\leq(a/2)\alpha(H)$ or $ab/2=\alpha(G)\alpha(H)\leq\alpha(G)b/2$. This makes it obvious that $\alpha(H)=b$ and $\alpha(G)=a/2$ or $\alpha(G)=a$ and $\alpha(H)=b/2$. From this it may be concluded that H is totally disconnected and G is very well covered or vice versa, as claimed.

Finally, if G is very well covered and H is totally disconnected (or G is totally disconnected and H is very well covered), then $\alpha(G)=a/2$, $\alpha(H)=b$ (or $\alpha(G)=a$, $\alpha(H)=b/2$) and $G[H]$ is well covered. Moreover, since $\alpha(G[H])=\alpha(G)\alpha(H)=ab/2=|V(G[H])|/2$, $G[H]$ is very well covered. □

Corollary 6. *The join $G+H$ of two nonempty graphs G and H is a well covered graph if and only if G and H are well covered graphs and*

$\alpha(G) = \alpha(H)$; $G + H$ is very well covered if and only if both G and H are totally disconnected and have the same number of vertices. □

Proof. The first part of the assertion immediately follows from Theorem 2, since $G + H$ is isomorphic to $K_2[\{G, H\}]$.

In order to prove the second part, assume first that G and H are totally disconnected and each of them has n vertices. Then $G+H$ is isomorphic to the bipartite complete graph $K_{n,n}$. Since $K_{n,n}$ is very well covered, $G + H$ is very well covered.

Now assume that $G+H$ is very well covered. Then at once $\alpha(G+H) = |V(G+H)|/2 = |V(G)|/2 + |V(H)|/2$ and $\alpha(G + H) = \alpha(G) = \alpha(H)$. Since $\alpha(G) \leq |V(G)|$ and $\alpha(H) \leq |V(H)|$, so we have $\alpha(G) = |V(G)| = |V(H)| = \alpha(H)$, and thus G and H are totally disconnected graphs of the same order. □

4. The disjunction of graphs

In this section the (very) well coveredness of a disjunction graph is established based upon the (very) well coveredness of the factors. The disjunction $G_1 \vee G_2$ of graphs G_1 and G_2 is the graph having vertex set $V(G_1 \vee G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \vee G_2$ are adjacent whenever $[v_1u_1 \in E(G_1)]$ or $[v_2u_2 \in E(G_2)]$. For a subset S of $V(G_1 \vee G_2)$, we denote by $\pi_{G_1}(S)$ and $\pi_{G_2}(S)$ the projections of S onto $V(G_1)$ and $V(G_2)$ respectively, so $\pi_{G_1}(S) = \{x \in V(G_1) : \exists_{y \in V(G_2)}(x, y) \in S\}$ and $\pi_{G_2}(S) = \{y \in V(G_2) : \exists_{x \in V(G_1)}(x, y) \in S\}$.

The next four properties of independent sets in a disjunction graph will help provide a well coveredness criterion for the disjunction of two graphs.

Proposition 7. *If $I_i \subseteq V(G_i)$ is an independent set in a graph G_i $(i = 1, 2)$, then $I_1 \times I_2$ is an independent set in $G_1 \vee G_2$.*

Proof. Since the set I_i is independent in G_i, $N_{G_i}(v_i) \subseteq V(G_i) - I_i$ for each vertex $v_i \in I_i$ $(i = 1, 2)$. Hence $N_{G_1 \vee G_2}((v_1, v_2)) = (N_{G_1}(v_1) \times V(G_2)) \cup (V(G_1) \times N_{G_2}(v_2)) \subseteq ((V(G_1) - I_1) \times V(G_2)) \cup (V(G_1) \times (V(G_2) - I_2)) = V(G_1 \vee G_2) - (I_1 \times I_2)$ for each $(v_1, v_2) \in I_1 \times I_2$, and therefore the set $I_1 \times I_2$ is independent in $G_1 \vee G_2$. □

Proposition 8. *If a set $I \subseteq V(G_1 \vee G_2)$ is independent in $G_1 \vee G_2$, then the set $\pi_{G_i}(I)$ is independent in G_i $(i = 1, 2)$.*

Proof. Let v_1, v_2 be any two vertices from $\pi_{G_1}(I)$. We claim that they are nonadjacent; for if not, then vertices $(v_1, v_1'), (v_2, v_2') \in I$ (for some

$v_1', v_2' \in V(G_2)$) would be adjacent in $G_1 \vee G_2$, which is impossible. This implies that the set $\pi_{G_1}(I)$ is independent in G_1. We conclude similarly that $\pi_{G_2}(I)$ is an independent set in G_2. □

Proposition 9. *If $I_i \subseteq V(G_i)$ is a maximal independent set in G_i $(i = 1, 2)$, then $I_1 \times I_2$ is a maximal independent set in $G_1 \vee G_2$.*

Proof. By Proposition 7, the set $I_1 \times I_2$ is independent in $G_1 \vee G_2$. Suppose to the contrary that $I_1 \times I_2$ is a proper subset of some independent set I in $G_1 \vee G_2$. Then the set $\pi_{G_i}(I)$ is independent in G_i (by Proposition 8) and $I_i \subseteq \pi_{G_i}(I)$ for $i = 1, 2$. Since $|I_i| \le |\pi_{G_i}(I)|$ $(i = 1, 2)$ and $|I_1 \times I_2| < |I| \le |\pi_{G_1}(I) \times \pi_{G_2}(I)|$, $|I_1| < |\pi_{G_1}(I)|$ or $|I_2| < |\pi_{G_2}(I)|$ and therefore at least one of the sets I_1 and I_2 is not a maximal independent set in G_1 and G_2, respectively, a contradiction. □

Proposition 10. *If $I \subseteq V(G_1 \vee G_2)$ is a maximal independent set in $G_1 \vee G_2$, then $I = \pi_{G_1}(I) \times \pi_{G_2}(I)$ and $\pi_{G_i}(I)$ is a maximal independent set in G_i $(i = 1, 2)$.*

Proof. Assume that I is a maximal independent set in $G_1 \vee G_2$. By Proposition 8, $\pi_{G_i}(I)$ is an independent set in G_i $(i = 1, 2)$. Let I_i be an independent set in G_i such that $\pi_{G_i}(I) \subseteq I_i$ $(i = 1, 2)$. Then $\pi_{G_1}(I) \times \pi_{G_2}(I)$ and $I_1 \times I_2$ are independent sets in $G_1 \vee G_2$ by Proposition 7. Since $I \subseteq \pi_{G_1}(I) \times \pi_{G_2}(I) \subseteq I_1 \times I_2$, from the maximality of I we have $I = \pi_{G_1}(I) \times \pi_{G_2}(I) = I_1 \times I_2$. In addition, $\pi_{G_1}(I) = I_1$ and $\pi_{G_2}(I) = I_2$. Consequently, $\pi_{G_1}(I)$ and $\pi_{G_2}(I)$ are maximal independent sets in G_1 and G_2, respectively. □

With the above, the main result of this section falls out quite quickly.

Theorem 3. *The disjunction $G_1 \vee G_2$ of graphs G_1 and G_2 is a well covered graph if and only if the graphs G_1 and G_2 are well covered.*

Proof. Assume G_1 and G_2 are well covered graphs. In order to prove the sufficiency, it is enough to show that every maximal independent set in $G_1 \vee G_2$ has $\alpha(G_1)\alpha(G_2)$ elements. Let $I \subseteq V(G_1 \vee G_2)$ be any maximal independent set in $G_1 \vee G_2$. Then by Proposition 10, $I = \pi_{G_1}(I) \times \pi_{G_2}(I)$, and $\pi_{G_1}(I)$ and $\pi_{G_2}(I)$ are maximal independent sets in G_1 and G_2, respectively. Consequently, by hypothesis, $|\pi_{G_1}(I)| = \alpha(G_1)$, $|\pi_{G_2}(I)| = \alpha(G_2)$ and therefore $|I| = \alpha(G_1)\alpha(G_2)$.

On the other hand assume that $G_1 \vee G_2$ is well covered and suppose to the contrary that G_1 or G_2 is not well covered. Without loss of generality, we may assume that G_1 is not well covered. Then G_1 has two maximal independent sets of different cardinality, say I_1 and I_1'. Let I_2 be a maximal independent set in G_2. Then by Proposition 9, $I_1 \times I_2$ and $I_1' \times I_2$ are maximal independent sets of different cardinality in $G_1 \vee G_2$, a contradiction. This proves the necessity and completes the proof of the theorem. □

Corollary 7. *If graphs G_1 and G_2 are nonempty and one of them is without isolated vertices, then the disjunction $G_1 \vee G_2$ is very well covered if and only if exactly one of G_1 and G_2 is very well covered and the second is totally disconnected.*

The proof of Corollary 7 is similar to the proof of the second part of Corollary 5, so it will be omitted.

5. The conjunction of graphs

The conjunction $G_1 \wedge G_2$ of graphs G_1 and G_2 is the graph having vertex set $V(G_1 \wedge G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \wedge G_2$ are adjacent if $[v_1 u_1 \in E(G_1)]$ and $[v_2 u_2 \in E(G_2)]$.

In this section we study conditions for the well coveredness of conjunction graphs. We begin with a simple observation.

Proposition 11. *Let G_1 and G_2 be graphs without isolated vertices. If I_1 and I_2 are maximal independent sets in G_1 and G_2 respectively, then $I_1 \times V(G_2)$ and $V(G_1) \times I_2$ are maximal independent sets in $G_1 \wedge G_2$.*

Proof. Assume that I_1 is a maximal independent set in G_1, and G_2 has no isolated vertex. Then $N_{G_1}(v) \cap I_1 = \emptyset$ ($\neq \emptyset$, resp.) if $v \in I_1$ ($v \in V(G_1) - I_1$, resp.), and $N_{G_2}(u) \neq \emptyset$ for $u \in V(G_2)$. Thus $N_{G_1 \wedge G_2}((v,u)) \cap (I_1 \times V(G_2)) = (N_{G_1}(v) \cap I_1) \times N_{G_2}(u) = \emptyset$ ($\neq \emptyset$, resp.) if $(v,u) \in I_1 \times V(G_2)$ ($(v,u) \notin I_1 \times V(G_2)$, resp.). Hence $I_1 \times V(G_2)$ is a maximal independent set in $G_1 \wedge G_2$. Likewise, $V(G_2) \times I_2$ is a maximal independent set in $G_1 \wedge G_2$. □

The next theorem gives necessary conditions for the conjunction of two graphs to be well covered.

Theorem 4. *If G_1 and G_2 are graphs without isolated vertices and $G_1 \wedge G_2$ is a well covered graph, then*
(1) G_1 and G_2 are well covered and
(2) $\alpha(G_1)|V(G_2)| = \alpha(G_2)|V(G_1)|$.

Proof. Let I_i be any maximal independent set in G_i ($i = 1, 2$). By Proposition 11, $I_1 \times V(G_2)$ and $V(G_1) \times I_2$ are maximal independent sets in $G_1 \wedge G_2$. Since $G_1 \wedge G_2$ is well covered, the sets $I_1 \times V(G_2)$ and $V(G_1) \times I_2$

have the same cardinality and therefore $|I_1||V(G_2)| = |I_2||V(G_1)|$. This implies that $|I_i| = \alpha(G_i)$ $(i = 1, 2)$ and then the result follows. □

The implication in Theorem 4 cannot be reversed. This can be seen with the aid of the cycle C_5 of length 5. The graphs $G_1 = G_2 = C_5$ have the properties (1) and (2) of Theorem 4, and it is easy to check that $C_5 \wedge C_5$ is not a well covered graph. However, for very well covered graphs the converse of Theorem 4 is true. The following proposition is useful in proving that fact.

Proposition 12. *Let $v_1, \ldots, v_{2n}$ and $u_1, \ldots, u_{2m}$ be the vertices of graphs G_1 and G_2, respectively. If the edges $v_{2i-1}v_{2i}$ $(i = 1, \ldots, n)$ and $u_{2j-1}u_{2j}$ $(j = 1, \ldots, m)$ form a perfect matching in G_1 and G_2 respectively, then the edges $(v_{2i-1}, u_{2j-1})(v_{2i}, u_{2j})$ and $(v_{2i}, u_{2j-1})(v_{2i-1}, u_{2j})$ $(i = 1, \ldots, n; j = 1, \ldots, m)$ form a perfect matching of the graph $G_1 \wedge G_2$.*

Proof. The proof is immediate. □

The following theorem and its corollaries will establish where the class of very well covered conjunction graphs belongs in the world of well covered graphs.

Theorem 5. *Let G_1 and G_2 be graphs without isolated vertices. Then the graph $G_1 \wedge G_2$ is very well covered if and only if G_1 and G_2 are very well covered.*

Proof. Let $G_1 \wedge G_2$ be a very well covered graph. By Theorem 4, G_1 and G_2 are well covered. Clearly, G_1 and G_2 are very well covered; for if not, there exists a maximal independent set I_1 in G_1 (or I_2 in G_2) such that $|I_1| \neq |V(G_1)|/2$ (or $|I_2| \neq |V(G_2)|/2$) and then $|I_1 \times V(G_2)| = |I_1||V(G_2)| \neq |V(G_1)||V(G_2)|/2 = |V(G_1 \wedge G_2)|/2$ (or $|V(G_1) \times I_2| \neq |V(G_1 \wedge G_2)|/2$), which is impossible since $I_1 \times V(G_2)$ (or $V(G_1) \times I_2$) is a maximal independent set in $G_1 \wedge G_2$. Hence, G_1 and G_2 are very well covered if $G_1 \wedge G_2$ is very well covered.

Conversely, assume that the graphs G_1 and G_2 are very well covered. For $i = 1, 2$, let M_i be a perfect matching of G_i that has the properties (1) and (2) of Proposition 3 in G_i. Assume that $M_1 = \{v_{2i-1}v_{2i} : i = 1, \ldots, n\}$ and $M_2 = \{u_{2j-1}u_{2j} : j = 1, \ldots, m\}$. By Proposition 12, $M = \{(v_{2i-1}, u_{2j-1})(v_{2i}, u_{2j}), (v_{2i}, u_{2j-1})(v_{2i-1}, u_{2j}) : i = 1, \ldots, n$ and $j = 1, \ldots, m\}$ is a perfect matching of $G_1 \wedge G_2$ and in order to prove that $G_1 \wedge G_2$ is very well covered it is enough to show that M satisfies the conditions of Proposition 3 in $G_1 \wedge G_2$.

First we claim that no edge of M belongs to a triangle in $G_1 \wedge G_2$. Let (v, u) be any vertex of $G_1 \wedge G_2$. It follows from the property (1) of M_1 and

M_2 that $\{v_{2i-1}, v_{2i}\} \not\subseteq N_{G_1}(v)$ $(i = 1, \ldots, n)$ and $\{u_{2j-1}, u_{2j}\} \not\subseteq N_{G_2}(u)$ $(j = 1, \ldots, m)$. Hence, neither $\{(v_{2i-1}, u_{2j-1}), (v_{2i}, u_{2j})\} \subseteq N_{G_1 \wedge G_2}((v, u))$ nor $\{(v_{2i-1}, u_{2j}), (v_{2i}, u_{2j-1})\} \subseteq N_{G_1 \wedge G_2}((v, u))$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ and therefore no edge of M belongs to a triangle in $G_1 \wedge G_2$.

Finally, we claim that the matching M has the property (2) (of Proposition 3) in $G_1 \wedge G_2$. Since M_1 and M_2 have the property (2) in G_1 and G_2 respectively, every vertex $v \in N_{G_1}(v_{2i-1})$ is adjacent to every vertex $v' \in N_{G_1}(v_{2i})$ $(i = 1, \ldots, n)$ in G_1, and every vertex $u \in N_{G_2}(u_{2j-1})$ is adjacent to every vertex $u' \in N_{G_2}(u_{2j})$ $(j = 1, \ldots, m)$ in G_2. This combined with the definition of the conjunction of graphs implies that every vertex $(v, u) \in N_{G_1 \wedge G_2}((v_{2i-1}, u_{2j-1}))$ is adjacent to every vertex $(v', u') \in N_{G_1 \wedge G_2}((v_{2i}, u_{2j}))$, and every $(v, u') \in N_{G_1 \wedge G_2}((v_{2i-1}, u_{2j}))$ is adjacent to every $(v', u) \in N_{G_1 \wedge G_2}((v_{2i}, u_{2j-1}))$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$. This implies the desired claim and finishes the proof. □

Corollary 8. *Let G_1 and G_2 be graphs without isolated vertices. If at least one of G_1 and G_2 is very well covered, then the following statements are equivalent:*

(1) *$G_1 \wedge G_2$ is well covered,*
(2) *$G_1 \wedge G_2$ is very well covered,*
(3) *both G_1 and G_2 are very well covered.*

Proof. We have already proved that (2) and (3) are equivalent, and since (2) trivially implies (1), it suffices to prove that (1) implies (3). Let us assume that $G_1 \wedge G_2$ is well covered and G_2 is very well covered. By Theorem 4, G_1 is well covered and $\alpha(G_1)|V(G_2)| = \alpha(G_2)|V(G_1)| = |V(G_1)||V(G_2)|/2$. Thus $\alpha(G_1) = |V(G_1)|/2$ and hence G_1 is very well covered. □

There is an analogous result for bipartite graphs.

Corollary 9. *Let G_1 and G_2 be graphs without isolated vertices. If at least one of G_1 and G_2 is bipartite, then the following statements are equivalent:*

(1) *$G_1 \wedge G_2$ is well covered,*
(2) *$G_1 \wedge G_2$ is very well covered,*
(3) *G_1 and G_2 are very well covered.*

Proof. By Theorem 5, (2) and (3) are equivalent. Our assumption on G_1 and G_2 imply that $G_1 \wedge G_2$ is a bipartite graph without isolated vertices, so (1) and (2) are equivalent by Corollary 2. □

Above results give rise to some interesting observations. For example, if both G_1 and G_2 are graphs without isolated vertices, then: (a) $G_1 \wedge G_2$

is very well covered if and only if both G_1 and G_2 are very well covered; (b) $G_1 \wedge G_2$ is not well covered if exactly one of G_1 and G_2 is very well covered; (c) G_1 and G_2 are well covered but not very well covered if $G_1 \wedge G_2$ is well covered but not very well covered.

As we have already admitted, it is possible that $G_1 \wedge G_2$ is not well covered while G_1 and G_2 are well covered. It appears difficult to find general theorems for the cases where each of the graphs G_1, G_2 and $G_1 \wedge G_2$ is well but not very well covered.

We conclude this section with well covered conjunctions of complete graphs and cycles.

Proposition 13. *The conjunction $K_n \wedge K_m$ of complete graphs K_n and K_m $(n, m \geq 2)$ is a well covered graph if and only if $n = m$; $K_n \wedge K_m$ is a very well covered graph if and only if $n = m = 2$.*

Proof. The necessity of the first part follows immediately by applying Theorem 4 to $K_n \wedge K_m$. On the other hand, assume that I is a maximal independent set in $K_n \wedge K_n$ and $(v, u) \in I$. Since $N_{K_n \wedge K_m}((v, u)) \cap I = \emptyset$ and $N_{K_n \wedge K_m}((v, u)) = (V(K_n) - \{v\}) \times (V(K_n) - \{u\})$, the maximality of I implies that either $I = \{v\} \times V(K_n)$ or $I = V(K_n) \times \{u\}$. Therefore every maximal independent set in $K_n \wedge K_n$ has exactly n elements, so $K_n \wedge K_n$ is well covered. Since K_2 is the only complete very well covered graph, Theorem 5 implies that $K_n \wedge K_m$ is very well covered if and only if $n = m = 2$. □

Proposition 14. *The conjunction $C_n \wedge C_m$ of cycles C_n and C_m is a well covered graph if and only if $n = m = 3$ or 4; $C_n \wedge C_m$ is a very well covered graph if and only if $n = m = 4$.*

Proof. It is clear that if n and k are integers such that $n \geq 3$ and $\lceil n/3 \rceil \leq k \leq \lfloor n/2 \rfloor$, then in the cycle C_n there exists a maximal independent set of cardinality k. This implies that the cycle C_n is well covered if and only if $\lceil n/3 \rceil = \lfloor n/2 \rfloor$, that is, if and only if $n = 3, 4, 5$ or 7.

Certainly, C_4 is the only very well covered cycle. Therefore, by Theorem 5, $C_n \wedge C_m$ is very well covered if and only if $n = m = 4$. This proves the second part of the theorem. The well coveredness of $C_3 \wedge C_3$ follows from Proposition 13, since $C_3 = K_3$.

On the other hand, assume that the conjunction $C_n \wedge C_m$ is well covered. Then C_n and C_m are well covered by Theorem 4; hence, $n, m \in \{3, 4, 5, 7\}$. Again, by Theorem 4, none of the six graphs $C_3 \wedge C_4$, $C_3 \wedge C_5$, $C_3 \wedge C_7$, $C_4 \wedge C_5$, $C_4 \wedge C_7$, $C_5 \wedge C_7$ is well covered. One can verify that neither $C_5 \wedge C_5$ nor $C_7 \wedge C_7$ is well covered. Thus, $C_3 \wedge C_3$ and $C_4 \wedge C_4$ are the only well covered conjunctions of cycles. □

Corollary 10. *The conjunction $C_n \wedge K_m$ of a cycle C_n $(n \geq 3)$ and a complete graph K_m $(m \geq 2)$ is a well covered graph if and only if $n = m = 3$ or $n = 4$ and $m = 2$; $C_n \wedge K_m$ is very well covered if and only if $n = 4$ and $m = 2$.*

Proof. This follows at once from the above results. □

6. The cartesian product of graphs

The cartesian product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph having vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices (v_1, v_2) and (u_1, u_2) of $G_1 \times G_2$ are adjacent if either $[v_1u_1 \in E(G_1)$ and $v_2 = u_2]$ or $[v_1 = u_1$ and $v_2u_2 \in E(G_2)]$.

We are not able to give a complete description of the relationship between the well coveredness of graphs formed by the cartesian product and their factors. However, we consider some special cases which seem interesting. Since the cartesian product $nK_1 \times G$ is isomorphic to nG, we may only consider the cartesian product of graphs which are not totally disconnected. We begin by proving that for such graphs, the cycle C_4 $(= K_2 \times K_2)$ is the only connected, bipartite, (very) well covered cartesian product of graphs.

Theorem 6. *If G_1, G_2 are connected bipartite graphs and each of them is different from K_1, then $G_1 \times G_2$ is well covered if and only if $G_1 = G_2 = K_2$.*

Proof. If $G_1 = G_2 = K_2$, then $G_1 \times G_2 = C_4$ is well covered. Conversely, assume that $G_1 \times G_2$ is a well covered graph. Since G_1, G_2 are bipartite, $G_1 \times G_2$ is bipartite. Thus, according to Proposition 4, $G_1 \times G_2$ has a perfect matching M such that for every edge $(x, y)(x', y') \in M$, the subgraph induced by $N_{G_1 \times G_2}((x, y)) \cup N_{G_1 \times G_2}((x', y'))$ is a complete bipartite graph. We claim that $G_1 = G_2 = K_2$. For if not, without loss of generality, let G_1 be a counterexample and let v be a vertex of degree at least two in G_1. Then for any $v' \in N_{G_1}(v)$, $v'' \in N_{G_1}(v) - \{v'\}$, $u \in V(G_2)$ and $u' \in N_{G_2}(u)$, the vertices (v'', u) and (v', u') are not adjacent in $G_1 \times G_2$ but each of them is adjacent to exactly one of the vertices incident with the edge $(v, u)(v', u)$ (and $(v, u)(v, u')$). Therefore neither the subgraph induced by $N_{G_1 \times G_2}((v, u)) \cup N_{G_1 \times G_2}((v', u))$ (for any $v' \in N_{G_1}(v)$) nor the subgraph induced by $N_{G_1 \times G_2}((v, u)) \cup N_{G_1 \times G_2}((v, u'))$ (for any $u' \in N_{G_2}(u)$) is complete bipartite. This implies that no edge incident with the vertex (v, u) belongs to M, contrary to the hypothesis that M is a perfect matching in $G_1 \times G_2$. □

Corollary 11. *If G_1, G_2 are connected very well covered graphs, then $G_1 \times G_2$ is very well covered if and only if $G_1 = G_2 = K_2$.*

Proof. Assume that G_1, G_2, and $G_1 \times G_2$ are very well covered graphs. Let I_i be a maximum independent set in G_i ($i = 1, 2$). It is then clear that the set $I_1 \times I_2$ is independent in $G_1 \times G_2$. Let I be a maximum independent superset of $I_1 \times I_2$ in $G_1 \times G_2$. Obviously, $|I| = |V(G_1 \times G_2)|/2 = |(I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))|$. By the maximality of I_i, every vertex $v_i \in V(G_i) - I_i$ is adjacent to some vertex of I_i in G_i ($i = 1, 2$). Thus every vertex $(v_1, v_2) \in ((V(G_1) - I_1) \times I_2) \cup (I_1 \times (V(G_2) - I_2))$ is adjacent to some vertex of $I_1 \times I_2$. Hence I is a subset of $(I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))$ and so $I = (I_1 \times I_2) \cup ((V(G_1) - I_1) \times (V(G_2) - I_2))$. By the independence of $(V(G_1) - I_1) \times (V(G_2) - I_2)$ in $G_1 \times G_2$, the sets $V(G_1) - I_1$ and $V(G_2) - I_2$ are independent in G_1 and G_2, respectively. This implies the bipartition of G_1 and G_2. The rest follows from Theorem 6 and Corollary 2. □

For the cartesian product of complete graphs we have

Proposition 15. *For all positive integers n and m, $K_n \times K_m$ is well covered.*

Proof. Assume $n \leq m$ and $V(K_n \times K_m) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$. Let I be any maximal independent set in $K_n \times K_m$. In order to prove that $K_n \times K_m$ is well covered, we shall show that $\alpha(K_n \times K_m) = n$ and $|I| = n$. It is easy to see that $|I \cap (\{x_i\} \times V(K_m))| \leq 1$ and $|I \cap (V(K_n) \times \{y_j\})| \leq 1$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Hence $|I| \leq n$ and therefore $\alpha(K_n \times K_m) \leq n$. On the other hand, since the set $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ has n elements and is independent in $K_n \times K_m$, $\alpha(K_n \times K_m) = n$. It remains only to show that $|I| = n$. Suppose indirectly that $|I| < n$. Then the sets $V(K_n) - \pi_{K_n}(I)$ and $V(K_m) - \pi_{K_m}(I)$ are nonempty, and for every $x \in V(K_n) - \pi_{K_n}(I)$ and $y \in V(K_m) - \pi_{K_m}(I)$, the proper superset $I \cup \{(x, y)\}$ of I is independent in $K_n \times K_m$, a contradiction. □

We now study the well coveredness of the cartesian product of two cycles. Let C_n and C_m be two cycles with $V(C_n) = \{x_1, \ldots, x_n\}$, $V(C_m) = \{y_1, \ldots, y_m\}$, $E(C_n) = \{x_i x_{i+1} : i = 1, \ldots, n-1\} \cup \{x_1 x_n\}$, and $E(C_m) = \{y_j y_{j+1} : j = 1, \ldots, m-1\} \cup \{y_1 y_m\}$. For the cartesian product $C_n \times C_m$ of the cycles C_n and C_m, we define $I_{n,m}$ to be the set of these vertices (x_i, y_j) of $C_n \times C_m$ that $i = 1, \ldots, 2\lfloor n/2 \rfloor$, $j = 1, \ldots, 2\lfloor m/2 \rfloor$, and $i + j$ is an even integer. Put $I^*_{n,m} = I_{n,m} \cup \{(x_n, y_m)\}$ if both n and m are odd, while $I^*_{n,m} = I_{n,m}$ in other cases. It is easy to check the following properties of the set $I^*_{n,m}$ in $C_n \times C_m$.

Proposition 16. *For all integers $n, m \geq 3$, the set $I^*_{n,m}$ is a maximal* *independent set in $C_n \times C_m$; in addition, $|I^*_{n,m}| = 2\lfloor n/2 \rfloor \lfloor m/2 \rfloor + 1$ if both n and m are odd, while $|I^*_{n,m}| = 2\lfloor n/2 \rfloor \lfloor m/2 \rfloor$ in other cases.* □

Proposition 17. *For every integer $m \geq 3$, the cartesian product $C_3 \times C_m$ is well covered.*

Proof. Let I be any maximal independent set in $C_3 \times C_m$. As in the proof of Proposition 15, it is enough to show that $\alpha(C_3 \times C_m) = m$ and $|I| = m$. Since $|I \cap (V(C_3) \times \{y_j\})| \leq 1$ for $j = 1, \ldots, m$, so $|I| \leq m$ and $\alpha(C_3 \times C_m) \leq m$. On the other hand, by Proposition 16, the set $I^*_{3,m}$ is independent in $C_3 \times C_m$ and $|I^*_{3,m}| = m$. Hence $\alpha(C_3 \times C_m) = m$. We now claim that $|I| = m$. For if not, then $|I| < m$ and therefore $I \cap (V(C_3) \times \{y_j\}) = \emptyset$ for some $j \in \{1, \ldots, m\}$, say $j = 2$. The maximality of I implies that $N_{C_3 \times C_m}((x_i, y_2)) \cap I \neq \emptyset$ for each $i = 1, 2, 3$. From this and from the structure of $C_3 \times C_m$ it follows that the subset $\bigcup_{i=1}^{3} N_{C_3 \times C_m}((x_i, y_2)) \cap I$ of $V(C_3) \times \{y_1, y_3\}$ has at least three vertices. Hence, $I \cap (V(C_3) \times \{y_1\})$ or $I \cap (V(C_3) \times \{y_3\})$ has at least two vertices, a contradiction. □

Proposition 18. *For all integers $n, m \geq 4$, the cartesian product $C_n \times C_m$ is not well covered.*

Proof. The result follows from Theorem 6 if both n and m are even. Thus it suffices to show that $C_n \times C_m$ is not well covered if n or m is odd. We consider two cases.

Case 1. n and m are odd. By Proposition 16, the set $I^*_{n,m}$ is a maximal independent set in $C_n \times C_m$. On the other hand, it is easy to check that the set

$$J_{n,m} = (I^*_{n,m} - \{(x_1, y_1), (x_1, y_3), (x_2, y_2)\}) \cup \{(x_1, y_2), (x_n, y_3)\}$$

is also a maximal independent set in $C_n \times C_m$. Since $|I^*_{n,m}| \neq |J_{n,m}|$, $C_n \times C_m$ is not well covered.

Case 2. Exactly one of n and m is odd. Since $C_n \times C_m$ is isomorphic to $C_m \times C_n$, we may assume that m is odd. An easy verification shows that the set

$$N_{n,m} = (I^*_{n,m} - \{(x_1, y_1), (x_1, y_3), (x_2, y_2), (x_n, y_2)\}) \cup \{(x_1, y_2), (x_1, y_m)\}$$

is a maximal independent set in $C_n \times C_m$. Since $N_{n,m}$ is smaller than $I^*_{n,m}$, $C_n \times C_m$ is not a well covered graph. This completes the proof. □

We summarize the above results in the following corollary.

Corollary 12. *The cartesian product $C_n \times C_m$ of cycles C_n and C_m is well covered if and only if $n = 3$ or $m = 3$.*

Staples [14] has observed that $K_2 \times G$ is a W_{n-1} graph if G is a W_n graph ($n \geq 2$). This implies that the cartesian products $K_2 \times C_3$ and $K_2 \times C_5$ are well covered. We conclude this section with the observation that the cycles C_3, C_5, and the graph $K_1 + (K_2 \cup nK_1)$ (for $n \geq 1$) are the only unicyclic graphs G for which the cartesian product $K_2 \times G$ is well covered. We begin by proving the following useful proposition.

Proposition 19. *Suppose that a connected graph G contains a bridge v_1v_2 such that v_1 is not an end vertex in G and the set $N_G(v_1)$ is independent. Then the cartesian product $K_2 \times G$ is not well covered.*

Proof. Let $V(K_2) = \{a, b\}$, $U = N_G(v_1) - \{v_2\}$, and let $G_i = G'_i - v_i$, where G'_i is the connected component of $G - v_1v_2$ that contains the vertex v_i ($i = 1, 2$). Let S be a maximal independent set in $G_1 - U$, let T be a maximal independent superset of U in $G_1 - S$, and let W be a maximal independent set in $K_2 \times G - N_{K_2 \times G}(\{b\} \times (V(G'_1) \cup \{v_2\}))$ $(= K_2 \times G_2 - N_{K_2 \times G}((b, v_2)))$. Observe that $W \cup (\{a\} \times S) \cup (\{b\} \times T) \cup \{(a, v_1), (b, v_2)\}$ and $W \cup (\{b\} \times S) \cup (\{a\} \times T) \cup \{(b, v_2)\}$ are maximal independent sets of different cardinality in $K_2 \times G$. Thus $K_2 \times G$ is not well covered. □

Proposition 20. *If G is a connected unicyclic graph, then the cartesian product $K_2 \times G$ is well covered if and only if $G = C_3$, $G = C_5$ or $G = K_1 + (K_2 \cup nK_1)$ for some positive integer n.*

Proof. We consider two cases.
Case 1. G is a cycle, $G = C_n$. Since $K_2 \times C_n$ is a cubic, planar, 3-connected graph, it follows from Proposition 5 that $K_2 \times C_n$ is well covered if and only if $n = 3$ or $n = 5$.
Case 2. G is not a cycle. Let C be the unique cycle of G, $V(K_2) = \{a, b\}$, and assume that $K_2 \times G$ is a well covered graph. Then it easily follows from Proposition 19 that C is a cycle of length three and each end vertex of G is adjacent to a vertex of C. Let $V(C) = \{v_1, v_2, v_3\}$ be the vertex set of C and denote $p_i = |N_G(v_i) - V(C)|$ ($i = 1, 2, 3$). We may assume that $p_1 \geq p_2 \geq p_3$. We claim that $p_2 = p_3 = 0$. For if not, then $p_2 > 0$ and the sets $I = (\{a\} \times (V(G) - V(C))) \cup \{(b, v_3)\}$ and $I' = (\{a\} \times (N_G(\{v_1, v_2\}) - \{v_1, v_2\})) \cup (\{b\} \times (N_G(v_3) - \{v_2\}))$ are maximal independent sets of different cardinality in $K_2 \times G$, a contradiction. Hence, $G = K_1 + (K_2 \cup nK_1)$ (for $n = p_1$) and it is easy to check that $K_2 \times (K_1 + (K_2 \cup nK_1))$ is well covered. □

7. Conclusion

There are a number of questions raised by the results presented here.

Unanswered in this paper is the problem of finding a "nice" characterization of well covered graphs G_1 and G_2 for which $G_1 \wedge G_2$ is well covered. The results of Section 6 indicate the difficulty in finding a characterization of graphs G_1 and G_2 for which $G_1 \times G_2$ is well covered. Finally, is it possible to find a pair of graphs, G_1 and G_2, for which $G_1 \times G_2$ is well covered but both G_1 and G_2 are not well covered?

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