Covering Morphisms Between Nets

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1. Introduction

1.1

In this paper we introduce an interesting class of functions between nets. By analogy with a classical notion from algebraic topology, we call them covering morphisms. The question of parametric feasibility is studied, and it is found that these morphisms must be of one of two possible types. An infinite series of Type 1 covering morphisms onto odd order affine planes is constructed. These can be used to construct certain mutiway designs which are optimal for statistical applications.

1.2 Preliminaries on nets

Recall [3] that a net of degree r and order k (in short, an (r, k) net) is a pair (P, L) where P is a finite set (its elements are called the points of the net) and L is a set of subsets of P (its elements are called the lines of the net) such that on each line lie k points, through each point pass r lines, any two lines have at most one point in common, and the system satisfies Playfair's axiom: given a point and a line, exactly one line through the given point is parallel to (= equal to or disjoint from) the given line. It follows that the set of lines of a net is partitioned into r parallel classes, where the lines in each class partition the point set and lines from different parallel classes intersect. If $X = (P, L)$ and $X' = (P, L')$ are two nets on the same point set, then we say that X' is a subnet of X if L' is a subset of L. Trivially, if X is an (r, k) net and $r' \leq r$ then one obtains the (r', k) subnets of X by deleting the lines from all but r' chosen parallel classes of X. For the existence of (r, k) nets it is necessary that $r \leq k + 1$ [3,5]. Nets with $r = k + 1$ are called affine planes of order k; these are necessarily 2-designs and indeed can be characterised as 2-$(k^2, k, 1)$ designs. If s is a prime power then the affine plane of order s arising from the field of order s is called the arguesian affine plane of order s and is denoted by $AG(2, s)$. Nets have been studied by many authors under several names; for instance, mutually orthogonal latin squares [2], partial geometries [1] (with $t = r - 1$) and (duals of) transversal designs [6].

1.3 Covering morphisms (definition)

Let $X_i = (P_i, L_i)$, $i = 1,2$, be two nets and let $\alpha \neq \beta$ be two non-negative integers. A function $\phi : P_1 \to P_2$ is called a covering morphism from X_1 to X_2 with parameters α, β provided conditions (1) and (2) stated below hold:

(1) For each line b of X_1 and for each point x of X_2, the number of points y in b with $\phi(y) = x$ is either α or β, and the set

$$\bar{b} = \{x \in P_2 : |\phi^{-1}(x) \cap b| = \alpha\}$$

is a line of X_2. The map $\bar{\phi}: L_1 \to L_2$ defined by $\bar{\phi}(b) = \bar{b}$ will be called the induced map.

(2) The restriction of the induced map $\bar{\phi}$ to each parallel class Π of X_1 is a bijection between Π and L_2.

1.4 Results

Say an (r, k) net is nontrivial if $r \geq 2$ and $k \geq 4$. Then our first result is:

Theorem 1. *Let X_i be a non-trivial (r_i, k_i) net for $i = 1,2$. Let ϕ be a covering morphism from X_1 to X_2 with parameters α, β. Then we have:*

(a) *The preimage of each point of X_2 under ϕ consists of exactly r_2^2 points of X_1.*

(b) *ϕ is of one of the following two types:*

Type 1 $\alpha = 2, \beta = 1, r_2 = k_2 + 1, k_1 = k_2(k_2 + 1)$,
Type 2 $\alpha = 0, \beta = 1, r_2 = k_2 - 1, k_1 = k_2(k_2 - 1)$.

Our second result is a construction of Type 1 covering morphisms into the arguesian affine planes $AG(2, s)$ of odd order:

Theorem 2. *Let s be an odd prime power and let n be such that an $(n+1, s+1)$ net exists. Then there is an $(n, s^2 + s)$ net X and a Type 1 covering morphism ϕ from X to $AG(2, s)$.*

Note: The hypothesis on n in Theorem 2 may be rephrased as $n \leq n(s) := N(s+1) + 1$, when $N(m)$ is the largest number of mutually orthogonal latin squares of order m. For a survey of what is known about the function $N(.)$ see [2]. In particular, we have $n(s) \geq 3$ for $s \neq 5$, $n(s) = s + 1$ if s is a Mersenne prime (i.e. a prime of the form $2^p - 1$) and $n(s) \geq s^c$ for all s, where $c > 0$ is a suitable constant.

1.5 Remarks

(a) Recall that a set of type (α, β) (in short an (α, β) set) in a linear incidence system is a point set which meets every line in α or β points. These have been studied extensively in projective geometries and Steiner systems. By part (1) in the definition of a covering morphism, each fibre (= pre-image of singleton) of such a morphism is an (α, β) set; hence it yields a partition of the point set of the domain net into (α, β) sets each of which has size r_2^2 by Theorem 1(a). In particular, Theorem 2 yields a partition of the point set of an $(n, s^2 + s)$ net into s^2 $(2, 1)$ sets of size $(s + 1)^2$ each. While the construction in [4] yields $(2^f, 0)$ sets in $AG(2, 2^e)$ for $e \geq f$, ours appears to be the first construction of non-trivial (α, β) sets in nets of order k where k is not a power of two.

(b) Recall that a covering morphism in algebraic topology is a continuous map between topological spaces with a technical requirement which forces, that (i) it is a local homeomorphism, and (ii) (when the range is path connected all the fibres are homeomorphic. Part (2) of our definition is an in-built analogue of local homeomorphism, which, together with part (1) ensures (as shown in Theorem 1(a)) that all the fibres have the same size. The analogy is imperfect, though suggestive.

(c) The case $n = 3$ (with s arbitrary prime power, not necessarily odd) of Theorem 2 is essentially contained in a construction in [8]. To see the relevance of [8], note that the point set of the net X of Theorem 2 may naturally be identified with the positions of a square array of order $s^2 + s$. We expect that Theorem 2 holds for $s = 2^e$ as well, though no neat construction is available. Construction of Type 2 covering morphisms is also open.

(d) The definition of covering morphism may painlessly be extended to (r, k, μ) nets (for definition see [5] for instance). We don't indulge in this generalisation since we have no non-trivial example with $\mu > 1$.

(e) Any covering morphism of Type 1 can be used to construct examples of the statistical designs in a mutiway setting which were proved to be optimal in the paper [7] by two of the present authors. Namely, fixing two of the parallel classes of the domain net X_1, the point set of X_1 may be identified with the positions of a square array in such a way that the lines in the two fixed parallel classes are the rows and columns of this array. The remaining lines of X_1 are distinguished transversals of this square, and the covering morphism is an assignment of the points of the range net to the positions of the square. It is easy to verify that this assignment is a balanced Youden hypercube as defined in [7].

2. Parametric Restrictions

In this section we prove Theorem 1. So the assumptions and notations are as in the statement of that theorem. Let $X_i = (P_i, L_i), i = 1, 2$.

2.0

Since the induced map $\bar{\phi}$ is a bijection between the set of k_1 lines in any parallel class of X_1 and the set of all $r_2 k_2$ lines of X_2:

$$k_1 = r_2 k_2 \tag{2.1}$$

The fibres of ϕ induce a partition of each line of X_1 into k_2 cells of size α and $k_2^2 - k_2$ cells of size β. So $r_2 k_2 = k_1 = \alpha k_2 + \beta(k_2^2 - k_2)$. Hence,

$$r_2 = \alpha + \beta(k_2 - 1) \tag{2.2}$$

2.1 Proof of Theorem 1(a)

Let F be a fibre of ϕ. That is, $F = \phi^{-1}(x)$ for some x in P_2. Fix a parallel class Π of X_1. Since $\bar{\phi}: \Pi \to L_2$ is a bijection, exactly r_2 of the lines b in Π satisfy $x \in \bar{\phi}(b)$, that is, $|b \cap F| = \alpha$. For the other $k_1 - r_2$ lines b in Π, $|b \cap F| = \beta$. Hence,

$$|F| = \alpha r_2 + \beta(k_1 - r_2) = r_2(\alpha + \beta(k_2 - 1)) = r_2^2$$

by (2.1) and (2.2).

2.2 Proof of Theorem 1(b)

First suppose, if possible, that $\beta = 0$. Then by (2.2) $\alpha = r_2$. Let F be as above, and let $\bar{B}$ be the set of nonempty intersections of the fibre F with lines of X_1. Then, clearly, we have: (i) any two elements of $\bar{B}$ have at most one point in common, (ii) through each point of F pass r_1 elements of $\bar{B}$, (iii) each element of $\bar{B}$ has size r_2, and (from the argument in 2.1 above), (iv) the partition of L_1 into r_1 parallel classes induces a partition of $\bar{B}$ into r_1 'parallel classes', where there are r_2 elements in each 'parallel class' and each 'parallel class' partitions F. By Theorem 1(a), $|F| = r_2$ hence by (ii) and (iii) we get $|\bar{B}| = r_1 r_2$. By (i), (ii) and (iii), given any element $\bar{b}$ of $\bar{B}$, exactly $(r_1 - 1)r_2$ other elements of $\bar{B}$ intersect $\bar{b}$ and hence $r_1 r_2 - (r_1 - 1)r_2 = r_2$ elements of $\bar{B}$ are equal to or disjoint from $\bar{b}$. Hence (iv) implies that elements of $\bar{B}$ from distinct 'parallel classes' intersect. Hence $(F, \bar{B})$ is an (r_1, r_2) net.

Let Π_1, Π_2 be two distinct parallel classes of X_1. These exist since $r_1 \geq 2$. Since the restriction of $\bar{\phi}$ to each of Π_1, Π_2 is onto L_2, there are lines b_1 in Π_1, b_2 in Π_2 such that $\bar{\phi}(b_1) = \bar{\phi}(b_2)$. Let F be any fibre of ϕ meeting b_1 and hence also b_2. Let $b_i^* = b_i \cap F$, $i = 1, 2$. Since all the lines of the net $(F, \bar{B})$ which are parallel to b_1^* are induced by lines from Π_1, b_1^* and b_2^* intersect, That is any fibre which meets b_1 passes through the unique point of intersection of b_1 and b_2; hence such a fibre is uniquely determined. But $k_2 = k_1/r_2$ fibres meet b_1. Hence $k_2 = 1$. Contradiction. $\beta \geq 1$.

Since $r_2 \leq k_2 + 1$, (2.2) implies $\alpha + (\beta - 1)(k_2 - 1) \leq 2$. Since $\beta \geq 1$ and $k_2 \geq 4$, it follows that $\beta = 1$ and $\alpha \leq 2$. As $\alpha \neq \beta$, this together with (2.1), (2.2) completes the proof.

3. The Construction

3.1 Notation and terminology

Throughout this section s is an odd prime power. F_s, $PG(1,s)$ and $AG(2,s)$ will denote the field, the projective line and the arguesian affine plane, respectively, of order s. Thus $PG(1,s) = F_s \cup \{\infty\}$ where ∞ is a symbol outside F_s. We adopt the usual conventions for algebraic manipulations involving ∞. $AG(2,s)$ is the $(s+1,s)$ net whose point set is $F_s \times F_s$ regarded as a two dimensional vector space over F_s and whose lines are the translates of the one dimensional subspaces of this vector space. For m in $PG(1,s)$ and c in F_s, the line of $AG(2,s)$ with slope m and intercept c is given by the equation $y = mx + c$ when $m \neq \infty$ and by the equation $x = c$ when $m = \infty$. The lines of $AG(2,s)$ with a given slope constitute a parallel class. Thus the parallel classes are naturally indexed by the points of the projective line.

3.2 A Product construction of nets

For $i = 1,2$, let $X_i = (P_i, L_i)$ be an (r, k_i) net with parallel classes Π_j^i; $1 \leq j \leq r$. For $1 \leq j \leq r$, let $f_j : P_1 \times \Pi_j^2 \to \Pi_j^2$ be functions such that, for each fixed x in P_1, $f_j(x, .) : \Pi_j^2 \to \Pi_j^2$ is a bijection. For b_1 in Π_j^1, b_2 in Π_j^2 let's put

$$b_1 * b_2 = \cup \{\{x\} \times f_j(x, b_2) : x \in b_1\}.$$

For $1 \leq j \leq r$, let $\Pi_j = \{b_1 * b_2 : b_1 \in \Pi_j^1, b_2 \in \Pi_j^2\}$. Finally, let $P = P_1 \times P_2$, $L = \cup\{\Pi_j : 1 \leq j \leq r\}$. Then one readily verifies that (P, L) is an $(r, k_1 k_2)$ net with parallel classes Π_j, $1 \leq j \leq r$. We shall denote this net by $X_1 * X_2$.

3.3 The domain net X

We now proceed to prove Theorem 2. Thus we are given an $(n+1, s+1)$ net $\overline{X}_1$. Without loss of generality, we assume that the point set of $\overline{X}_1$ is

$$P_1 = PG(1,s) \times PG(1,s),$$

and that

$$\Pi = \{\{x\} \times PG(1,s) : x \in PG(1,s)\} \qquad 3.1$$

is a parallel class of $\overline{X}_1$.

Let X_1 be the $(n, s+1)$ subnet of $\overline{X}_1$ obtained by deleting the lines in the parallel class Π. Let X_2 be any (n, s) subnet of $AG(2,s)$. Then the parallel classes of X_2 inherit the natural indexing from $AG(2,s)$. Let us say that the parallel classes of X_2 are Π_m^2, $m \in T$, where T is a subset of size n of $PG(1,s)$ and Π_m^2 consists of the lines of $AG(2,s)$ of slope m. Let us also index the parallel classes of X_1 arbitrarily by the same subset T of $PG(1,s)$. Thus the parallel classes of X_1 are Π_m^1, $m \in T$.

We shall take the product net $X = X_1 * X_2$ to be the domain of the covering morphism under construction. To complete the description of X we must specify the functions (see 3.2) $f_m: P_1 \times \Pi_m^2 \to \Pi_m^2$, $m \in T$. If $m \neq 1$, take $f_m(x,b) = b$. If $m = 1$, $x = (\alpha,\beta) \in P_1 = (PG(1,s) \times PG(1,s)$ and b is the unique line in Π_1^2 of intercept $c \in F_s$, let $f_1(x,b)$ be the unique line in Π_1^2 of intercept c', where we set $c' = c + \alpha$ if $\alpha \neq \infty$, and $c' = c$ if $\alpha = \infty$.

Note that if $n \neq s+1$, we may choose the set T so that 1 does not belong to T. Such a choice simplifies our construction considerably.

3.4 The covering morphism

We now define a function $\phi: P_1 \times P_2 \to P_2$ such that ϕ is a covering morphism from X to $AG(2,s)$. Here $P_1 = PG(1,s) \times PG(1,s)$ and $P_2 = F_s \times F_s$.

Fix a **nonsquare** element u of F_s. This exists since s is odd. For $\alpha, \beta \in PG(1,s)$, $x, y \in F_s$, we define:

$$\phi(\alpha,\beta,x,y) = \begin{cases} (x+\alpha, y+\alpha) & \text{if } \alpha \neq \infty \\ (x, u^{-1}y) & \text{if } \alpha = \beta = \infty \\ (u\beta x + y, x + \beta y) & \text{if } \alpha = \infty, \beta \neq \infty. \end{cases} \tag{3.2}$$

3.5 Proof of Theorem 2

To verify that ϕ is indeed a covering morphism we fix m in $T \subseteq PG(1,s)$ and examine the action of ϕ on the lines of Π_m. Take b in Π_m. Then $b = b_1 * b_2$ with b_i in Π_m^i, $i = 1,2$. Let the intercept of b_2 be c. Note that since b_1 is a line of $\bar{X}_1$ outside Π, (3.1) implies:

For each h in $PG(1,s)$ there is a unique k in $PG(1,s)$ with $(h,k) \in b_1$.(3.3)

In particular, let k be the uniue point of $PG(1,s)$ such that $(\infty,k) \in b_1$. Note that the function

$$b \to (k,c)$$

is a bijection from Π_m onto $PG(1,s) \times F_s$. (3.4)

In view of (3.2), (3.3) and the definition of $*$, it is immediate that the restriction of ϕ to $b - \{(\infty,k)\} \times b_2$ is a bijection onto $P_2 = F_s \times F_s$. Also, ϕ maps $\{(\infty,k)\} \times b_2$ bijectively onto the line of $AG(2,s)$ of slope m' and intercept c', where

$$m' = \begin{cases} (mk+1)/(uk+m) & \text{if } m \neq \infty \\ k & \text{if } m = \infty, \end{cases}$$

$$c' = \begin{cases} c(uk^2-1)/(uk+m) & \text{if } k \neq \infty, m \neq -uk, m \neq \infty \\ c & \text{if } k \neq \infty, m = -uk \\ c(1-uk^2) & \text{if } k \neq \infty, m = \infty \\ c/u & \text{if } k = \infty, m \neq \infty \\ c & \text{if } k = \infty, m = \infty. \end{cases}$$

This verifies part (1) of the definition of covering morphism with $\alpha = 2, \beta = 1$.

The fact that u is a fixed non-square, together with the above formulae, shows that for each fixed m the map

$$(k, c) \rightarrow (m', c')$$

is a bijection of $PG(1, s) \times F_s$ onto itself. From this fact and (3.4) we deduce that $\bar{\phi}$ restricted to Π_m is a bijection from Π_m onto the set of all lines of $AG(2, s)$. This verifies part (2) of the definition of covering morphism.

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