On the complexity of coloring areflexive relations

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Abstract. Let ρ be an h-ary areflexive relation. We study the complexity of finding a strong h-coloring for ρ, which is defined in the same way for h-uniform hypergraphs. In particular we reduce the H-coloring problem for graphs to this problem, where H is a graph depending on ρ. We also discuss the links of this problem with the problem of finding a completeness criterion for finite algebras.

1. Motivation and Preliminaries

Let A be a finite nonempty set. For a positive integer n a *partial* n-ary *operation* on A is a map $f: D_f \to A$ where $D_f \subseteq A^n$. Let $\mathbf{P}^{(n)}$ denote the set of all partial n-ary operations on A and let $\mathbf{P} = \bigcup_{n\geq 1} \mathbf{P}^{(n)}$. A subset $C \subseteq \mathbf{P}$ is a *partial clone* on A if C contains all projections and is closed under arbitrary superpositions (for more details see [10]). Let $F \subseteq \mathbf{P}$. We denote by $\bar{F}$ the partial clone generated by F (i.e. the smallest partial clone containing F) and we say that the set F is *complete* (or primal) if $\bar{F} = \mathbf{P}$.

Finding a general completeness criterion is a fundamental problem in Universal Algebra. Since every proper partial clone is contained in a maximal one ([8]) such a criterion may be based on the knowledge of all maximal partial clones (i.e. the dual atoms in the lattice of partial clones, recall that the partial clones on A, ordered by inclusion, form an algebraic lattice). The maximal partial clones are determined in [4] for $|A| = 2$. The following concept is used for the general case:

Let ρ be an h-ary relation on A (i.e. $\rho \subseteq A^h$) and let f be a partial n-ary operation with domain $D_f \subseteq A^n$. We say that f *preserves* ρ if for every $h \times n$ matrix $A = [A_{ij}]$ whose columns $A_{*j} \in \rho$ ($j = 1, \ldots, n$) and whose rows $A_{i*} \in D_f$ ($i = 1, \ldots, h$) we have $(f(A_{1*}), \ldots, f(A_{h*})) \in \rho$. Let $\mathrm{Pol}\,\rho = \{f \in \mathbf{P} : f$ preserves $\rho\}$ and it is well known that $\mathrm{Pol}\,\rho$ is a partial clone.

Example: Let $0 \in A$. Then

$$\mathrm{Pol}\,\{0\} = \bigcup_{n\geq 1} \{f \in \mathbf{P}^{(n)} : (0,0,\ldots,0) \in D_f \Rightarrow f(0,\ldots,0) = 0\}.$$

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Now for $|A| \geq 3$, the maximal partial clones are described in [9]. With one exception, each of them is determined by an h-ary relation on A (where $1 \leq h \leq |A|$) admitting a special coloring. The study has been separated into 3 cases, one of them deals with areflexive h-ary relations having a kind of symmetry:

Definition: The h-ary relation ρ on A is said to be *areflexive* if for every $(x_0, \ldots, x_{h-1}) \in \rho$ and all $0 \leq i < j \leq h-1$, we have $x_i \neq x_j$.

Let S_h be the symmetric group on $\underline{h} = \{0, \ldots, h-1\}$ and let $\pi \in S_h$. For an h-ary relation ρ let

$$\rho^{(\pi)} := \{(x_{\pi(0)}, \ldots, x_{\pi(h-1)}) : (x_0, \ldots, x_{h-1}) \in \rho\}.$$

Definition: Let ρ be an h-ary areflexive relation on A and let $\pi \in S_h$. We say that ρ is *symmetric* with respect to π (respectively *asymmetric* with respect to π) if $\rho = \rho^{(\pi)}$ (resp. $\rho \cap \rho^{(\pi)} = \emptyset$).

Assume that there exists a subgroup G_ρ of S_h for which the h-ary areflexive relation ρ is symmetric with respect to each $\pi \in G_\rho$ and asymmetric with respect to each $\alpha \in S_h \backslash G_\rho$, ($G_\rho$ is called the symmetry group of ρ).

Definitions:

1) The *model* of ρ is the h-ary relation

$$M_\rho := \{(\pi(0), \ldots, \pi(h-1)) : \pi \in G_\rho\}$$

on the set $\underline{h} = \{0, \ldots, h-1\}$.

2) A *strong h-coloring* of ρ is a map

$$\phi : A \to \underline{h}$$

which is a relational homomorphism from ρ to M_ρ (i.e. for every $(x_0, \ldots, x_{h-1}) \in \rho$, $(\phi(x_0), \ldots, \phi(x_{h-1})) \in M_\rho$). Note that a strong h-coloring is surjective.

Example: Let $A = \{0, 1, 2, 3, 4\}$ and $\rho = \{(0,1,2), (1,0,2), (1,3,4), (3,1,4), (0,1,4), (1,0,4)\}$. Then $G_\rho = \{I_3, (01)\}$ (where I_3 is the identity permutation on $\underline{3} = \{0, 1, 2\}$) and $M_\rho = \{(0,1,2), (1,0,2)\}$. Define $\phi : A \to \{0, 1, 2\}$ by $\phi(0) = \phi(3) = 0$, $\phi(1) = 1$ and $\phi(2) = \phi(4) = 2$, then clearly ϕ is a strong 3-coloring of ρ.

In [10] it is shown that the h-ary areflexive relations which determine maximal partial clones on A are exactly those which have a symmetry group G and admit a strong h-coloring. The question arises naturally is under which conditions (especially concerning the group G_ρ) does there exist an efficient algorithm deciding whether an h-ary areflexive relation ρ has a strong h-coloring and under which conditions is the problem NP-complete.

In the sequel we assume $A = \underline{k} = \{0, \ldots, k-1\}$.

2. The Strong Coloring Problem (SCP)

Instance. $h \geq 2, k \geq 2, G$ a subgroup of S_h, ρ an h-ary areflexive relation on $\underline{k} = \{0, \ldots, k-1\}$, symmetric with respect to each $\pi \in G$ and asymmetric with respect to each $\alpha \in S_h \backslash G$.

Question. Does ρ have a strong h-coloring?

Examples:

1) If $H = (V, E)$ is a simple graph, then $G_H = S_2$, $M_H = \{(0,1),(1,0)\}$ and H has a strong 2-coloring if and only if H is bipartite.
2) If $D = (V, E)$ is a digraph with no cycles of length 2, then $G_D = \{I_2\}$, $M_D = \{(0,1)\}$ and D has a strong 2-coloring if and only if D has no two consecutive arcs (i.e. if $(x, y) \in E$ then $(y, z) \notin E \forall z \in V$).

Therefore the case $h = 2$ is trivial. Note that in general the SCP is in NP. For a given subgroup G_1 of S_h we denote by SCP_{G_1} the restriction of the SCP to those instances (h, k, ρ, G) such that $G = G_1$. We want to determine the complexity of the SCP.

Definition: An h-ary areflexive relation is *totally symmetric* if it is symmetric with respect to every $\pi \in S_h$.

Now from [2] we deduce

Proposition 1. *Determine whether a tenary areflexive totally symmetric relation has a strong 3-coloring is NP-complete. (i.e. the SCP_{S_3} is NP-complete).* ∎

Suppose there is a partition $B_1, \ldots, B_s$ of $\underline{h}$ such that $\pi(B_i) \subseteq B_i$ for every $\pi \in G$ and every $i = 1, \ldots, s$. Put

$$G_i := G|_{B_i} = \{\pi|_{B_i} : \pi \in G\}.$$

Proposition 2. *There is a polynomial reduction from the SCP_{G_i} to the SCP_G.*

Hence

1) If the SCP_G is in P, then so are the SCP_{G_i} $(i = 1, \ldots, s)$.
2) If for some $i = 1, \ldots, s$, the SCP_{G_i} is NP-complete, then so is the SCP_G.

Proof: It is enough to prove the result for $s = 2$. We rearrange the elements of $\underline{h} = \{0, \ldots, h-1\}$ such that $B_1 = \{0, \ldots, t-1\}$, where $0 < t < h$.

Let λ be a t-ary areflexive relation on $\underline{k}$ such that $G_\lambda = G_1$. Hence there are n different t-tuples

$$\underset{\sim}{x}^1 = (x_0^1, \ldots, x_{t-1}^1), \ldots, \underset{\sim}{x}^n = (x_0^n, \ldots, x_{t-1}^n)$$

such that

$$\lambda = \bigcup_{i=1}^{n} \{(x_{\pi(0)}^i, \ldots, x_{\pi(t-1)}^i) : \pi \in G_1\}.$$

For each $\underset{\sim}{x}^i$ put

$$\underset{\sim}{y}^i := (x_0^i, \ldots, x_{t-1}^i, a_t^i, \ldots, a_{h-1}^i)$$

where $a_t^\ell \neq a_j^i$ whenever $\ell \neq i$ or $t \neq j$.

Let

$$\rho := \bigcup_{i=1}^{n} \left\{ (x_{\pi(0)}^i, \ldots, x_{\pi(t-1)}^i, a_{\pi(t)}^i, \ldots, a_{\pi(h-1)}^i) : \pi \in G \right\}.$$

Clearly ρ is an h-ary areflexive relation on the set

$$B = \underline{k} \cup \left\{ a_j^i : 1 \leq i \leq n, t \leq j \leq h-1 \right\}$$

whose group of symmetry is G.

We show that ρ is strongly h-colorable if and only if λ is strongly t-colorable.
($\Leftarrow$) Let $\phi: \underline{k} \to \underline{t}$ be a strong t-coloring of λ. We extend ϕ to a mapping ψ

$$\psi : B \to \underline{h}$$

by the following way:
Let $\underset{\sim}{y}^i = (x_0^i, \ldots, x_{t-1}^i, a_t^i, \ldots, a_{h-1}^i) \in \rho$. Since ϕ is a strong t-coloring of λ we have

$$(\phi(x_0^i), \ldots, \phi(x_{t-1}^i)) = (\alpha(0), \ldots, \alpha(t-1)) \text{ for some } \alpha \in G_1.$$

Let $\beta \in G$ be such that $\beta|_{\underline{t}} = \alpha$ and put

$$\psi(a_j^i) := \beta(j) \text{ for all } 1 \leq i \leq n \text{ and } t \leq j \leq h-1.$$

We show that ψ is a strong h-coloring of ρ. Indeed let $\underset{\sim}{y} = (x_{\pi(0)}^i, \ldots, x_{\pi(t-1)}^i, a_{\pi(t)}^i, \ldots, a_{\pi(h-1)}^i) \in \rho$ where $\pi \in G$. Then $\psi(\pi_{\pi(s)}^i) = \alpha \circ \pi(s)$ for $s = 0, \ldots, t-1$ and $\alpha \in G_1$ is such that $(\phi(x_0^i), \ldots, \phi(x_{t-1}^i)) = (\alpha(0), \ldots, \alpha(t-1))$.

Now $\psi(a_j^i) = \beta(j)$ implies $\psi(a_{\pi(j)}^i) = \beta \circ \pi(j)$ for all $1 \leq i \leq n$ and $t \leq j \leq h-1$.

Since β is such that $\beta|_{\underline{t}} = \alpha$, we have

$$\begin{aligned}(\psi(x_{\pi(0)}^i), \ldots, \psi(x_{\pi(t-1)}^i), \psi(a_{\pi(t)}^i), \ldots, \psi(a_{\pi(h-1)}^i)&\\ = (\alpha \circ \pi(0), \ldots, \alpha \circ \pi(t-1), \beta \circ \pi(t), \ldots, \beta \circ \pi(h-1))&\\ = (\beta \circ \pi(0), \ldots, \beta \circ \pi(h-1)),&\end{aligned}$$

where $\beta \circ \pi \in G$. Hence ψ is a relational homomorphism from ρ to M_ρ and thus ρ is strongly h-colorable.

($\Rightarrow$) If $\psi: a \to \underline{h}$ is a relational homomorphism from ρ to M_ρ, then clearly $\phi = \psi|_{\underline{k}}$ is a relational homomorphism from λ to M_λ. ∎

Remark: The converse of Proposition 2 seems to be false, but a counter example is not known. It will be interesting to find a subgroup G of S_h such that the SCP_G is NP-complete and for which there is a partition B_1, B_2 of $\underline{h}$ such that the SCP_{G_i} is in P for $i = 1,2$ (clearly G should not be the direct product of G_1 and G_2. Since in such a case, if both SCP_{G_i} are in P, $i = 1,2$, then so is SCP_G).

Definition: A subgroup G of S_h is *sharply transitive* (or regular) if for all $i, j \in \underline{h}$, $i \neq j$, there is a unique $\pi \in G$ such that $\pi(i) = j$.

Theorem 3. *If G is a sharply transitive subgroup of S_h then the SCP_{G_i} is in P.*

Proof: Let G be a sharply transitive subgroup of S_h. We give an efficient algorithm to decide whether an h-ary areflexive relation ρ such that $G_\rho = G$ has a strong h-coloring. In fact our algorithm deals with a component of ρ (a component of an h-ary relation is defined in a similar way as for a hypergraph).

Fix $(x_0, \ldots, x_{h-1}) \in \rho$. We define by induction on $n \geq 1$ a family of subrelations ρ_n of ρ and subsets Γ_n of $\underline{k}$, as follows:

Put $\rho_1 := \{(x_0, \ldots, x_{h-1})\}$ and $\Gamma_1 := \{x_0, \ldots, x_{h-1}\}$. For $n > 1$, put $\rho_n := \{(y_0, \ldots, y_{h-1}) \in \rho : y_i \in \Gamma_{n-1}$ for some $i = 0, \ldots, h-1\}$ and let Γ_n be the set covered by ρ_n (or the set of vertices of ρ_n). Clearly the chain $\rho_1 \subset \rho_2 \subset \cdots \subset \rho_i \subset \rho_{i+1} \subset \ldots$ is finite. Let $t \geq 1$ be such that ρ_t is a component of ρ and let Γ_t be the subset of $\underline{k}$ covered by the component ρ_t. We define a correspondance $\alpha: \Gamma_t \to \underline{h}$ and we show that α is a mapping if and only if ρ_t is strongly h-colorable.

Let α be defined on Γ_1 by $\alpha(x_i) = i, i = 0, \ldots, h-1$. Now assume that α is defined on Γ_{n-1} and let $(y_0, \ldots, y_{h-1}) \in \rho_n$. Then $y_s \in \Gamma_{n-1}$ for some $s \in \underline{h}$.

Let $\alpha(y_s) r \in \underline{h}$. There is a unique $\pi \in G$ such that $\pi(s) = r$. Put $\alpha(y_i) := \pi(i)$ for $i = 0, \ldots, h-1$.

Claim: $\alpha: \Gamma_t \to \underline{h}$ is a mapping if and only if ρ_t is strongly h-colorable.

Proof (of the claim):

($\Rightarrow$) Assume α is a mapping, hence each $x \in \Gamma_t$ receives a unique color and by construction it is a relational homomorphism from ρ_t to M_ρ, thus ρ has a strong h-coloring.

($\Leftarrow$) Let $\phi: \Gamma_t \to \underline{h}$ be a strong h-coloring of ρ_t and $\pi \in G$ be such that $(\phi(x_0), \ldots, \phi(x_{h-1})) = (\pi(0), \ldots, \pi(h-1))$. Then clearly $\psi := \pi^{-1} \circ \phi$ is a strong h-coloring of ρ_t such that $(\psi(x_0), \ldots, \psi(x_{h-1})) = (0, \ldots, h-1)$. We show by induction on $n \geq 1$ that $\alpha|_{\Gamma_n} = \psi|_{\Gamma_n}$.

Clearly $\alpha|_{\Gamma_0} = \psi|_{\Gamma_0}$. Assume that $\alpha|\Gamma_{n-1} = \psi|\Gamma_{n-1}$ for some $1 \leq n \leq t-1$ and let $(y_0, \ldots, y_{h-1}) \in \rho_n$. Since ϕ is a strong h-coloring of ρ_t we have

$$(\phi(y_0), \ldots, \phi(y_{h-1})) = (\pi_1(0), \ldots, \pi_1(h-1)) \text{ for some } \pi_1 \in G,$$

hence $(\psi(y_0),\ldots,\psi(y_{h-1})) = (\pi^{-1}\circ\pi_1(0),\ldots,\pi^{-1}\circ\pi_1(h-1))$. Now $(y_0,\ldots,y_{h-1}) \in \rho_n$. Therefore there exists $j \in \underline{h} = \{0,\ldots,h-1\}$ such that $y_j \in \Gamma_{n-1}$. Hence $\alpha(y_j) = \pi^{-1}\circ\pi_1(j) \in \underline{h}$. By construction of α we have $\alpha(y_i) = \pi^{-1}\circ\pi_1(i) = \psi(y_i)$ for all $i = 0,\ldots,h-1$. Hence $\alpha|\Gamma_n = \psi|\Gamma_n$.

We have shown that $\alpha = \psi$ and as ψ is a mapping then so is α. ∎

In the following we show that, for a suitable orbit H of G on $\underline{h}\times\underline{h}$, the H-coloring problem for digraphs is reducible to the SCP_G (actually if H is an undirected graph, then the same holds for the H-coloring problem for graphs).

Let H be a fixed digraph (respectively an undirected graph).

Definition: An *H-coloring* of a digraph D (resp. of a graph D) is a mapping $\phi: V(D) \to V(H)$ such that $(\phi(x),\phi(y))$ is an edge of H whenever (x,y) is an edge of D.

The H-coloring problem is described as follows.

Instance. A digraph D (resp. a graph D).

Question. Does there exist an H-coloring of D?

The complexity of the H-coloring problem has been studied by several authors (clearly the problem is in NP for any fixed H). In particular Hell and Nesetril proved the following result for undirected graphs.

Theorem 4 [11]. *The H-coloring problem is in P if H is bipartite and is NP-complete if H is not bipartite.* ∎

At the present time, no similar results are known for the directed case. However cases such as tournaments and semicomplete digraphs are solved:

Theorem 5 [6]. *Let T be a semicomplete digraph.*

1) *If T has more than one directed cycle, then the T-coloring problem for digraphs is NP-complete.*
2) *If T is acyclic or has a unique cycle, then the T coloring problem is in P.*

Note that the same holds for a tournament T.

Also results about the effect of two cycles on the complexity of the H-coloring problem (for digraphs) are shown in [5]. We return to the SCP_G.

Definition: Let G be a subgroup of S_h and $0 \le i \le j \le h-1$. Put

$$R_{ij} := \{(\pi(i),\pi(j)) : \pi \in G\}$$

Clearly R_{ij} is an orbit of G on $\underline{h}\times\underline{h}$.

Theorem 6. *If for some $0 \le i \le j \le h-1$, the R_{ij}-coloring problem for digraphs is NP-complete, then so is the SCP_G.*

Proof: Let $D = (V(D),E(D))$ be a digraph with $E(D) = \{\vec{u}_1,\ldots,\vec{u}_n\}$ where $\vec{u}_t = (x_t,y_t)$ and let $M = \{(\pi(0),\ldots,\pi(h-1)) : \pi \in G\}$. On the top of each edge $\vec{u}_t$ we construct a copy of M by the following way:

For $t = 1, \dots, n$ let

$$\underset{\sim}{x}^t := (x_0^t, \dots, x_{h-1}^t) \text{ where } (x_i^t, x_j^t) = (x_t, y_t) = \vec{u}_t,$$

$x_s^t \notin V(D)$ for $s \notin \{i, j\}$ and $x_s^t = s_{s'}^{t'} \Rightarrow x_s^t \in V(D)$ (hence if two edges $\vec{u}_t$ and $\vec{u}_{t'}$ have no common vertices, then the copies of M constructed on $\vec{u}_t$ and $\vec{u}_{t'}$ are disjoint). Put

$$M^t := \{(x_{\pi(0)}^t, \dots, x_{\pi(h-1)}^t) : \pi \in G\}$$

and

$$\rho := \bigcup_{t=1}^{n} M^t.$$

Clearly ρ is an h-ary areflexive relation whose group of symmetry is G (hence $M_\rho = \{(\pi(0), \dots, \pi(h-1) : \pi \in G\}$.

Claim: D is R_{ij}-colorable if and only if ρ is strongly h-colorable.

Proof (of the claim): Let A be the set covered by ρ.

($\Rightarrow$) Let $\phi: V(D) \to V(R_{ij})$ be an R_{ij}-coloring of D. We extend ϕ to $\psi: A \to \underline{h}$ in the following way: Let $\underset{\sim}{x}^t = (x_0^t, \dots, x_i^t, \dots, x_j^t, \dots, x_{h-1}^t)$. Since $(x_i^t, x_j^t) = (x_t, y_t) \in E(D)$ we have that $(\phi(x_i^t), \phi(x_j^t)) \in R_{ij}$. Choose a $\pi_1 \in G$ such that $(\phi(x_i^t), \phi(x_j^t)) = (\pi_1(i), \pi_1(j))$ (note that π_1 is not necessarily unique) and define $\psi(x_s^t) := \pi_1(s)$ for $s = 0, \dots, h-1$. Now for $\underset{\sim}{x} = (x_{\pi(0)}^t, \dots, x_{\pi(h-1)}^t) \in \rho$ put $\psi(x_{\pi(s)}^t) := \pi \circ \pi_1(s)$ for $s = 0, \dots, h-1$. Here $(\psi(x_{\pi(0)}^t), \dots, \psi(x_{\pi(h-1)}^t)) = (\pi \circ \pi_1(0), \dots, \pi \circ \pi_1(h-1)) \in M_\rho$ since $\pi, \pi_1 \in G$ (hence $\pi \circ \pi_1 \in G$). Thus ψ is a strong h-coloring of ρ proving the first part of the Theorem.

($\Leftarrow$) Let $\psi: A \to \underline{h}$ be a strong h-coloring of ρ and let $\vec{u}_t = (x_t, y_t) \in E(D)$. Hence $(x_0^t, \dots, x_i^t, \dots, x_j^t, \dots, x_{h-1}^t) \in \rho$ where $x_i^t = x_t, x_j^t = y_t$ and $(\psi(x_0^t), \dots, \psi(x_{h-1}^t)) = (\pi(0), \dots, \pi(h-1))$ for some $\pi \in G$. Then $(\psi(x_t), \psi(y_t)) = (\psi(x_i^t), \psi(x_j^t)) = (\pi(i), \pi(j)) \in R_{ij}$. Therefore the restriction of ψ to $V(D)$ is an R_{ij}-coloring of D. ∎

Note that Theorem 6 holds if the orbit R_{ij} is an undirected graph.

Corollary 1. *If for some* $0 \leq i < j \leq h-1$, R_{ij} *is a nonbipartite undirected graph, then the* SCP_{G_i} *NP-complete.* ∎

Applying Corollary 1 to $G = S_h$ with $h \geq 3$, we obtain the following known result:

Corollary 2. [2]. *Determining whether a partial* $(v, k, 1)$ *system has a strong* k*-coloring is NP-complete for any fixed* $k \geq 3$. ∎

We can also deduce the following:

Corollary 3. *Let G be a subgroup of S_h.*

1) *If G is n-fold transitive for $n > 1$, then the SCP_G is NP-complete.*
2) *If G is the alternating group A_h, then the SCP_G is NP-complete for $h{>}4$.*

∎

Proof: The first part follows from the fact that if G is n-fold transitive on $\underline{h}$ with $n > 1$, then $R_{ij} = K_h$ for all $0 \leq i < j \leq h - 1$.

Now the second part follows from the first since A_h is $(h-2)$-fold transitive on $\underline{h}$ for $h \geq 3$ (see [13] Theorem 9.7). ∎

Moreover combining Theorems 5 and 6 we can deduce many results such as:

Corollary 4. *Let $\bar{G}$ be the group of degree 10 constructed in the remark preceding Theorem 31.1 of [13]. Then the $SCP_{\bar{G}}$ is NP-complete.*

Proof: By a direct verification, R_{12} is a tournament with more than one cycle. ∎

Remark: $\bar{G}$ is non-doubly transitive. Actually $p = 5$ is the only prime for which a non-doubly transitive group of degree $2p$ is known to exist (see [13] p. 94).

By way of conclusion, we raise the following problem: Let $G < G'$ be two subgroups of S_h. Assume that the SCP_G is NP-complete. Is it true that the same holds for G'?

If so, then we will have a "cut" in the lattice of subgroups of S_h where the subgroups G for which the SCP_G is in P are below those for which the SCP is NP-complete (since the SCP_{I_h} is in P and the SCP_{S_h} is NP-complete). Of course if it were possible to construct a group as described in the remark following Proposition 2, then the answer to our problem would be no.

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