ON COVERING DESIGNS WITH BLOCK SIZE 5 AND INDEX 4

Ahmed M. Assaf
Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48859
U.S.A.

N. Shalaby
Department of Mathematics
University of Toronto
Toronto, Ontario, M5A 1A1
CANADA

Abstract. A (v,k,λ) covering design of order v, block size k, and index λ is a collection of k-element subsets, called blocks of a set V such that every 2-subset of V occurs in at least λ blocks. The covering problem is to determine the minimum number of blocks in a covering design. In this paper we solve the covering problem with $k = 5$ $\lambda = 4$ and all positive integers v with the possible exception of $v = 17, 18, 19, 22, 24, 27, 28, 78, 98$.

1. Introduction.

A (v,k,λ) covering design of order v, block size k, and index λ is a collection β of k-element subsets, called blocks, of a v-set V such that every 2 subset of V occurs in at least λ blocks.

Let $\alpha(v,k,\lambda)$ denote the minimum number of blocks in a (v,k,λ) covering design. A (v,k,λ) covering design with $|\beta| = \alpha(v,k,\lambda)$ will be called a minimum covering design.

Schönheim [13] has shown that

$$\alpha(v,k,\lambda) \geq \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \right\rceil \lambda \right\rceil = \phi(v,k,\lambda)$$

where $\lceil x \rceil$ is the smallest integer satisfying $\lceil x \rceil \geq x$. Hanani [5] has sharpened this bound in certain cases by proving the following result.

Theorem 1.1. *If* $\lambda(v-1) \equiv 0 \bmod (k-1)$ *and* $\lambda v(v-1)/(k-1) \equiv -1 \pmod{k}$ *then* $\alpha(v,k,\lambda) \geq \phi(v,k,\lambda) + 1$.

The value of $\alpha(v,3,\lambda)$ for all v and λ has been determined by Hanani [5]. The value of $\alpha(4,1,2)$ has been determined by Mills [9, 10]. The value of $\alpha(4,\lambda,v)$ for all v and $\lambda > 1$ has been determined by Assaf [1] and Hartman [6].

The value of $\alpha(5,1,v)$ $v \not\equiv 0 \pmod 4$ has been determined by Lamken, Mills, Mullin, Vanstone [8] and by Mills and Mullin [11], and $\alpha(5,2,v)$ for v even has been determined by Horton and Mullin [7].

In order to state the result known about $\alpha(v,k,\lambda)$ we need the following definition. A balanced incomplete block design, $B(v,k,\lambda)$ is a (v,k,λ) covering design where every 2-subset of points is contained in precisely λ blocks. If a $B(v,k,\lambda)$ exists, then it is clear that $\alpha(v,k,\lambda) = \lambda v(v-1)/k(k-1) = \phi(v,k,\lambda)$ and Hanani [5] has proved the following existence theorem for $B(5,v,\lambda)$.

Theorem 1.2. *Necessary and sufficient conditions for the existence of a* $B(v,5,\lambda)$ *are that* $\lambda(v-1) \equiv 0 \pmod 4$ *and* $\lambda v(v-1) \equiv 0 \pmod{20}$ *and* $(v,\lambda) \neq (15,12)$.

This theorem implies that $\alpha(v,5,1) = \phi(v,5,1)$ for all $v \equiv 1,2,5,6 \pmod{20}$ by taking a balanced incomplete block design when $v \equiv 1,5 \pmod{20}$. For $v = 2,6 \pmod{(20)}$ take a $B(v-1,5,1)$ design and partition the $v-1$ points into $\lceil v-1/4 \rceil$ blocks and add a point to these blocks.

In this paper we are interested in determining the values of $\alpha(v,5,4)$. Our goal is to prove that $\alpha(v,5,4) = \phi(v,5,4)$ for all v with some few possible exceptions. Specifically we prove the following.

Theorem 1.3. *For all positive integers* v *we have* $\alpha(v,5,4) = \phi(v,5,4)$ *with the possible exceptions of* $v = 17,18,19,22,24,27,28,78,98$.

2. Recursive construction of covering design.

In order to describe our recursive constructions we need the notions of designs with a hole, transversal designs and truncated transversal designs.

Let (V,β) be a (v,k,λ) covering design, and let H be a subset of V of cardinality h. We shall say that (V,β) is an exact covering design with a hole of size h if no 2-subset of H appears in any block, and every other 2-subset of V appears in precisely λ blocks.

Lemma 2.1 (Assaf-Hartman [2]).

(i) *Let* $v \equiv 2,4 \pmod 5$. *An exact* $(v,5,4)$ *covering design with a hole of size 2 exists for all* $v \neq 7$.

(ii) *Let* $v \equiv 3 \pmod 5$. *An exact design with a hole of size 3 exists for all* $v \neq 8$ *and possible exceptions of* $v = 43,68$.

Let k, λ and w be positive integers. A transversal design $T(k,\lambda,w)$ is a triple (V,β,γ) where V is a set of points with $|V| = kw$, and $\gamma = \{G_1,\dots,G_k\}$ is a partition of V into k sets of size w. The parts, G_i, of the partition are called groups. The collection β consists of k-subsets of V, called blocks with the following properties:

1. $|B \cap G_i| = 1$ for all $B \in \beta$ and $G_i \in \gamma$;
2. every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

It is well known that a $T(k, 1, w)$ is equivalent to $k-2$ mutually orthogonal Latin squares of side w.

In the sequel we shall use the following existence theorems for transversal designs. The proofs of these results may be found in [3], [4], [5], [12] and [14].

Theorem 2.1. *There exists a $T(6, 1, w)$ for all positive integers w with the exception of $w \in \{2, 3, 4, 6\}$ and the possible exception of $w \in \{10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.*

Theorem 2.2. *There exists a $T(7, \lambda, w)$ for all positive integers w and all integers $\lambda \geq 2$.*

We now give the definition of truncated transversal design. Let k, λ and w be positive integers, and let u be a non-negative integer. A truncated transversal design $TT(k, \lambda, w, u)$ is a triple (V, β, γ) where V is a set of points with $|V| = (k-1)w + u$, and $\gamma = \{G_1, \ldots, G_k\}$ is a partition of V into $k-1$ sets of size w and one set G_k of size u. G_i are called the groups of the truncated transversal design. The collection β consists of k-subsets and $(k-1)$-subsets of V, called blocks, with the following properties:

1. $|B \cap G_i| = 1$ for all $B \in \beta$ and $1 \leq i < k$;
2. $|B \cap G_k| = 1$ for all $B \in \beta$ such that $|B| = k$;
3. every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

Clearly, a $TT(k, \lambda, w, 0)$ is equivalent to a $T(k-1, \lambda, w)$. Furthermore, if $0 \leq u \leq w$ then one may construct a $TT(k, \lambda, w, u)$ from a transversal design $T(k, \lambda, w)$ by removing points from the last group, and from all the blocks which contain them. Thus, we have the following existence results which are in the form most useful to us.

Theorem 2.3. *There exists a $TT(6, 1, w, u)$ for all integers $0 \leq u \leq w$ and for all positive integers w with the exception of $w \in \{2, 3, 4, 6\}$ and the possible exception of $w \in \{10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.*

Theorem 2.4. *There exists a $T(5, 4, w)$ for all positive integers w.*

We can now give the recursive constructions used in the proof of our main theorem.

Theorem 2.5. *If there exists a $TT(6, 1, w, u)$ with $w \equiv 0$ or $1 \pmod 5$ and $\alpha(u, 5, 4) = \phi(u, 5, 4)$ then $\alpha(5w + u, 5, 4) = \phi(5w + u, 5, 4)$.*

Proof: On the blocks and the groups of size w of the truncated transversal design construct balanced incomplete block design $B(v,5,4)$ with $v = 5,6$ and w. On the groups of size u construct a $(u,5,4)$ covering design with $\phi(u,5,4)$ blocks. This gives us a $(5w+u,5,4)$ covering design with $\phi(5w+u,5,4)$ blocks. ▮

Let us add h points to the groups of a $TT(6,1,w,u)$. On the blocks construct a $B(v,5,4)$ with $v = 5,6$. On the groups of size w we construct a $(w+h,5,4)$ covering design with hole of size h and index 4, and on the last group we construct a $(u+h,5,4)$ covering design (we assume that the last two designs exist). The resultant design is a $(5w+u+h,5,4)$ covering design. We may write the above observation as the following theorems.

Theorem 2.6. *If there exists a* $TT(6,1,w,u)$ *with* $w \equiv 0$ *or* $4 \pmod 5$ *and* $\alpha(u+1,5,4) = \phi(u+1,5,4)$ *then* $\alpha(5w+u+1,5,4) = \phi(5w+u+1,5,4)$.

Theorem 2.7. *If there exists a* $TT(6,1,w,u)$ *with* $w \equiv 0$ *or* $2 \pmod 5$ *and* $\alpha(u+2,5,4) = \phi(u+2,5,4)$ *then* $\alpha(5w+u+2,5,4) = \phi(5w+u+2,5,4)$.

Theorem 2.8. *If* $w \equiv 0$ *or* $2 \pmod 5$ *and* $\alpha(w+2,5,4) = \phi(w+2,5,4)$ *then* $\alpha(5w+2,5,4) = \phi(5w+2,5,4)$.

Theorem 2.9. *If* $w \equiv 0 \pmod 5$, *and there exists an exact* $(w+3,5,4)$ *covering design with a hole of size 3, then there exists an exact* $(5w+3,5,4)$ *covering design with a hole of size 3, and hence* $\alpha(5w+3,5,4) = \phi(5w+3,5,4)$.

The following lemma is very useful for constructing covering designs with a hole.

Lemma 2.2 (Assaf-Hartman [2]). *Let* r *be a positive integer such that* $3r+1$ *is a prime power. Then there exists an exact* $(4r+1,5,4)$ *covering design with a hole of size* r.

Lemma 2.3. *If there exists a* $TT(6,1,w,u)$ *then there exists a* $TT(5,4,2w,2u)$.

Proof: Let X be the pointset of a $TT(6,1,w,u)$ and construct a $TT(5,4,2w,2u)$ by replacing each point $x \in X$ by two points $\{x_0, x_1\}$ so the groups are of size $2w$ and $2u$. On each block B of size five construct a $GD[5,4,2,10]$ in such a way that it has groups $\{b_0, b_1\}$ for $b \in B$. Such design exists by Theorem 2.4; and on each block B of size six construct a $GD[5,4,2,12]$ where the groups are $\{b_0, b_1\}$ for $b \in B$. [Note that a $GD[5,4,2,12]$ can be constructed as follows.] Let the pointset be $Z_2 \times Z_5 \cup \{a,b\}$ then the required blocks are:

$$\langle (0,0)(0,1)(0,2)(0,3)(0,4) \rangle$$
$$\langle (1,0)(1,1)(1,2)(1,3)(1,4) \rangle$$
$$\langle (0,0)(0,2)(1,3)(1,4),a \rangle \quad (\mathrm{mod}(-,5))$$
$$\langle (0,0)(0,1)(1,2)(1,4),b \rangle \quad (\mathrm{mod}(-,5))$$

each block taken twice. ∎

The following theorem is the last recursive construction we need to prove Theorem 1.3.

Theorem 2.10. *If there exists a* $TT(5,4,w,u)$ *then*

(1) *If* $w \equiv 0$ *or* $4 \pmod 5$ *and* $\alpha(u+1,5,4) = \phi(u+1,5,4)$ *then* $\alpha(5w+u+1,5,4) = \phi(5w+u+1,5,4)$.

(2) *If* $w \equiv 0$ *or* $2 \pmod 5$ *and* $\alpha(w+2,5,4) = \phi(w+2,5,4)$ *then* $\alpha(5w+u+2,5,4) = \phi(5w+u+2,5,4)$.

(3) *If* $w \equiv 0 \pmod 5$ *and there exist an exact* $(w+3,5,4)$ *covering design with a hole of size 3, and* $\alpha(u+3,5,4) = \phi(u+3,5,4)$ *then* $\alpha(5w+u+3,5,4) = \phi(5w+u+3,5,4)$.

3. The main theorem.

Before giving an induction proof of Theorem 1.3, we need the following construction of covering designs with small values of v.

Lemma 3.1. $\alpha(v,5,4) = \phi(v,5,4)$ *for* $v = 7,8,9,12,13,14$.

Proof: For $v = 7$ let $X = Z_6 \cup \{a\}$ then the blocks are:

$$\langle 0,1,2,4,a\rangle \pmod 6$$
$$\langle 0,1,2,3,5\rangle$$
$$\langle 1,2,3,4,5\rangle$$
$$\langle 0,1,3,4,5\rangle.$$

For $v = 8$ let $X = Z_7 \cup \{a\}$ then the blocks are:

$$\langle 0,2,3,4,a\rangle \pmod 7$$
$$\langle 1,2,4,5,6\rangle + i, \quad i \in Z_4$$
$$\langle 2,1,3,4,5\rangle.$$

For $v = 9$ let $X = Z_2 \times Z_3 \cup \{a,b,c\}$ then the blocks are:

$$\langle (0,0),(0,1),a,b,c\rangle \pmod{(-,3),}$$
$$\langle (1,0),(1,1),a,b,c\rangle \pmod{(-,3),}$$
$$\langle (0,0),(0,1),(1,0),(1,2),a\rangle \pmod{(-,3)}$$
$$\langle (0,0),(0,1),(1,0),(1,1),b\rangle \pmod{(-,3)}$$
$$\langle (0,0),(0,2),(1,1),(1,0),c\rangle \pmod{(-,3)}.$$

For $v = 12$ let $X = Z_3 \times Z_3 \cup \{a, b, c\}$ then the required blocks are:

$$\langle (0,0),(0,1),(0,2),(1,2),(2,2)\rangle \ (\text{mod } (-,3))$$
$$\langle (1,0),(1,1),(1,2),(0,1),(2,1)\rangle \ (\text{mod } (-,3),)$$
$$\langle (2,0),(2,1),(2,2),(0,0),(1,0)\rangle \ (\text{mod } (-,3))$$
$$\langle (0,0),(1,1),(2,2),a,b\rangle \ (\text{mod } (-,3)) \text{ twice}$$
$$\langle (0,0),(1,2),(2,1),b,c\rangle \ (\text{mod } (-,3)) \text{ twice}$$
$$\langle (0,0),(1,0),(2,0),a,c\rangle \ (\text{mod } (-,3))$$
$$\langle (0,0),(0,1),(0,2),a,c\rangle \ (\text{mod } (-,3)).$$

For $v = 13$ let $X = Z_{13}$, then the required blocks are:

$\langle 1,4,9,12,13\rangle$	$\langle 1,3,6,8,9\rangle$
$\langle 2,3,6,9,10\rangle$	$\langle 1,4,5,9,11\rangle$
$\langle 4,5,7,10,13\rangle$	$\langle 1,6,7,10,12\rangle$
$\langle 5,6,7,9,13\rangle$	$\langle 1,2,10,11,13\rangle$
$\langle 3,8,11,12,13\rangle$	$\langle 1,2,4,7,8\rangle$
$\langle 2,5,9,11,12\rangle$	$\langle 1,6,7,11,13\rangle$
$\langle 3,7,8,9,12\rangle$	$\langle 4,7,8,11,12\rangle$
$\langle 2,6,11,12,13\rangle$	$\langle 4,6,7,9,11\rangle$
$\langle 5,8,9,10,13\rangle$	$\langle 3,4,10,12,13\rangle$
$\langle 2,7,9,10,12\rangle$	$\langle 3,4,9,10,11\rangle$
$\langle 2,3,5,7,13\rangle$	$\langle 1,5,8,10,12\rangle$
$\langle 5,6,8,10,11\rangle$	$\langle 2,4,6,8,10\rangle$
$\langle 2,4,5,6,12\rangle$	$\langle 1,3,7,10,11\rangle$
$\langle 3,4,6,8,13\rangle$	$\langle 1,2,3,4,5\rangle$
$\langle 1,2,8,9,13\rangle$	$\langle 3,5,7,8,11\rangle$
$\langle 1,3,5,6,12\rangle$	$\langle 2,3,7,8,11\rangle$.

For $v = 14$ let $X = Z_{14}$ then the required blocks are:

$\langle 1,3,9,11,13 \rangle$	$\langle 3,4,5,10,12 \rangle$
$\langle 1,3,7,10,14 \rangle$	$\langle 1,3,4,5,6 \rangle$
$\langle 5,8,9,11,14 \rangle$	$\langle 2,3,7,9,14 \rangle$
$\langle 2,6,7,8,13 \rangle$	$\langle 2,5,6,9,11 \rangle$
$\langle 1,2,10,11,12 \rangle$	$\langle 1,4,11,13,14 \rangle$
$\langle 3,7,8,11,12 \rangle$	$\langle 4,7,9,12,14 \rangle$
$\langle 1,4,8,9,12 \rangle$	$\langle 6,10,11,12,14 \rangle$
$\langle 1,5,7,8,11 \rangle$	$\langle 6,8,12,13,14 \rangle$
$\langle 5,7,10,11,14 \rangle$	$\langle 2,4,11,12,13 \rangle$
$\langle 1,6,10,13,14 \rangle$	$\langle 3,4,8,10,11 \rangle$
$\langle 1,6,7,8,12 \rangle$	$\langle 1,5,6,9,12 \rangle$
$\langle 1,2,4,7,13 \rangle$	$\langle 4,6,7,9,10 \rangle$
$\langle 4,8,9,10,13 \rangle$	$\langle 2,4,6,7,11 \rangle$
$\langle 2,3,9,12,14 \rangle$	$\langle 3,6,9,11,13 \rangle$
$\langle 2,5,10,12,13 \rangle$	$\langle 2,4,5,8,14 \rangle$
$\langle 5,7,9,10,13 \rangle$	$\langle 3,5,7,12,13 \rangle$
$\langle 1,2,8,9,10 \rangle$	$\langle 3,5,8,13,14 \rangle$
$\langle 2,3,6,8,10 \rangle$	$\langle 1,2,3,5,14 \rangle$
	$\langle 3,4,5,6,14 \rangle$

■

We now give a table describing the construction of some exact $(v,5,4)$ covering designs with a hole of size n. In general the construction is as follows. Let $X = Z_{u-n} \cup H_n$ where $H_n = \{h_0, h_1, \ldots, h_{n-1}\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks under the action of the cyclic group generated by the permutation which fixes the elements of H_n and sent $i \longrightarrow i+1 \pmod{v-n}$ for each $i \in Z_{v-n}$.

Figure 1.

Exact $(v, 5, 4)$ covering designs with a hole of size n.

v	n	Point Set	Base Blocks
29	7	$Z_{22} \cup H_7$	$\langle 0,3,8,12,h_0 \rangle$ $\langle 0,2,6,11,h_1 \rangle$ $\langle 0,1,8,10,h_2 \rangle$ $\langle 0,2,3,10,h_3 \rangle$ $\langle 0,5,6,12,h_4 \rangle$ $\langle 0,3,4,11,h_5 \rangle$ $\langle 0,3,5,9,h_6 \rangle$
32	7	$Z_{25} \cup H_7$	$\langle 0,5,10,15,20 \rangle$ $+i,\ i \in Z_5$ $\langle 0,4,11,17,h_0 \rangle$ $\langle 0,2,3,9,h_1 \rangle$ $\langle 0,3,11,12,h_2 \rangle$ $\langle 0,4,6,13,h_3 \rangle$ $\langle 0,3,4,11,h_4 \rangle$ $\langle 0,2,4,14,h_5 \rangle$ $\langle 0,1,6,9,h_6 \rangle$
34	7	$Z_{27} \cup H_7$	$\langle 0,1,4,10,15 \rangle$ $\langle 0,2,7,8,h_0 \rangle$ $\langle 0,3,7,16,h_1 \rangle$ $\langle 0,2,8,12,h_2 \rangle$ $\langle 0,1,3,11,h_3 \rangle$ $\langle 0,7,12,13,h_4 \rangle$ $\langle 0,2,9,13,h_5 \rangle$ $\langle 0,9,12,17,h_6 \rangle$
38	8	$Z_{30} \cup H_8$	$\langle 0,1,6,9,17 \rangle$ $\langle 0,10,12,16,h_0 \rangle$ $\langle 0,3,7,18,h_1 \rangle$ $\langle 0,9,10,17,h_2 \rangle$ $\langle 0,2,7,13,h_3 \rangle$ $\langle 0,1,5,15,h_4 \rangle$ $\langle 0,2,3,12,h_5 \rangle$ $\langle 0,4,7,16,h_6 \rangle$ $\langle 0,2,8,13,h_7 \rangle$
39	7	$Z_{32} \cup H_7$	$\langle 0,1,5,11,18 \rangle$ $\langle 0,7,9,12,20 \rangle$ $\langle 0,6,9,10,h_0 \rangle$ $\langle 0,14,16,24,h_1 \rangle$ $\langle 0,3,15,19,h_2 \rangle$ $\langle 0,1,6,15,h_3 \rangle$ $\langle 0,2,7,11,h_4 \rangle$ $\langle 0,6,14,21,h_5 \rangle$ $\langle 0,1,3,13,h_6 \rangle$
47	9	$Z_{38} \cup H_9$	$\langle 0,1,4,14,20 \rangle$ $\langle 0,2,7,19,27 \rangle$ $\langle 0,9,15,24,h_0 \rangle$ $\langle 0,1,4,12,h_1 \rangle$ $\langle 0,5,7,17,h_2 \rangle$ $\langle 0,2,10,13,h_3 \rangle$ $\langle 0,1,7,17,h_4 \rangle$ $\langle 0,14,18,23,h_5 \rangle$ $\langle 0,1,5,27,h_6 \rangle$ $\langle 0,2,15,22,h_7 \rangle$ $\langle 0,3,9,17,h_8 \rangle$
48	8	$Z_{40} \cup H_8$	$\langle 0,2,9,13,25 \rangle$ $\langle 0,5,8,22,30 \rangle$ $\langle 0,6,19,20,26 \rangle$ $\langle 0,1,4,13,h_0 \rangle$ $\langle 0,1,5,14,h_1 \rangle$ $\langle 0,1,5,16,h_2 \rangle$ $\langle 0,2,7,17,h_3 \rangle$ $\langle 0,2,8,18,h_4 \rangle$ $\langle 0,2,8,19,h_5 \rangle$ $\langle 0,3,10,21,h_6 \rangle$ $\langle 0,3,12,24,h_6 \rangle$
59	12	$Z_{47} \cup H_{12}$	see [2]

The following lemma is our main reason for constructing exact covering designs with holes.

Lemma 3.2. *If an exact* $(v, 5, 4)$ *covering with a hole of size* $h \geq 5$ *exists and* $\alpha(h, 5, 4) = \phi(h, 5, 4)$ *then* $\alpha(v, 5, 4) = \phi(v, 5, 4)$.

Proof: Take the blocks of an $(h, 5, 4)$ covering design with $\phi(h, 5, 4)$ blocks and the blocks of the exact covering design. ∎

As a result of the above lemma $\alpha(v, 5, 4) = \phi(v, 5, 4)$ for all the values of v appearing in Figure 1.

Lemma 3.3. *For all* $v \equiv 3 \pmod{20}$, $\alpha(v, 5, 4) = \phi(v, 5, 4)$.

Proof: Let $X = \{1, \ldots, v\}$, then the blocks of a $(v, 5, 4)$ covering design can be constructed as follows:

(1) Take the blocks of a $B(v-2, 5, 2)$.
(2) Take the blocks of a $B(v+2, 5, 1)$ and assume that no triple of the points $\{v-1, v, v+1, v+2\}$ appears in one block. Further assume we have the two blocks $\langle 1, 2, 3, v-1, v+1\rangle$ $\langle 4, 5, 6, v, v+2\rangle$. In the first block change $v+1$ to v and in the second change $v+2$ to $v-1$. In the remaining blocks of $B(v+2, 5, 1)$ change $v+2$ to v and $v+1$ to $v-1$.
(3) Take $B(v+2, 5, 1)$ one more time and assume that no triple of $\{v-1, v, v+1, v+2\}$ appears in one block. Further assume we have the two blocks $\langle 1, 2, 3, v, v+2\rangle$ $\langle 4, 5, 6, v-1, v+1\rangle$. In the first block change $v+2$ to $v-1$ and in the second change $v+1$ to v. In all other blocks of $B(v+2, 5, 1)$ change $v+2$ to v and $v+1$ to $v-1$. ∎

We are now able to prove our main theorem, which is restated below for the reader's convenience.

Theorem 1.3. *For all positive integers* v *we have* $\alpha(v, 5, 4) = \phi(v, 5, 4)$ *with the possible exceptions of* $v = 17, 18, 19, 22, 24, 27, 28, 47, 48, 78$.

Proof: For $v \equiv 0$ or $1 \pmod 5$ there exists a $B(v, 5, 4)$; and for $v \equiv 3 \pmod{20}$ it follows from the above lemma. For other values of v we consider the following cases:

Case 1. $v \equiv 2, 3, 4 \pmod{25}$.

In this case $v = 5w + u + 1$ where $w \equiv 4 \pmod 5$ and $u \in \{6, 7, 8\}$. By theorem 2.3 there exists a $TT(6, 1, w, u)$ for all the relevant pairs (w, u) with the exception of $w \in \{4, 14, 34, 44\}$. So for $v \neq 27, 28, 29, 77, 78, 179, 227, 228, 229$, apply Theorem 2.6 to give the result. For $v = 29$ see Figure 1. For $v = 77$, 79, 177, 178, 179, 227, 228, 229, see Figure 2.

Figure 2

Exceptional constructions for $(v, 5, 4)$ covering design.

v	w	u	Theorem
57	10	6	2.10
58	10	6	2.10
59	10	8	2.10
73	12	11	2.7
77	14	6	2.10
79	14	8	2.10
117	20	7	2.5
118	20	8	2.5
119	20	9	2.5
122	20	0	2.8
123	20	20	2.9
124	22	12	2.10
137	25	12	2.5
138	25	13	2.5
139	25	14	2.5
157	29	11	2.6
158	29	12	2.6
159	29	13	2.6

v	w	u	Theorem
173	30	22	2.10
177	34	6	2.10
178	31	23	2.5
179	34	8	2.10
187	35	12	2.5
188	35	13	2.5
189	35	14	2.5
217	41	12	2.5
218	41	13	2.5
219	41	14	2.5
222	42	10	2.10
224	42	12	2.10
227	44	6	2.10
228	41	23	2.5
229	44	8	2.10
269	51	14	2.5
272	47	35	2.8
274	47	37	2.8

Case 2. $v \equiv 7, 8, 9 \pmod{25}$.

In this case $v = 5w + u$ where $w \equiv 0 \pmod 5$ and $u \in \{7, 8, 9\}$. By Theorem 2.3 there exists a $TT(6, 1, w, u)$ for all relevant pairs (w, u) with the exception of $w \in \{10, 30\}$. So for $v \neq$ 32, 33, 34, 57, 58, 59, 157, 158, 159, apply Theorem 2.6 to give the result. For $v = 32, 34, 59$, see Figure 1. For $v = 33$ apply Lemma 2.2 and Lemma 3.2. For the remaining values see Figure 2.

Case 3. $v \equiv 12, 13, 14 \pmod{25}$.

In this case $v = 5w + u$ where $w \equiv 1 \pmod 5$ and $u \in \{7, 8, 9\}$. By Theorem 2.3 there exists a $TT(6, 1, w, u)$ for all relevant pairs (w, u) with the exception of $w \in \{26, 36\}$. So for $v \neq$ 38, 39, 137, 138, 139, 187, 188, 189, apply Theorem 2.6 to give the result. For $v = 38$, 39, see Figure 1. For the remaining values see Figure 2.

Case 4. $v \equiv 17, 18, 19, 22, 23, 24 \pmod{25}$.

In this case $v = 5w + u + 2$ where $w \equiv 2 \pmod 5$ and $u \in \{5, 6, 7, 10, 12\}$. By Theorem 2.3 there exists a $TT(6, 1, w, u)$ for all relevant pairs (w, u) with the exception of $w \in \{22, 42, 52\}$. So for $v \neq$ 117, 118, 119, 122, 124, 217, 218, 219, 224, 267, 268, 269, 272, 274, apply Theorem 2.6 to give the result. For $v = 47$ see Figure 1. For $v = 49$ apply Lemma 2.2 and Lemma 3.4. For the other values of v, see Figure 2.

Case 5. $v \equiv 23 \pmod{25}$.

In this case $v = 5w + 23$ where $w \equiv 0 \pmod 5$ and $u = 23$. By Theorem 2.3 there exist a $TT(6, 1, w, u)$ for all relevant pairs (w, u) with the exception of $w \in \{10, 15, 20, 30\}$. So for $v \neq 48, 73, 98, 123, 173$, apply Theorem 2.6 to give the result. For $v = 48$ see Figure 1. For $v = 73, 123, 173$, see Figure 2. ∎

References

1. A.M. Assaf, *On the covering of pairs by quadruples*, Discrete Math. **61** (1986), 119-132.
2. A.M. Assaf, A. Hartman, *On packing designs with block size 5 and index 4*, Discrete Math. (to appear).
3. T. Beth, D. Jungnickel, H. Lenz, "Design Theory", Bibl. Inst. Manheim, 1985.
4. A.E. Brouwer, *The number of mutually orthogonal Latin squares — a table up to order 10000*, Math. Centrum **ZW 123** (1979), 2-31, Amsterdam.
5. H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. **11** (1975), 225-369.
6. A. Hartman, *On small packing and covering designs with block size 4*, Discrete Math. **59** (1986), 275-281.
7. J.D. Horton, R.C. Mullin, *Bicovers of pairs by quintuples: v even.* (to appear).
8. E.R. Lamken, W.H. Mills, R.C. Mullin, S.A. Vanstone, *Covering of pairs by quintuples*, J. Comb. Theory ser A **44** (1987), 49-68.
9. W.H. Mills, *On the covering of pairs by quadruples I*, J. Comb. Theory ser A **13** (1972), 55-78.
10. W.H. Mills, *On the covering of pairs by quadruples II*, J. Comb. Theory ser A **15** (1973), 138-166.
11. W.H. Mills, R.C. Mullin, *Covering pairs by quintuples: the case v congruent to 3 (mod 4)*. (to appear).
12. R. Roth, M. Peters, *Four pairwise orthogonal Latin squares on side 24*, J. Comb. Theory ser V **44** (1987), 152-155.
13. J. Schönheim, *On covering*, Pacific J. Math. **14** (1964), 1405-1411.
14. D.T. Todorov, *Four mutually orthogonal Latin squares of order 20*, Ars. Combinatoria. (to appear).