Super-free Graphs

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Abstract. Using the definition of k-free, a known result can be re-stated as follows: If G is not edge-reconstructible then G is k-free, for all even k. To get closer, therefore, to settling the Edge-Reconstruction Conjecture, an investigation is begun into which graphs are, or are not, k-free (for different values of k, in particular for $k = 2$); we also discuss which graphs are k-free, for all k.

1. Introduction

The graphs in this paper are connected simple graphs with n vertices. Such a graph G will be considered as a spanning subgraph of K_n.

Definition: Suppose that G is a graph and that $1 \leq k \leq |E(G)|$. Then G is k-**free** if, for every subset A of $E(G)$ with $|A| = |E(G)| - k$, there exists an automorphism ϕ of K_n such that $E(G) \cap E(\phi(G)) = A$. A graph is **even-free** if it is k-free, for all even k. A graph is **super-free** if it is k-free, for all k.

Now the following result is known (see Nash-Williams [3]):

Lemma 1. *If G is not edge-reconstructible then, for every subset A of $E(G)$ such that $|A| \equiv |E(G)|$ (modulo 2), there exists an automorphism ϕ of K_n such that $E(G) \cap E(\phi(G)) = A$.*

This lemma says that if a graph is not edge-reconstructible then it is even-free, and so a graph that is not k-free, for some even value of k, is edge-reconstructible. An investigation into which graphs are, or are not, k-free may therefore bring closer the settling of the Edge-Reconstruction Conjecture. In subsequent sections of this paper, this question is considered for different classes of graph.

The above definition, with some related remarks, was presented in [5], and Theorems 1 and 3 below were announced there, without proofs.

The concept is related to that of fixing subgraph introduced in [4]. If G has a fixing subgraph U then G is not k-free, for all $k \leq |E(G)| - |E(U)|$. This is elaborated upon in Section 5 below.

The following method is used in practice to determine whether or not a graph is k-free. Let $\{e_1, e_2, \dots, e_k\}$ be a set of edges of a graph G. A **replacing set** is any set of edges that can be added to $G - \{e_1, e_2, \dots, e_k\}$ to form a graph isomorphic to G. The replacing set $\{f_1, f_2, \dots, f_k\}$ is called a **disjoint replacing set** if $\{e_1, e_2, \dots, e_k\} \cap \{f_1, f_2, \dots, f_k\} = \emptyset$. If there is a disjoint replacing set for $\{e_1, e_2, \dots, e_k\}$, we shall say that the set $\{e_1, e_2, \dots, e_k\}$ is **replaceable**.

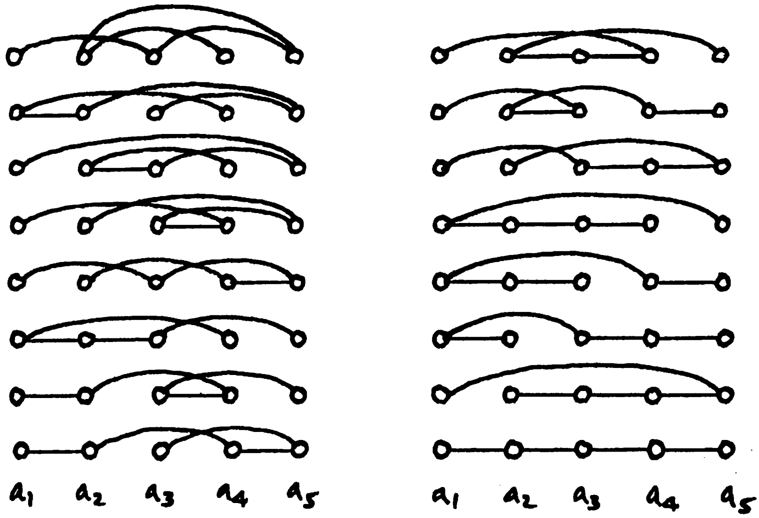


Figure 1

Lemma 2. *The graph G is k-free if and only if every set of k edges is replaceable.*

Proof: Suppose that G is k-free. Let $\{e_1, e_2, \ldots, e_k\}$ be a set of k edges and let $A = E(G) - \{e_1, e_2, \ldots, e_k\}$. Then there is an automorphism ϕ such that $E(G) \cap E(\phi(G)) = A$. Now $E(G) = A + \{e_1, e_2, \ldots, e_k\}$ so $E(\phi(G)) = A + \{f_1, f_2, \ldots, f_k\}$, where $\{e_1, e_2, \ldots, e_k\} \cap \{f_1, f_2, \ldots, f_k\} = \emptyset$. Since $\phi(G)$ is isomorphic to G, $\{f_1, f_2, \ldots, f_k\}$ is a disjoint replacing set for $\{e_1, e_2, \ldots, e_k\}$.

The converse is equally straightforward.

2. Paths

The path P_3, with 3 vertices, is 1-free but not 2-free. But we have the following:

Theorem 1. *The path P_n $(n \geq 4)$ is super-free.*

Proof: The result can be verified for $n = 4$. The proof for $n \geq 5$ is by induction and proves the stronger result:

Let G be the path P_n $(n \geq 5)$, with vertices $a_1, a_2, \ldots, a_n$ and edges $a_i a_{i+1}$ $(i = 1, \ldots, n-1)$. Let A be a subset of $E(G)$. Then there exists an automorphism ϕ of K_n such that $E(G) \cap E(\phi(G)) = A$. Moreover, if $a_1 a_2 \notin A$, there exists such a ϕ such that a_1 has degree 1 in $\phi(G)$.

(i) Let G be the path P_5. The result can be checked and is illustrated in the figure. For each of the 2^4 subsets A, a diagram showing $\phi(G)$ is shown in Figure 1. The vertex a_1 is on the left-hand end and it can be seen that, when $a_1 a_2 \notin A$, a_1 has degree 1 in $\phi(G)$.

(ii) Now let G be the path P_n ($n > 5$) and A any subset of $E(G)$. Suppose first that $a_1a_2 \notin A$. Let P' be the path (a subgraph of G) with vertices $a_2, \ldots, a_n$. Then $A \subseteq E(P')$. By the induction hypothesis, there is an automorphism ϕ' such that $E(P') \cap E(\phi'(P')) = A$. Now $\phi'(P')$ has two vertices of degree 1; one of them, b say, is not a_2. Then $\phi'(P') + a_1b$ is isomorphic to G and so is equal to $\phi(G)$, for some ϕ. It is clear that $E(G) \cap E(\phi(G)) = A$ and that a_1 has degree 1 in $\phi(G)$.

Suppose now that $a_1a_2 \in A$. Let j be the smallest value such that $a_{j-1}a_j \in A$ and $a_ja_{j+1} \notin A$. (If $j = n$, take the identity automorphism for ϕ.) Let P' be the path (a subgraph of G) with vertices $a_j, a_{j+1}, \ldots, a_n$.

First, suppose that $n-j \geq 4$. Let $A' = A \cap E(P')$. By the induction hypothesis, there exists a ϕ' such that $E(P') \cap E(\phi'(P')) = A'$ and a_j has degree 1 in $\phi'(P')$. Then $\phi'(P') + \{a_1a_2, a_2a_3, \ldots, a_{j-1}a_j\}$ is isomorphic to G and so is equal to $\phi(G)$, for some ϕ, with the right property.

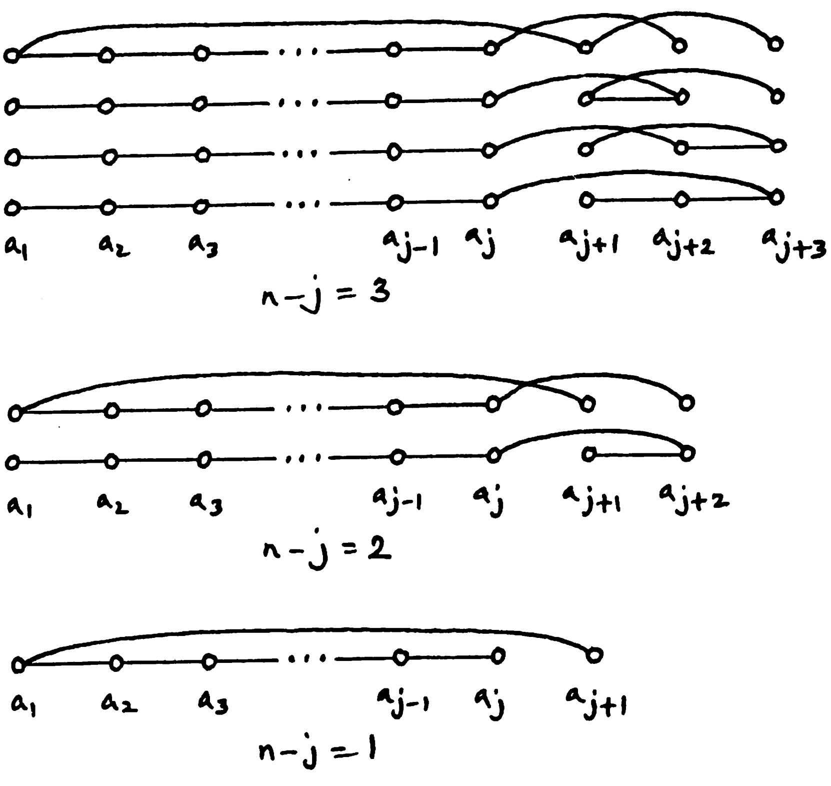


Figure 2

Suppose that $n - j = 3$. There are 4 possibilities for A and the existence of ϕ is illustrated in Figure 2. Also shown are the two cases when $n - j = 2$ and the case when $n - j = 1$. ∎

3. Trees

The only 1-free trees (apart from paths) are the trees T_1 and T_2, given in Theorem 10 of [4], and shown in Figure 3. It is easy to check that these two are, in fact, super-free. So they are the only super-free trees (apart from paths).

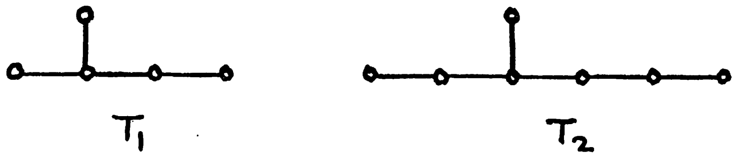


Figure 3

We are at present determining all 2-free trees; it seems likely that there are just a finite number of them (apart from paths). Here is a first result on 2-free trees:

Theorem 2. *The only 2-free bidegreed trees (apart from paths) are the three trees shown in Figure 4.*

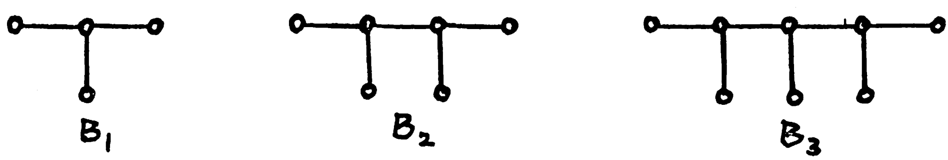


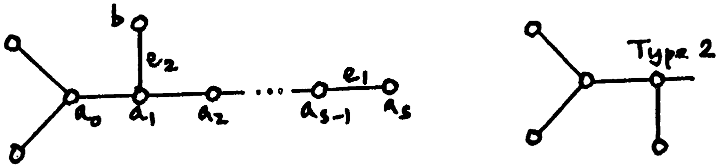
Figure 4

Proof: Let G be a bidegreed tree (not a path). Then G has at least two vertices of degree 1. Suppose that the vertices of G have degrees 1 and Δ (≥ 3).

Case (i): $\Delta \geq 4$. In G, there is a vertex of degree Δ adjacent to at least three vertices of degree 1. (If not, the deletion of all the pendant edges gives a subgraph with degrees ≥ 2, which thus has a cycle—a contradiction.) So let a_0 be a vertex of degree Δ with adjacent vertices a_1, a_2 and a_3 of degree 1. Then $\{a_0a_1, a_0a_2\}$ is clearly not replaceable. So there are no such 2-free bidegreed trees.

Case (ii): $\Delta = 3$. In G, there is a vertex of degree 3 adjacent to at least two vertices of degree 1. (Otherwise, the deletion of all pendant edges again would give a subgraph which would have to have a cycle.) Let a vertex of degree 3 adjacent to exactly two vertices of degree 1 be called a 'type 1' vertex. If there is no type 1 vertex, G is the tree B_1. Otherwise, consider a shortest path, of length s (≥ 2), from a type 1 vertex to a non-adjacent vertex of degree 1.

If $s \geq 3$, let $a_0, a_1, \ldots, a_s$ be the vertices of such a shortest path, where a_0 is the type 1 vertex and a_s has degree 1. Let b be the other vertex adjacent to a_1, as shown in Figure 5(a). For the set $\{a_{s-1}a_s, a_1b\}$ it can be seen that the only disjoint replacing set could be $\{a_1a_s, a_{s-1}b\}$; but then in $G - \{a_{s-1}a_s, a_1b\} + \{a_1a_s, a_{s-1}b\}$, which is isomorphic to G, there is a path of length 2 from a type 1 vertex to a non-adjacent vertex of degree 1, a contradiction.



(a) (b)

Figure 5

So $s = 2$. Therefore, there is a vertex adjacent to both a type 1 vertex and at least one vertex of degree 1. Let a vertex adjacent to both a type 1 vertex and exactly one vertex of degree 1 be called a 'type 2' vertex. (See Figure 5(b).) If there is no type 2 vertex then G is the tree B_2. Otherwise, it can be shown, in the same way, that the shortest distance from a type 2 vertex to a non-adjacent vertex of degree 1 is equal to 2, and the argument is continued. Thus G is one of the family of graphs B_i, in which, for $i \geq 2$, B_i has i vertices of degree 3 forming a path, the end-vertices of which are each adjacent to two vertices of degree 1, and the others each adjacent to one vertex of degree 1.

It is easy to show that the first three of this family, which are shown above, are indeed 2-free and that the others are not. ∎

We note that B_1 is exactly 2-free (i.e., 2-free but not k-free for $k \neq 2$); B_2 and B_3 are k-free, for all $k \geq 2$.

We conjecture that these are the only 2-free bidegreed graphs (that are not paths). A proof of this conjecture—namely that all other bidegreed graphs (apart from paths) are not 2-free—would, by Lemma 1 above, give the result of Myrvold, Ellingham and Hoffman [2], that bidegreed graphs are edge-reconstructible.

4. Cycles

For cycles, we have the following result:

Theorem 3. *The cycle C_n is not k-free, for $k \leq 4$. The cycle C_n ($n \geq 5$) is k-free, for all $k \geq 5$.*

Proof: In the cycle C_n, it is clear that, for $1 \leq k \leq 4$, k consecutive edges are not replaceable. So C_n is not k-free, for these values of k.

Now let G be the cycle C_n ($n \geq 5$). Let A be a subset of $E(G)$ such that $|A| = |E(G)| - k$, with $k \geq 5$. If $A = \emptyset$, there certainly exists an automorphism ϕ

such that $E(G) \cap E(\phi(G)) = A$, since, for $n \geq 5$, there is a packing of C_n and C_n.

So suppose that $A \neq \emptyset$. Label the vertices of G (= C_n) as $a_1, a_2, \ldots, a_n$, with edges $a_1a_2, \ldots, a_{n-1}a_n, a_na_1$, in such a way that $a_na_1 \notin A$, $a_1a_2, \ldots, a_{i-1}a_i \in A$ and $a_ia_{i+1} \notin A$. Let P_{n-i} be the path (a subgraph of G) with vertices $a_{i+1}, a_{i+2}, \ldots, a_n$. There are at least 3 more edges in $E(G) - A$ and these belong to $E(P_{n-i})$, so $|E(P_{n-i})| \geq 3$, i.e., $n - i \geq 4$. Let $A' = A \cap E(P_{n-i})$. Since P_{n-i} is super-free, by Theorem 1, there is an automorphism ϕ' such that $E(P_{n-i}) \cap E(\phi'(P_{n-i})) = A'$. Let b and c be the two vertices of $\phi'(P_{n-i})$ of degree 1; label them so that b is not a_n and c is not a_{i+1}. Then $\phi'(P_{n-i}) + \{ba_1, a_1a_2, \ldots, a_{i-1}a_i, a_ic\}$ is isomorphic to G (= C_n) and so is $\phi(G)$, for some ϕ, and $E(G) \cap E(\phi(G)) = A$. ∎

5. Regular graphs

There are no super-free regular graphs, because regular graphs are clearly not 1-free. Indeed, we have the following:

Theorem 4. *Let G be a regular graph of degree r. Then G is not k-free, for all $k \leq 2r$.*

Proof: If G is a complete graph then any set of edges is not replaceable, so G is not k-free, for all k.

If G is not complete, let u and v be two vertices such that $d(u, v) = 2$. Let $e_1, \ldots, e_r$ be the edges incident with u and $e_{r+1}, \ldots, e_{2r}$ the edges incident with v. Then, for all $k \leq 2r$, the set $\{e_1, e_2, \ldots, e_k\}$ is not replaceable. ∎

Results about other regular graphs also follow from [4]. A spanning subgraph U of G is a **fixing subgraph** of G if, whenever $E(G) \cap E(\phi(G)) \supseteq E(U)$, for some automorphism ϕ of K_n, $E(G) \cap E(\phi(G)) = E(G)$. Moreover, if U is a fixing subgraph of G and $U \subseteq V \subseteq G$ then V is also a fixing subgraph of G. Consequently, if G has a fixing subgraph U then G is not k-free, for all $k \leq |E(G)| - |E(U)|$.

In particular, consider the Tutte cages. In [4], it was shown that each i-cage ($i = 3, 4, 5, 6, 8$) has a fixing subgraph (in fact, a tree) with $n - 1$ edges. Since $|E(G)| = \frac{3}{2}n$, it follows that the i-cage is not k-free, for $k \leq \frac{3}{2}n - (n-1) = \frac{1}{2}n + 1$.

6. Self-complementary graphs

A graph is $|E(G)|$-free if and only if there is a packing of G and $\bar{G}$. Let G be a graph with $|E(G)| = \frac{1}{4}n(n-1)$, where $n \equiv 0$ or $1 \pmod 4$. Then it is immediate that G is self-complementary if and only if G is $|E(G)|$-free. We also have the following result for self-complementary graphs:

Theorem 5. *Let G be a self-complementary graph, with n vertices and q ($= \frac{1}{4}n(n-1)$) edges. Then G is k-free if and only if it is $(q-k)$-free.*

Proof: Suppose that G is k-free. Let A be a subset of $E(G)$ such that $|A| = k$. Let $B = E(G) - A$, so that $|B| = q - k$.

Since G is k-free, there is an automorphism ϕ with $E(G) \cap E(\phi(G)) = B$. Let $D = E(K_n) - (E(G) \cup E(\phi(G)))$. A Venn diagram is shown in Figure 6.

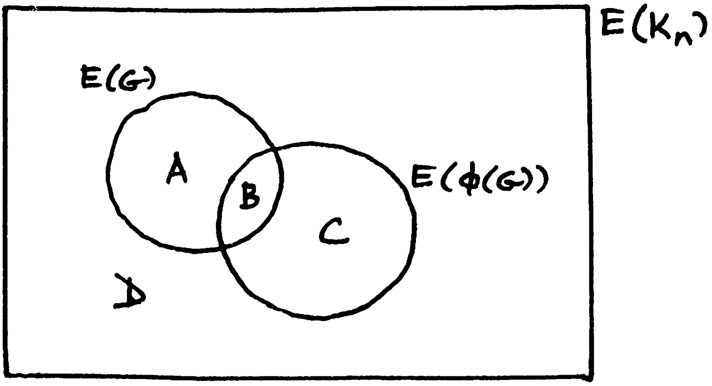


Figure 6

Since $E(K_n) = 2q$, $|D| = 2q - (q + q - (q-k)) = q - k$. Now $\phi(G)$ is self-complementary so there is a mapping σ that maps $\phi(G)$ onto its complement, that is, $E(\sigma(\phi(G))) = A \cup D$. So $E(G) \cap E(\sigma\phi(G)) = A$. Since such a mapping (namely, $\sigma\phi$) exists for all A, G is $(q-k)$-free. ∎

7. Conclusion

The interesting problem remains of determining all super-free graphs.

A bound on the number of edges is known. For if G is super-free then, for every subset A of $E(G)$, there exists an automorphism ϕ of K_n such that $E(G) \cap E(\phi(G)) = A$. So, as in Müller [1], it follows that $2^{|E(G)|} \leq n!$ and we obtain the bound $|E(G)| \leq n \log_2 n$.

The class of unicyclic graphs was considered in [5]. There it was explained how, from results in [4], the 1-free unicyclic graphs are known and how it follows that the only super-free unicyclic graphs are the three, M_0, M_2 and M_3, shown in Figure 7.

Thus the super-free graphs known so far are: (i) the paths P_n ($n \geq 4$), (ii) the trees T_1 and T_2 given in Section 3, and (iii) the unicyclic graphs M_0, M_2 and M_3.

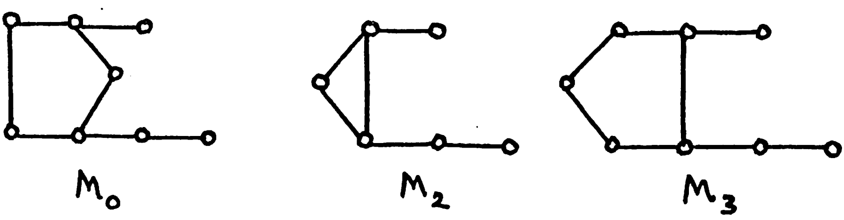


Figure 7

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