RECTANGULAR DESIGNS WITH VARYING REPLICATES

Kishore Sinha
Department of Statistics
Birsa Agricultural University
Ranchi - 834006, India

A. D. Das
Department of Statistics
Bidhan Chandra Krishi Vishwavidyalaya
Cooch-Behar - 736101, India

Sanpei Kageyama
Department of Mathematics
Hiroshima University
Shinonome, Hiroshima 734, Japan

Abstract. The concept of rectangular designs with varying replicates is introduced. A class of such designs is constructed with an example.

1. Introduction.

The balanced bipartite block designs studied among others by Corsten (1962), and Kageyama and Sinha (1988) may be considered as an extension of group divisible designs which are 2-associate partially balanced incomplete block (PBIB) designs. Here, the concept of rectangular designs with varying replicates is newly introduced by extending the concept of the usual rectangular designs which are 3-associate PBIB designs. These designs may be useful for $m \times n$ factorial experiments with varying replicates.

A rectangular design with varying replicates is an incomplete block design with $v(= mn)$ treatments arranged into b blocks each of size $k(< v)$ such that

(i) the mn treatments can be arranged into m rows and n columns;
(ii) each treatment in the ith column is replicated r_i times for $i = 1, 2, \ldots, n$;
(iii) with each treatment in the ith column $(i = 1, 2, \ldots, n)$, the treatments (a) in the same row occur λ_1 times (that is, they are called first associates), (b) in the same column occur $\lambda_{i(2)}$ times (that is, they are called second associates), (c) others occur λ_3 times (that is, they are called third associates).

Let n_i be the number of the ith associates of any treatment. Then $n_1 = n - 1$, $n_2 = m - 1$, and $n_3 = (m-1)(n-1)$. The following conditions hold among the parameters of a rectangular design with varying replicates:

$$v = mn, \sum_{j=1}^{3} n_j = v - 1, m \sum_{i=1}^{n} r_i = bk, n_1\lambda_1 + n_2\lambda_{i(2)} + n_3\lambda_3 = r_i(k-1).$$

2. Construction.

Theorem. *There exists a class of rectangular designs with varying replicates and having parameters*

$$v = 2(4t+1) = b,\ r_1 = 2(2t+1),\ r_2 = 4t,\ k = 4t+1,$$
$$\lambda_1 = 4t,\ \lambda_{1(2)} = 2t+1,\ \lambda_{2(2)} = 2t-1,\ \lambda_3 = 2t;$$
$$n_1 = 1,\ n_2 = n_3 = 4t,\ m = 4t+1,\ n = 2,$$

where $4t+1$ *is a prime or a prime power and* t *is a positive integer.*

Proof: Let $4t+1$ be a prime or a prime power for a positive integer t. It is known (*cf.* Raghavarao (1971; Theorem 5.7.5)) that by developing $(x^0, x^2, \ldots, x^{4t-2})$ and $(x, x^3, \ldots, x^{4t-1})$ mod $(4t+1)$, where x is a primitive element of $GF(4t+1)$, a balanced incomplete block (BIB) design with parameters

$$v = 4t+1,\ b = 2(4t+1),\ r = 4t,\ k = 2t,\ \lambda = 2t-1 \qquad (2.1)$$

and its complementary BIB design with parameters

$$v = 4t+1,\ b = 2(4t+1),\ r = 2(2t+1),\ k = 2t+1,\ \lambda = 2t+1 \qquad (2.2)$$

can be constructed. Then, by developing the initial blocks

$$\begin{matrix} (0, x^0, x^0+\infty, x^2, x^2+\infty, \ldots, x^{4t-2}, x^{4t-2}+\infty) \\ (0, x, x+\infty, x^3, x^3+\infty, \ldots, x^{4t-1}, x^{4t-1}+\infty) \end{matrix} \quad (\text{mod } 4t+1)$$

where x is a primitive element of $GF(4t+1)$ and ∞ $(= 4t+1)$ will remain constant during the development, we can obtain a rectangular design with varying replicates and having the required parameters. Here, let the $(4t+1) \times 2$ array (after reducing the power cycle expression of x in $GF(4t+1)$) be given as follows.

$$\begin{matrix} 0 & 4t+1 \\ 1 & 4t+2 \\ 2 & 4t+3 \\ \vdots & \vdots \\ 4t & 8t+1 \end{matrix}.$$

Since the elements of the first column occur $2t+1$ times in the blocks developed from each of the initial blocks, we have $r_1 = 2(2t+1)$; also since the elements of the second column, denoted as $x^i + \infty$, occur $2t$ times in the blocks developed from each of the initial blocks, we have $r_2 = 4t$. Now, since the treatments in the first group alone form a BIB design with parameters (2.2), we have $\lambda_{1(2)} = 2t + 1$, and the treatments in the second group alone form a BIB design with parameters (2.1), we have $\lambda_{2(2)} = 2t - 1$. Furthermore, since an x^i and $x^i + \infty$ occur together $2t$ times in blocks generated from each of the initial blocks, $\lambda_1 = 4t$, and any two treatments neither in the same row nor column occur together $\lambda_3 = (2 \times 2t + 4t(2t-1))/(4t) = 2t$ times. ∎

Example: The above theorem for $t = 1$ yields a plan for a rectangular design with parameters

$$v = b = 10,\ r_1 = 6,\ r_2 = 4,\ k = 5,\ \lambda_1 = 4,\ \lambda_{1(2)} = 3,$$
$$\lambda_{2(2)} = 1,\ \lambda_3 = 2;\ n_1 = 1,\ n_2 = n_3 = 4,\ m = 5,\ n = 2,$$

whose blocks are given by

(0 1 4 6 9), (0 1 2 5 7), (1 2 3 6 8), (2 3 4 7 9),
(0 3 4 5 8); (0 2 3 7 8), (1 3 4 8 9), (0 2 4 5 9),
(0 1 3 5 6), (1 2 4 6 7),

where the 5×2 rectangular array is expressed as

$$\begin{matrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{matrix}.$$

Here, 2 is a primitive element of $GF(5)$.

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