On the Chromatic Number of Some Rational Spaces

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Let Q^d denote the collection of all the rational points of the d-space E^d, and let $G(Q^d)$ denote the graph, obtained by taking Q^d as its vertex set, and connecting two points if they are at distance one.

It is well known that $\chi(G(Q^2)) = 2$ [7], and that $\chi(G(Q^3)) = 2$ and $\chi(G(Q^4)) = 4$ [1]; Benda and Perles asked in [1] for the value of $\chi(G(Q^5))$. Clearly, $\chi(G(Q^5)) \geq 4$; we [8] showed that $\chi(G(Q^6)) \geq 6$, $\chi(G(Q^7)) \geq 8$ and $\chi(G(Q^8)) \geq 9$. These results follow also from [2], where the maximum clique number $\omega(G(Q^d))$ of $G(Q^d)$ is determined, and from the fact that $\chi(G) \geq \omega(G)$ for all graphs G.

The purpose of this note is to prove

Theorem 1. $\chi(G(Q^5)) \geq 5$, $\chi(G(Q^6)) \geq 7$, $\chi(G(Q^7)) \geq 9$, *and* $\chi(G(Q^8)) \geq 10$.

We need the following.

Lemma 1. *If A and B are two different points of Q^d, then the reflection of E^d with respect to the hyperplane which is the perpendicular bisector of the segment AB, is a rational transformation (that is, takes rational points to rational points).*

Proof of Lemma 1: The reflection $f: E^d \to E^d$ with respect to the hyperplane which is the perpendicular bisector of the segment AB, is given by

$$f(x) = x - 2\frac{\langle (x - \frac{A+B}{2}) \cdot (B - A) \rangle}{\langle (B - A) \cdot (B - A) \rangle} \cdot (B - A).$$

f is clearly a rational transformation, $f: Q^d \to Q^d$.

Notice that the said hyperplane contains the origin if and only if A and B are at equal distances from the origin, which happens if and only if $\langle (A + B) \cdot (B - A) \rangle = 0$; in this case $f(x)$ is given by

$$f(x) = x - 2\frac{\langle x \cdot (B - A) \rangle}{\langle (B - A) \cdot (B - A) \rangle} (B - A).$$

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Lemma 2. *Let O (the origin), $F_1, \ldots, F_k$ be a unit-distance k-simplex in Q^d, $k \geq 2$, and let $A = (a_1 \ldots a_d) = \frac{2}{k}\sum_{i=1}^{k} F_i$. If there exists a* rational *solution to the system*

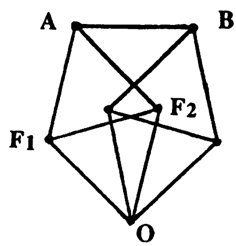
$$\sum_{i=1}^{d} x_i^2 = \sum_{i=1}^{d} a_i^2 \tag{1}$$

$$\sum_{i=1}^{d} (x_i - a_i)^2 = 1 \tag{2}$$

then $\chi(G(Q^d)) \geq k + 2$.

Proof of Lemma 2: A is chosen such that the points $A, F_1, \ldots, F_k$ form another unit-distance k-simplex; if $B = (x_1 \ldots x_d)$ is the said solution to the system (1,2) then $d(O,B) = d(O,A)$ and $d(A,B) = 1$. Thus, if $\chi(G(Q^d)) = k + 1$, then the points O and A must have the same colour. Let f be the reflection of E^d with respect to the hyperplane which is the perpendicular bisector of the segment AB. By Lemma 1, f is a rational transformation, $O = f(O)$ and $f(A) = B$. It follows by considering $\{O, f(F_1), \ldots, f(F_k), B\}$ that O and B have the same colour, therefore, A and B have the same colour, which is a contradiction, since $d(A,B) = 1$. Thus, $\chi(G(Q^d)) \geq k + 2$. ∎

The idea of the proof is based on L. Moser and W. Moser [6] claim that $\chi(G(E^2)) \geq 4$ (see also [**4, 5**]); this claim is proved by using the following well-known configuration, in which all the segments have unit length.



Proof of the Theorem: It suffices to give the points $F_1, \ldots, F_k$ A and B in each case, as follows:

To show $\chi(G(Q^5)) \geq 5$, take

$F_1 = (0,0,0,-1,0)$, $F_2 = \frac{1}{2}(-1,1,1,-1,0)$, $F_3 = \frac{1}{2}(-1,1,-1,-1,0)$, $k = 3$, $A = \frac{2}{3}(-1,1,0,-2,0)$ and $B = \frac{1}{12}(-15,2,3,-11,5)$.

To show $\chi(G(Q^6)) \geq 7$, take

$F_1 = (0, \ldots, 0, -1)$, $F_2 = \frac{1}{2}(-1, 1, 0, 0, 1, -1)$,
$F_3 = \frac{1}{2}(-1, 1, 0, 0, -1, -1)$, $F_4 = \frac{1}{2}(-1, 0, 1, 1, 0, -1)$,
$F_5 = \frac{1}{2}(-1, 0, 1, -1, 0, -1)$, $k = 5$,
$A = \frac{2}{5}(-2, 1, 1, 0, 0, -3)$ and $B = \frac{1}{20}(-28, -1, 1, 2, 1, -13)$.

To show $\chi(G(Q^7)) \geq 9$, take

$F_1 = (-1, 0, \ldots, 0)$, $F_2 = \frac{1}{2}(-1, 1, 0, 0, 1, -1, 0)$,
$F_3 = \frac{1}{2}(-1, -1, 0, 0, 1, -1, 0)$, $F_4 = \frac{1}{2}(-1, 0, 1, 1, 0, -1, 0)$,
$F_5 = \frac{1}{2}(-1, 0, -1, 1, 0, -1, 0)$, $F_6 = \frac{1}{2}(-1, 0, 0, 1, 1, 0, 1)$,
$F_7 = \frac{1}{2}(-1, 0, 0, 1, 1, 0, -1)$, $k = 7$,
$A = \frac{4}{7}(-2, 0, 0, 1, 1, -1, 0)$ and $B = \frac{1}{56}(-84, 1, 1, 10, 0, 3, 1)$.

Finally, to show $\chi(G(Q^8)) \geq 10$, take

$F_1 = (-1, 0, \ldots, 0)$, $F_2 = \frac{1}{2}(-1, 1, 0, 0, 0, 0, 1, -1)$,
$F_3 = \frac{1}{2}(-1, -1, 0, 0, 0, 0, 1, -1)$, $F_4 = \frac{1}{2}(-1, 0, 1, 0, 0, 1, 0, -1)$,
$F_5 = \frac{1}{2}(-1, 0, -1, 0, 0, 1, 0, -1)$, $F_6 = \frac{1}{2}(-1, 0, 0, 1, 1, 0, 0, -1)$,
$F_7 = \frac{1}{2}(-1, 0, 0, -1, 1, 0, 0, -1)$, $F_8 = \frac{1}{2}(-1, 0, 0, 0, 1, 1, 1, 0)$, $k = 8$,
$A = \frac{3}{8}(-3, 0, 0, 0, 1, 1, 1, -2)$ and $B = \frac{1}{6}(-8, 1, 1, 3, 1, 0, -1, -2)$.

This completes the proof of the theorem. ∎

Notice that we have used maximum cliques [2] of $G(Q^d)$ $5 \leq d \leq 8$, taken from ([8], see also [2]) and shifted them to the origin. The technique is useless for $d \geq 10$, since $\chi(G(Q^d)) \geq d + 2$ for all $d \geq 10$ (see [3]).

K.B. Chilakamarri has another proof that $\chi(G(Q^5)) \geq 5$, based on a 10-point configuration (private communication).

References

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