Domination Sequences of Graphs

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Abstract. A dominating set X of a graph G is a k-minimal dominating set of G iff the removal of any $\ell \leq k$ vertices from X followed by the addition of any $\ell-1$ vertices of G results in a set which does not dominate G. The k-minimal domination number $\Gamma_k(G)$ of G is the largest number of vertices in a k-minimal dominating set of G. The sequence $R: m_1 \geq m_2 \geq \cdots \geq m_k \geq \cdots \geq n$ of positive integers is a domination sequence iff there exists a graph G such that $\Gamma_1(G) = m_1, \Gamma_2(G) = m_2, \ldots, \Gamma_k(G) = m_k, \ldots$, and $\gamma(G) = n$, where $\gamma(G)$ denotes the domination number of G. We give sufficient conditions for R to be a domination sequence.

1. Introduction

A set X of vertices of a graph $G = (V, E)$ is a *dominating set* of G iff each vertex in $V - X$ is adjacent to at least one vertex in X. The *domination number* $\gamma(G)$ (*upper domination number* $\Gamma(G)$) of G is the smallest (largest) number of vertices in a minimal dominating set of G.

The concepts of minimality and maximality in general were extended in [1]. The application of this generalisation of minimality to domination parameters in particular results in the following definitions: A dominating set $X \subseteq V$ is a *k-minimal dominating set* of G iff for all $\ell \in \{1, \ldots, k\}$ for all ℓ-subsets Q of X and all $(\ell - 1)$ subsets R of V, $(X - Q) \cup R$ is not a dominating set of G. The *k-minimal domination number* $\Gamma_k(G)$ of G is the largest cardinality of a k-minimal dominating set of G. The k-minimal domination numbers of all paths and cycles are determined in [1] and [3] respectively, while the product of k-minimal domination numbers of a graph and its complement is discussed in [2].

It is clear that for any graph G,

$$\Gamma(G) = \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \geq \Gamma_k(G) \geq \cdots \geq \gamma(G).$$

The sequence

$$A : \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \geq \Gamma_k(G) \geq \cdots \geq \gamma(G)$$

is called the *domination sequence* of G. Further, the sequence

$$R : m_1 \geq m_2 \geq \cdots \geq m_k \geq \cdots \geq n$$

of positive integers is a *domination sequence* iff there exists a graph G such that

$$\Gamma_1(G) = m_1, \Gamma_2(G) = m_2, \ldots, \Gamma_k(G) = m_k, \ldots, \gamma(G) = n.$$

In this paper we begin the study of domination sequences of graphs. We first deduce a simple necessary condition for a sequence R to be a domination sequence and then find a sufficient condition for R to be such a sequence by explicitly constructing a graph having a given R as domination sequence. This supplies a partial solution to the problem of characterising domination sequences, which seems to be a very difficult problem in general.

2. Preliminary Results

We begin by stating a classical result of Ore [4, p. 206] which provides us with a useful method of determining when a dominating set is a minimal dominating set. Let $N_G(U)$ ($N_G[U]$) denote the neighbourhood (closed neighbourhood) in a graph G of a subset U of $V(G)$. If $U = \{u\}$, we also write $N_G(u)$ for $N_G(\{u\})$ and $N_G[u]$ for $N_G[\{u\}]$.

Proposition 1. *A dominating set X of a graph $G = (V, E)$ is a minimal dominating set of G iff for each $x \in X$ one of the following two conditions hold:*

(i) *x is an isolated vertex of $\langle X \rangle$;*

(ii) *there exists a vertex $y \in V - X$ such that $N_G(y) \cap X = \{x\}$.*

Although $\Gamma_k(G)$ is defined for all positive integers k it is clear from the finiteness of G that the sequence A above can contain only finitely many distinct integers. In our next result we show that the maximum number of distinct integers in A depends on the value of $\gamma(G)$.

Proposition 2. *If $R : m_1 \geq m_2 \geq \cdots \geq m_k \geq \cdots \geq n$ is a domination sequence, then $m_i = n$ for all $i \geq n+1$.*

Proof: Let G be a graph having R as its domination sequence and let X be a dominating set of G with $|X| = n$. If Y is a dominating set of G with $|Y| \geq n+1$ and Y' is any $(n+1)$-subset of Y, then $(Y - Y') \cup X$ dominates G, showing that Y is not $(n+1)$-minimal and consequently not i-minimal for $i \geq n+1$. It follows that $\Gamma_i(G) \leq n$ for all $i \geq n+1$ and since $\gamma(G) = n$, $\Gamma_i(G) = n$ for all $i \geq n+1$. ∎

We now devote the rest of this paper to the following question: *For which positive integers k and $m_1, \ldots, m_k, m_{k+1}$ with $m_1 \geq \ldots, \geq m_k \geq m_{k+1} = k$ does there exist a graph G such that*

$$\gamma(G) = k \text{ and } \Gamma_i(G) = m_i, \quad i = 1, \ldots, k?$$

We shall also need the following definition: An ℓ-subset Q of a dominating set X of a graph G is said to be *stable* (*unstable*) iff there does not exist (there exists) an $(\ell-1)$-subset R of $V - X$ such that $(X - Q) \cup R$ is dominating. Notice that a dominating set is k-minimal iff for each $1 \leq \ell \leq k$, all ℓ-subsets of X are stable.

3. Main Construction

Consider the sequence $R : m_1 \geq \cdots \geq m_k \geq m_{k+1} = k$ as above. Let $K = \{1, \ldots, k\}$ and $I' = \{i \in K - \{1\} : m_i > m_{i+1}\}$. If $I' = \phi$ define I as

$$I = \begin{cases} \phi & \text{if } m_1 = m_2 \\ \{1\} & \text{otherwise.} \end{cases}$$

If $I' \neq \phi$, let $a' = \max_{i \in I'}\{i\}$; we assume $m_1 \geq m_2 + a' - 2$ since this is required by the sufficiency condition, and define I as

$$I = \begin{cases} I' & \text{if } m_1 = m_2 + a' - 2 \\ I' \cup \{1\} & \text{otherwise.} \end{cases}$$

Suppose that $|I| = q$ and $I = \{a_1, \ldots, a_q\}$ with $a_1 \leq \cdots \leq a_q = a'$ and define $Q = \{1, \ldots, q\}$. For each $i \in Q$ for which $a_i \geq 2$, let $n_i = m_{a_i} - m_{a_i+1} + a_i - a_{i-1}$, where a_o is defined to be 0. If $a_1 = 1$ let $n_1 = m_1 - m_2 - a_q + 3$ if $a_q \geq 2$, and $n_1 = m_1 - m_2 + 1$ if $a_q = 1$. Note that $m_{a_i+1} = m_{a_i+2} = \cdots = m_{a_{i+1}}$ (where $m_{a_{q+1}}$ denotes $m_{k+1} = k$), and that $n_i \geq 2$ for all $a_i \in I$.

For each $i \in Q$ define $H_i = \overline{K}_{n_i}$ and construct $\widetilde{H}_i$ as follows: Take a_i copies $H_{i1}, \ldots, H_{ia_i}$ of H_i with $V(H_{ij}) = \{v_{ij\lambda} | \lambda = 1, \ldots, n_i\}$ and join the vertices of each H_{ij} to the vertices of each $H_{i\ell}$, $j \neq \ell$; $j, \ell = 1, \ldots, a_i$, by the 1-factor $\{v_{ijr}v_{i\ell r} | r = 1, \ldots, n_i\}$. Note that in this way n_i disjoint copies of K_{a_i} are formed. Add a set $S_i = \{s_{i1}, \ldots, s_{ia_i}\}$ of a_i independent vertices and join each s_{ij} to all vertices of H_{ij} as well as to each vertex $v_{i\ell 1}$, $\ell \neq j$; $\ell, j = 1, \ldots, a_i$. The resulting graph is $\widetilde{H}_i$.

Now form G_R by joining the graphs $\widetilde{H}_i$ recursively as follows: Let $G_1 = \widetilde{H}_1$ and let G_i be the graph obtained by joining $\widetilde{H}_i$ to G_{i-1} with the edges $s_{ir}s_{jr}$ where $j < i$ and $r \leq a_j$, and all edges from s_{ir} to H_{jr} where $j < i$ and $r \leq a_j$. Then

$$G_R = \begin{cases} G_q & \text{if } a_q = k \\ G_q \cup \overline{K}_{k-a_q} & \text{if } a_q < k \\ \overline{K}_k & \text{if } I = \phi. \end{cases}$$

The construction of G_R is illustrated in Figure 1.

4. Lower Bounds for $\Gamma_i(G_R)$

Our purpose is to impose suitable conditions on R so that it will be the domination sequence of G_R. In this section we formulate and prove various lemmas concerning dominating sets of G_R and its subgraphs and finally establish that $\Gamma_i(G_R) \geq m_i$ for all $i \geq 2$. It is significant that no restrictions on the m_i's are necessary to obtain this result.

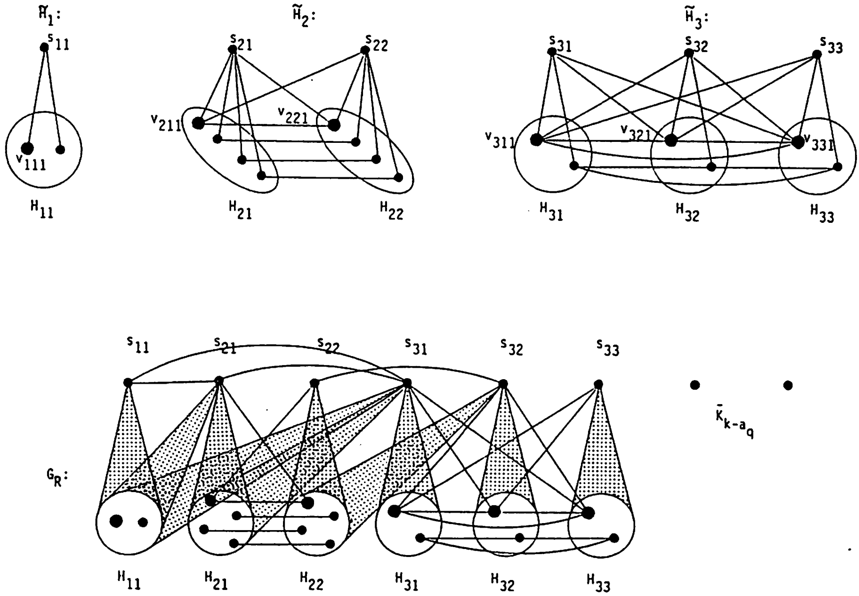

Figure 1: Illustration of construction with $m_1 = 11, m_2 = 9,$
$m_3 = 6, m_4 = m_5 = m_6 = k = 5$

Lemma 1. *For any* $i \in Q$ *and any* $j = 1, \ldots, a_i$, $V(H_{ij})$ *is an* a_i*-minimal, but, if* $n_i > a_i$, *not an* $(a_i + 1)$*-minimal dominating set of* $\widetilde{H}_i$.

Proof: It is clear that $D = V(H_{ij})$ dominates $V(H_{i\ell}), \ell = 1, \ldots, a_i$ as well as S_i. Moreover, if $|D| > a_i$ then, since $|S_i| = a_i$ and S_i dominates $\widetilde{H}_i$, no $(a_i + 1)$-subset of D is stable. We now show that D is a_i-minimal.

For $r \leq a_i$, let X be any r-subset of D and suppose, contrary to the required result, that there exists an $(r-1)$-subset $Y \cup Y'$ of $V(\widetilde{H}_i)$ such that

$$Y \subseteq S_i,$$
$$Y' \subseteq \cup_{\ell=1}^{a_i} V(H_{i\ell}),$$
$$Y \cup Y' \cup (D - X) \text{ dominates } \widetilde{H}_i.$$

If $Y = \phi$, in order to dominate $\cup_{\ell=1}^{a_i} V(H_{i\ell})$ we must have $|Y'| \geq r$, a contradiction. If $Y \neq \phi$ then at least $r-1$ vertices of at least one H_{ij} are undominated by $Y \cup (D - X)$. Therefore $|Y'| \geq r - 1$ and $|Y \cup Y'| \geq r$, a contradiction. ∎

It is important to realise that a dominating set $X \subseteq \cup_{j=1}^{a_i} V(H_{ij})$ of $\widetilde{H}_i$ is not necessarily contained in $V(H_{ij})$ for some j. Some properties of such dominating sets are given in Lemma 2 while further properties are discussed in Lemma 4(ii).

Lemma 2. *For any $i \in Q$ and any dominating set X of $\widetilde{H}_i$ with $X \subseteq \cup_{j=1}^{a_i} V(H_{ij})$, the following conditions hold:*

(i) $|X| \geq n_i$;
(ii) X *is* a_i*-minimal if and only if* $|X| = n_i$;
(iii) *if* $|X| > n_i$ *then* X *is not a minimal dominating set.*

Proof: Clearly, (i) and (iii) follow directly from the construction of $\widetilde{H}_i$; the sufficiency of (ii) can be proved similarly to Lemma 1 while (iii) implies the necessity of (ii). ∎

In order to find lower bounds for the upper domination numbers of G_R, we first prove that $\Gamma_{a_i}(G_i) \geq m_{a_i} - m_{a_{i+1}} + a_i$.

Lemma 3. *For any $i \in \{2, \ldots, q\}$ and any $j = 1, \ldots, a_{i-1}$, let $D = V(H_{ij})$ and $D_i = D \cup S_{i-1}$. Further, let $D_1 = V(H_{1j})$ for any $j = 1, \ldots, a_1$. Then each D_i is an a_i-minimal but not $(a_i + 1)$-minimal dominating set of G_i containing, if $a_i \geq 2$, precisely $m_{a_i} - m_{a_{i+1}} + a_i$ vertices.*

Proof: It is easy to see that $|D_i| = m_{a_i} - m_{a_{i+1}} + a_i$ if $a_i \geq 2$. Also, referring to the construction of G_R, it is clear that S_{i-1} dominates G_{i-1} and hence D_i dominates G_i. Since S_i dominates G_i and $|D_i| \geq a_i + 1$, the removal of any $a_i + 1$ vertices from D_i followed by the addition of S_i yields a dominating set D' of G_i with $|D'| = |D_i| - 1$, showing that D_i is not $(a_i + 1)$-minimal. We now show by induction over i that D_i is an a_i-minimal dominating set of G_i.

If $i = 1$, then $S_{i-1} = \phi$ and D_1 is a_1-minimal by Lemma 1. For $i \geq 2$ suppose that D_{i-1} is an a_{i-1}-minimal dominating set of G_{i-1} for any set D_{i-1} satisfying the hypothesis of the lemma and consider any dominating set of G_i of the form $D_i = D \cup S_{i-1}$. For $p \leq a_i$ let X be any p-subset of D_i and suppose $|X \cap D| = r$, $|X \cap S_{i-1}| = t$. The following vertices are not dominated by $Z = D_i - X$:

(i) r vertices of H_{ij} for each $j = 1, \ldots, a_i$, and possibly vertices of S_i;
(ii) all vertices of t copies of H_{i-1} if $t = a_{i-1}$, or the vertices other than $v_{(i-1)j1}$ of t copies of H_{i-1} if $t < a_{i-1}$;
(iii) the vertices in at least $a_{i-\ell} - a_{i-1} + t$ copies of $H_{i-\ell}, \ell \geq 2$, (since $a_{i-1} - t$ vertices of S_{i-1} remain in D_i and they dominate at most $a_{i-1} - t$ copies of $H_{i-\ell}$);
(iv) t vertices of S_{i-1} and at least $a_{i-\ell} - a_{i-1} + t$ vertices of $S_{i-\ell}, \ell \geq 2$.

Let $Y \subseteq V(G_i)$ be such that $Z \cup Y$ dominates the vertices in (ii)-(iv) above and such that Y is minimal w.r.t. this property. Then $Y \cap (\cup_{j=1}^{a_i} V(H_{ij})) = \phi$, $Y \cap \{s_{i(a_{i-1}+1)}, \ldots, s_{ia_i}\} = \phi$ and for each $j = 1, \ldots, a_{i-1}$, at most one of s_{ij} and $s_{(i-1)j}$ is contained in Y. We now consider two cases.

Case 1: $|Y \cap (S_i \cup S_{i-1})| = c \geq t$.
Then $|Y \cap S_i| = b$, $|Y \cap S_{i-1}| = b_1$ with $b + b_1 = c$. In order to dominate the vertices in (i) above with as few vertices as possible, a set Y' has to be added

to $Z \cup Y$, where Y' contains d vertices of S_i, $d \in \{0, \ldots, a_i - b\}$ and 0, at least $r-1$, or r vertices in $\cup_{j=1}^{a_i} V(H_{ij})$ depending on whether $b + d = a_i$, $1 \leq b + d < a_i$ or $b + d = 0$ respectively. Clearly, if $b + d = 0$ or $b + d = a_i$, then $|Y \cup Y'| = |Y| + |Y'| \geq t + r = p$. If $1 \leq b + d < a_i$ and $d \neq 0$, then $|Y \cup Y'| \geq t + d + r - 1 \geq t + r$. Further, if $d = 0$ and $1 \leq b < a_i$, then, since $r \leq a_i - t$ and $D = V(H_{ij})$ for some $j = 1, \ldots, a_{i-1}$, at least one vertex $s_{i\ell}$ with $\ell \in \{a_{i-1}+1, \ldots, a_i\}$ or a vertex of $\cup_{j=1}^{a_i} V(H_{ij})$ is not dominated by $Z \cup Y \cup Y'$ if $|Y' \cap (\cup_{j=1}^{a_i} V(H_{ij}))| = r - 1$. Hence in this case Y' needs to contain r vertices in $\cup_{j=1}^{a_i} V(H_{ij})$ so that $|Y \cup Y'| \geq t + r = p$. It follows that X is stable, thus settling this case.

Case 2: $|Y \cap (S_i \cup S_{i-1})| = c < t$.
Then Y contains at least $n_{i-1} - 1$ vertices of $\cup_{j=1}^{a_{i-1}} V(H_{(i-1)j})$. Let Y' be the set obtained by substituting each vertex s_{ij} of Y with $s_{(i-1)j}$; note that $|Y'| = |Y|$. Furthermore, let Y'' be the set obtained by substituting $n_{i-1} - 1$ vertices of $Y' \cap (\cup_{j=1}^{a_{i-1}} V(H_{(i-1)j})$ with $n_{i-1} - 1$ corresponding vertices of $H_{(i-1)1}$. Again $|Y''| = |Y'| = |Y|$. Note that $(Z \cap S_{i-1}) \cup Y'' = B$ is a dominating set of G_{i-1}. Let $D_{i-1} = V(H_{(i-1)1}) \cup S_{i-2}$, where $S_o = \phi$. By the induction hypothesis, D_{i-1} is an a_{i-1}-minimal dominating set of G_{i-1}. But then $|B| \geq |D_{i-1}|$, for if $|B| < |D_{i-1}|$, then B contains at most a_{i-2} vertices not in D_{i-1} and can be obtained by removing $a_{i-2} + 1 \leq a_{i-1}$ vertices from D_{i-1} and adding a_{i-2} vertices, contradicting the a_{i-1}-minimality of D_{i-1}. But

$$|B| = a_{i-1} - t + |Y''|$$

and

$$|D_{i-1}| = m_{a_{i-1}} - m_{a_{i-1}+1} + a_{i-1}.$$

It follows that

$$\begin{aligned} |Y''| &\geq m_{a_{i-1}} - m_{a_{i-1}+1} + t \\ &\geq t + 1. \end{aligned}$$

Hence $|Y| \geq t+1$ and as in Case 1, if Y^* is a set such that $Z \cup Y \cup Y^*$ dominates G_i, then $Y \cup Y^*| \geq t + r = p$ so that X is stable. This completes the proof of the lemma. ∎

If we take $t = p \leq a_{i-1}$ in the proof of the above lemma and define D_{q+1} to be S_q, we get the following result:

Corollary 1. *For any $i \in Q$, the set S_i is an a_i-minimal dominating set of G_i.* ∎

Corollary 2. $\gamma(G_R) = k$.

Proof: By Corollary 1, S_q is an a_q-minimal dominating set of G_q and it contains a_q vertices. Hence $\gamma(G_q) = a_q$. Clearly then, S_q together with the $k - a_q$ isolated vertices of G_R is a k-minimal dominating set of G_R containing k vertices so that no set with fewer than k vertices can dominate G_R. ∎

We are now ready to establish lower bounds for the upper domination numbers of G_R. Let $L = V(\overline{K}_{k-a_q})$ if $k > a_q$ and $L = \phi$ if $k = a_q$.

Theorem 1. *For any $i \in Q$, the set*

$$D_i^* = V(H_{q1}) \cup V(H_{(q-1)1}) \cup \cdots \cup V(H_{i1}) \cup S_{i-1} \cup L$$

is an a_i-minimal but not $(a_i + 1)$-minimal dominating set of G_R containing, if $a_i \geq 2$, precisely $k + \sum_{j=i}^{q}(m_{a_j} - m_{a_j+1})$ vertices.

Proof: It is easy to see that D_i^* dominates G_R, and simple arithmetic shows that $|D_i^*| = k + \sum_{j=i}^{q}(m_{a_j} - m_{a_j+1})$ if $a_i \geq 2$. In Lemma 3 it is shown that D_i is not an $(a_i + 1)$-minimal dominating set of G_i and it follows that D_i^* is not an (a_i+1)-minimal dominating set of G_R. Furthermore, by extending the techniques employed in the proof of Lemma 3, it can be shown that D_i^* is a_i-minimal. ∎

Corollary 3. *For each $i \geq 2, \Gamma_i(G_R) \geq m_i$.*

Proof: Let $i \geq 2$. For each $i = a_q + 1, \ldots, k$, $S_q \cup L$ is an i-minimal dominating set of G_R with $k = m_{a_q+1} = \cdots = m_k$ vertices and therefore $\Gamma_i(G_R) \geq m_i$.

For each $j = 1, \ldots, q$ and each $i = a_{j-1} + 1, \ldots, a_j$, D_j^* is an i-minimal dominating set of G_R with

$$\begin{aligned}
|D_j^*| &= k + \sum_{\ell=j}^{q}(m_{a_\ell} - m_{a_\ell+1}) \\
&= k + m_{a_j} - m_{a_j+1} + \cdots + m_{a_q} - m_{a_q+1} \\
&= k + m_{a_j} - m_{a_{j+1}} + \cdots + m_{a_q} - k \\
&= m_{a_j}.
\end{aligned}$$

Hence $\Gamma_i(G_R) \geq m_i = m_{a_j}$ for each $i = a_{j-1} + 1, \ldots, a_j$ and each $j = 1, \ldots, q$. ∎

5. Upper Bounds for $\Gamma_i(G_R)$

In this section we show that subject to certain conditions, $\Gamma_i(G_R) \leq m_i$ for each $i = 2, \ldots, k$. We begin by considering, for each $i \in Q$, all possible types of minimal dominating sets of $\widetilde{H}_i$ as listed below; although tedious it is not hard to see that these cases exhaust all possibilities. An example of each type of dominating set is given in Figure 2. Note that here $D_1, \ldots, D_4$ denote different sets than in Section 4.

Lemma 4. *For each $i \in Q$ the minimal dominating set of $\widetilde{H}_i$ can be divided into the following types:*

(i) S_i;

(ii) $D_1 \subseteq \cup_{j=1}^{a_i} V(H_{ij})$ *with* $|D_1| = n_i$ *— note that in this case* D_1 *consists of exactly one vertex of each of* n_i *copies of* K_{a_i} *and is* a_i*-minimal (Lemma 2(ii)) but not necessarily* $(a_i + 1)$*-minimal (as in Lemma 1);*

(iii) D_2 *consisting of* $n_i - 1$ *vertices of* $M_i = \cup_{j=1}^{a_i}(V(H_{ij}) - \{v_{ij1}\})$ *such that* D_2 *contains exactly one vertex of each of* $n_i - 1$ *copies of* K_{a_i} *and* $D_2 \cap V(H_{ij}) \neq \phi$ *for at least* $a_i - 1$ j*'s, together with one vertex from* S_i *(representing the copy, if necessary, of* H_i *not already represented) — in this case* D_2 *is* $(a_i - 1)$*-minimal but not necessarily* a_i*-minimal (similar to the proof of lemma 1);*

(iv) D_3 *satisfying the same conditions as* D_2 *above except that* $D_3 \cap V(H_{ij}) \ / = \phi$ *for exactly* $a_i - t$ j*'s,* $t = 2, \ldots, a_i - 1$*, and that* D_3 *contains* t *vertices from* S_i *corresponding to copies of* H_i *not yet represented — note that* $|D_3| = n_i - 1 + t$ *and* D_3 *is not 2-minimal.*

(v) $D_4 = S_i' \cup M_i' \cup M_i''$ *(disjoint union), where* S_i' *is a* t*-subset of* S_i *for some* $t \in \{1, \ldots, a_i - 1\}$ *and* M_i' *is an* $(a_i - t)$*-subset of* M_i *chosen from* ℓ *different copies of* K_{a_i}*,* $\ell \leq a_i - t$*, and such that exactly one element of* M_i' *is in each copy of* H_i *which corresponds to a vertex of* $S_i - S_i'$*.*

Finally, M_i'' *is an* $(n_i - 1 - \ell)$*-subset of* M_i *containing one vertex from each of the remaining* $(n_i - 1 - \ell)$ *copies of* K_{a_i}*. Further, each vertex of* M_i'' *must be from a copy* H_{ij} *of* H_i *such that*

(a) $s_{ij} \in S_i - S_i'$ *and the vertex already elected in* $H_{ij} \cap M_i'$ *is the only vertex of* M_i' *in its copy of* K_{a_i}*, or*

(b) $t = 1$ *and* H_{ij} *is the unique copy of* H_i *corresponding to the vertex of* S_i'*.*

Here $|D_4| = a_i + n_i - 1 - \ell \leq a_i + n_i - 2$*, and* D_4 *is not 2-minimal if* $t \geq 2$ *or if* $|D_4 \cap M_i| > n_i - 1$. ∎

Note that the choices of M_i' and M_i'' are interdependent, i.e. depending upon the choice of M_i' it may be impossible to choose a set M_i'' satisfying the conditions of D_4.

The following two lemmas and their corollaries contain information on the structure of dominating and especially minimal dominating sets of G_R.

Lemma 5. *Let* X *be a minimal dominating set of* G_R*. Let* $W = \{1, \ldots, a_i\}$ *and let* $T \subseteq W$*,* $|T| = t$*. If there exists, for each* $\ell \in W - T$*, an integer* $j \geq i$ *such that* $s_{j\ell} \in X$*, and* $|X \cap (V(G_i) - S_i)| > t$*, then* X *is not* $(t+1)$*-minimal.*

Proof: Suppose X satisfies the conditions above and let $Y \subseteq X \cap (V(G_i) - S_i)$ with $|Y| > t$. Let $Y' = \{s_{i\ell} | \ell \in T\}$ and consider $Z = (X - Y) \cup Y'$. Then,

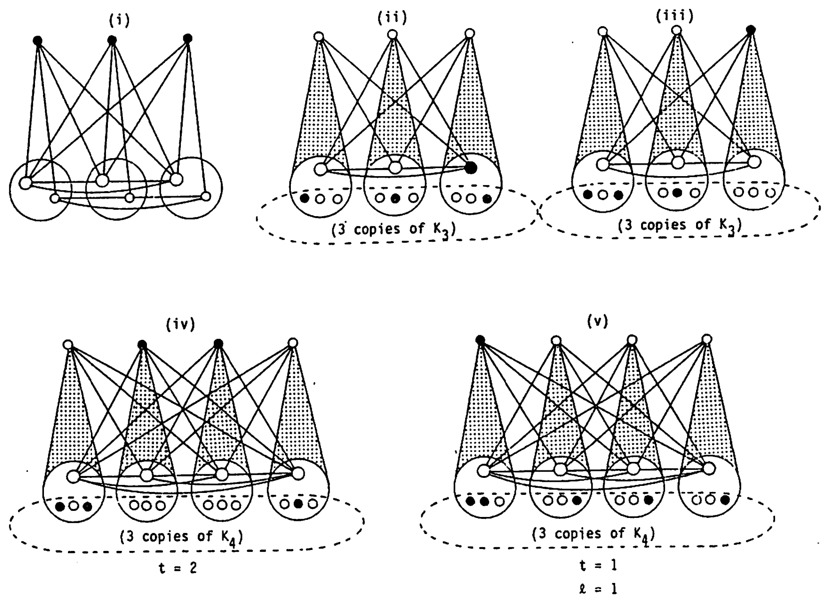

Figure 2: Minimal dominating sets (indicated by black vertices) of $\widetilde{H}_i$

for each $\ell \in W$ there exists an integer $j \geq i$ such that $s_{j\ell} \in Z$. We show that Z dominates G_R.

Firstly, let $v \in V(G_i) - S_i$. If $v = s_{j'\ell}$ for some $j' < i$, then $\ell < a_i$ and there exists a $j \geq i$ such that $u = s_{j\ell} \in Z$; note that u and v are adjacent. If $v = v_{j'\ell r}$ for some $j' \leq i$ and some ℓ and r, then $\ell \leq a_i$ and $u = s_{j\ell} \in Z$ for some $j \geq i$ dominates v. Hence each $v \in V(G_i) - S_i$ is dominated by Z. Now let $v \in (V(G_R) - V(G_i)) \cup S_i$ be adjacent to a vertex in Y. Then $v = s_{j'\ell}$ for some $j' \geq i$ and some $\ell \leq a_i$. But then $u = s_{j\ell} \in Z$ for some $j \geq i$ and either $v = u$ or v is adjacent to u. Hence each $v \in V(G_R)$ is dominated by Z. But $|Z| \leq |X| - |Y| + |Y'| \leq |X| - t - 1 + t < |X|$, proving that X is not $(t + 1)$ minimal. ∎

The following result is a direct corollary of Lemma 5 using $t = 0$, whereas Corollary 5 follows from the proof of Lemma 5.

Corollary 4. *Let X be a minimal dominating set of G_R. If, for some $i \in Q$, there exists for each $\ell = 1, \ldots, a_i$ an integer $j \geq i$ such that $s_{j\ell} \in X$, then $X \cap (V(G_i) - S_i) = \phi$.* ∎

Corollary 5. *If X is a dominating set of G_R which contains more than a_i vertices of G_i for some $i \in Q$, then X is not $(a_i + 1)$-minimal.*

Proof: Let $Y \subseteq X \cap V(G_i)$ with $|Y| = a_i + 1$ and let $Z = (X - Y) \cup S_i$. As in the proof of Lemma 5, Z dominates G_R and since $|Z| = |X| - 1$, X is not $(a_i + 1)$-minimal. ∎

Let $S = \cup_{i=1}^{q} S_i$. In Lemma 6 we show that a minimal dominating set X of G_R can contain only certain subsets of S. Using this we obtain the upper bounds for $|X \cap S|$ in Corollary 6.

For each $v \in X$, a dominating set of a graph G, define U_v by $U_v = \{u \in V(G) - X | N_G(u) \cap X = \{v\}\}$.

Lemma 6. *Let X be a minimal dominating set of G_R.*

(i) *If $v = s_{i\ell} \in X$ and $u = v_{ir\ell} \in U_v$ for some r, then $X \cap S_i = \{s_{i\ell}\}$ and $s_{jr} \notin X$ for all $j > i$.*

(ii) *If $s_{j\ell}, s_{j'\ell} \in X$ for $j \neq j'$, then each vertex $v \in X$ with $v = s_{i\ell}, i \in Q$, is a non-isolate of $\langle X \rangle$ such that $U_v \subseteq \{v_{i11}, \ldots, v_{ia_i1}\}$ and $U_v - \{v_{i\ell 1}\} \neq \phi$.*

Proof:

(i) If $u = v_{ir\ell}$ is not dominated by $X - \{v\}$ then, by the construction of G_R, $(X - \{v\}) \cap S_i = \phi$ and $s_{jr} \notin X$ for all $j > i$.

(ii) We prove the contrapositive of (ii). Let $v = s_{i\ell} \in X$. If v is an isolated vertex of $\langle X \rangle$ then clearly $s_{j\ell} \notin X$ for all $j \neq i$. If v is a non-isolate of $\langle X \rangle$ then (by assumption) there exists a vertex $u \in (U_v - \{v_{i11}, \ldots, v_{ia_i1}\}) \cup \{v_{i\ell 1}\}$. If $u \in S$, then $u = s_{r\ell}$ for some $r \neq i$, implying that $s_{j\ell} \notin X$ for each $j = 1, \ldots, q, j \neq i$. If $u \notin S$, then $u = v_{r\ell t}$ for some $r \leq i$. Clearly, $s_{j\ell} \notin X$ for each $j = r, \ldots, q, j \neq i$. Since u is not dominated by $X - \{v\}$, $v_{rpt} \notin X$ for each $p = 1, \ldots, a_r$. But each $v_{rpt}, p \neq \ell$, is dominated by $X - \{v\}$ and therefore for each such p there exists an integer $p' \geq r$ such that $s_{p'p} \in X$. By Corollary 4, $X \cap (V(G_r) - S_r) = \phi$ so that $s_{j\ell} \notin X$ for each $j = 1, \ldots, r-1$ and the result follows. ∎

Corollary 6. *If X is minimal dominating set of G_R and there exists an integer ℓ such that $s_{i\ell}, s_{j\ell} \in X$ for $i \neq j$, then $|X \cap S| \leq a_q + q - 2$ for $a_q \geq 2$; otherwise $|X \cap S| \leq a_q$.*

Proof: If there is no integer ℓ such that $s_{i\ell}, s_{j\ell} \in X$ for $i \neq j$, then obviously $|X \cap S| \leq a_q$. Suppose there are $r \geq 1$ such integers ℓ. If $s_{i\ell}, s_{j\ell} \in X$ for $i \neq j$, then by Lemma 6(ii), $v_{ix1} \in U_v$ (where $v = s_{i\ell}$, say) for some x. By Lemma 6(i), $X \cap S_i = \{s_{i\ell}\}$. It therefore follows that $|X \cap S_i| = 1$ for at least r i's and hence $r \leq q$. We now consider two cases.

Case 1: $|X \cap S_i| = 1$ for q i's.
Then $|X \cap S| = q \leq a_q + q - 2$ if $a_q \geq 2$.

Case 2: $|X \cap S_i| = 1$ for t i's where $r \leq t \leq q - 1$.
In this case there is a non-empty set $J = \{i \in Q || X \cap S_i| \neq 1\}$ such that $|X \cap (\cup_{i \in J} S_i| \leq a_q - r$. Hence $|X \cap S| \leq t + a_q - r \leq q + a_q - 2$. ∎

We are now ready to formulate conditions that will ensure that $\Gamma_i(G_R) \leq m_i$ for all $i = 2, \ldots, k$.

Theorem 2. *Let $a_i \geq 2$ and suppose $m_{a_j} > m_{a_{j+1}} + a_{j-1}$ for all $2 \leq a_j \in I - \{a_q\}$. If X is a minimal domination set of G_R containing more than m_{a_i} vertices, then X is not d-minimal, where $d = \max\{2, a_{i-1} + 1\}$.*

Proof: Let X be a minimal dominating set of G_R with more than m_{a_i} vertices and let $Y = X \cap V(G_{i-1})$ (where $V(G_{i-1}) = \phi$ per definition if $i = 1$). If $|Y| > a_{i-1}$ then, by Corollary 5, X is not $(a_{i-1} + 1)$-minimal. Hence we may assume that $|Y| = a_{i-1} - r$, where $0 \leq r \leq a_{i-1}$, so that

$$\begin{aligned} |X - Y| &\geq m_{a_i} - a_{i-1} + r + 1 \\ &= \sum_{j=i}^{q} n_j + (k - a_q) + r + 1. \end{aligned} \tag{1}$$

Assume first that $Y \neq \phi$ and suppose there is no r-subset W of $X - Y$ such that $W \cup Y$ dominates G_{i-1}. Let $W' \subseteq X - Y$ with $|W'| = r' > r$, be a minimal subset of $X - Y$ such that $W' \cup Y$ dominates G_{i-1}. (Note that in this case, $r < a_{i-1}$.) Then W' consists of vertices $s_{j\ell}$, where $j \geq i$ and $1 \leq \ell \leq a_{i-1}$. Let $Y \cap S_{i-1} = W''$ with $|W''| = r''$, $0 \leq r'' \leq a_{i-1} - r$. By the minimality of W', at most one of $s_{j\ell}$ and $s_{j'\ell}$, $j \neq j'$ $(j, j' \geq i - 1)$ is in $W' \cup W''$ and it follows that $r' + r'' \leq a_{i-1}$. We may now apply Lemma 5 with $t = a_{i-1} - r' - r''$. Note that

$$|X \cap (V(G_{i-1}) - S_{i-1})| = a_{i-1} - r - r'' > t,$$

so that X is not $(t + 1)$-minimal. But

$$\begin{aligned} t + 1 &= a_{i-1} - r' - r'' + 1 \\ &\leq a_{i-1} - r - r'' \\ &\leq a_{i-1}, \end{aligned}$$

hence X is not a_{i-1}-minimal. We may therefore assume that there exists an r-subset W of $(X - Y) \cap S$ such that $W \cup Y$ dominates G_{i-1}.

Next, if $Y = \phi$, let i' be the largest integer with the property that for each $\ell = 1, \ldots, a_{i'}$, there exists an integer $\ell' \geq i'$ such that $s_{\ell'\ell} \in X$. Note that $i - 1$ has this property and therefore $i - 1 \leq i' \leq q$. By Corollary 4, $X \cap (V(G_{i'}) - S_{i'}) = \phi$ and if $i' = i - 1$, then $X \cap V(G_{i-1}) = \phi$. If $i' = q$ then $X = S_q$. But then $|X| = a_q \leq k \leq m_{a_i}$ for all $i \in Q$, contrary to assumption. Consequently $i - 1 \leq i' < q$. Let T with $|T| = a_{i'}$ denote the set of vertices $s_{\ell'\ell} \in X$ as defined above. Now, from (1),

$$\begin{aligned}|X| &\geq \sum_{j=i}^{q} n_j + (k - a_q) + a_{i-1} + 1 \\ &\geq \sum_{j=i'+1}^{q} n_j + \sum_{j=i}^{i'} n_j + a_{i-1} + 1 \\ &\geq \sum_{j=i'+1}^{q} n_j + a_{i'} + 1 \text{ (by hypothesis and the definition of the } n_j\text{'s).}\end{aligned} \tag{2}$$

We now define Z as

$$Z = \begin{cases} W \cup Y & \text{if } Y \neq \phi \\ T & \text{if } Y = \phi, \end{cases}$$

and, for $j = i, \ldots, q$, we let $X_j = (X - Z) \cap V(\widetilde{H}_j)$. Note that

$$|X| = \begin{cases} \sum_{j=i}^{q} |X_j| + a_{i-1} & \text{if } Y \neq \phi \\ \sum_{j=i'+1}^{q} |X_j| + a_{i'} & \text{if } Y = \phi. \end{cases} \tag{3}$$

Since $|X| = |X - Y| + |Y|$, it now follows from (1), (2) and (3) that there exists an integer $j \geq i$ if $Y \neq \phi$ ($j \geq i' + 1$ if $Y = \phi$) such that $|X_j| > n_j$.

Now suppose that for each integer ℓ with $1 \leq \ell \leq a_j$, a vertex of the form $s_{j'\ell}$ for some $j' \geq j$ is contained in X. If $Y = \phi$, this contradicts the choice of i' since $j \geq i' + 1$. If $Y \neq \phi$, then this implies that $X - Y$ dominates G_R since $j \geq i$ and $Y \subseteq V(G_{i-1})$, contradicting the minimality of X. Hence there exists an integer ℓ with $1 \leq \ell \leq a_j$ such that $s_{j'\ell} \notin X$ for all $j' \geq j$. Since $V(H_{j\ell})$ is dominated, X_j contains at least $n_j - 1$ vertices of $M_j = \bigcup_{x=1}^{a_j} (V(H_{jx}) - \{v_{jx1}\})$ (i.e., at least one vertex of each of $n_j - 1$ copies of K_{a_j}). We consider two cases, depending on whether $v_{jx1} \in X_j$ for some $x = 1, \ldots, a_j$ or not.

Case 1: $v_{jx1} \in X_j$ for some $x = 1, \ldots, a_j$.

Then X_j contains a dominating set D of $\widetilde{H}_j$ of the form D_1 as described in Lemma 4(ii). Since $|X_j| > n_j$, there exists a vertex $v \in X_j - D$.

Subcase 1(a): $v = s_{jb}$ for some $b = 1, \ldots, a_j, b \neq \ell$.

If $b > a_{q-1}$ (which is only possible if $j = q$), then X is not minimal since $\widetilde{H}_q$ is dominated by D and s_{qa_q} is not adjacent to any vertex of G_{q-1}; hence $b \leq a_{q-1}$. (4)

As before, let $U_v = \{u \in V(G_R) - X | N_{G_R}(u) \cap X = \{v\}\}$. Since D dominates $\widetilde{H}_j$, $U_v \cap V(\widetilde{H}_j) = \phi$, and since Z dominates G_{i-1} if $Y \neq \phi$ ($G_{i'}$ if $Y = \phi$), $U_v \cap V(G_{i-1}) = \phi$ if $Y \neq \phi$ ($U_v \cap V(G_{i'}) = \phi$ if $Y = \phi$). (5)

Now suppose $u = v_{\alpha\beta\gamma} \in U_v$ for some $\alpha = 1, \ldots, q$; $\beta = 1, \ldots, a_\alpha$ and $\gamma = 1, \ldots, n_\alpha$. Then $\alpha < j$ and $\beta = b$ by the construction of G_R. But then $v_{\alpha\beta'\gamma} \notin X$ for all $\beta' = 1, \ldots, a_\alpha$ so that for each $\beta' = 1, \ldots, a_\alpha$ there exists an $\alpha' \geq \alpha$ such that $s_{\alpha'\beta'} \in X$. But since (by (5)) $\alpha > i - 1$ if $Y \neq \phi$ ($\alpha > i'$ if $Y = \phi$), this contradicts the minimality of X if $Y \neq 0$ (the choice of i' if $Y = \phi$) and hence U_v consists entirely of vertices of the form s_{yb}, $y = 1, \ldots, q$, $y \neq j$. Let g be the smallest integer larger than $i - 1$ if $Y \neq \phi$ (i' if $Y = \phi$) for which s_{gb} is defined. If $g = q$, then by (4), $i - 1 = q - 1$ if $Y \neq \phi$ ($i' = q - 1$ if $Y = \phi$) and so by (5), $U_v = \phi$ in which case X is not minimal. Therefore $g \leq q - 1$. By the minimality of X or the choice of i' there exists at least one integer $c = 1, \ldots, a_g$, $b \neq c$, such that $s_{g'c} \notin X$ for all $g' \geq g$. Since $V(H_{gc})$ is dominated, X contains at least $n_g - 1$ vertices of M_g, and if $X \cap S_g = \phi$ then X also contains v_{gf1} for some $f = 1, \ldots, a_g$. By the choice of g and the hypothesis of the theorem, $2 \leq a_g \in I - \{a_q\}$ and hence

$$m_{a_g} > m_{a_{g+1}} + a_{g-1},$$

i.e.,

$$n_g > a_g.$$

Therefore, $|X \cap V(H_{gg'})| \geq 2$ for at least one g', or $|X \cap V(H_{gg'})| = 1$ for all g' and $s_{gf'} \in X$ for at least one f'. In the former case, if $v_{gg'x'} \in X \cap V(H_{gg'})$, then $(X - \{v, v_{gg'x'}\}) \cup \{v_{gbx'}\}$ dominates G_R, proving that X is not 2-minimal. In the latter case, $(X - \{v, v_{gf'x'}\}) \cup \{v_{gbx'}\}$ dominates G_R.

Subcase 1(b): $v = v_{jbc}$ for some $b = 1, \ldots, a_j$ and some $c = 1, \ldots, n_j$.
As in Subcase 1(a), $U_v \cap V(\widetilde{H}_j) = \phi$ and so by the construction of G_R, U_v consists of vertices s_{yb}, $y = j+1, \ldots, q$. Again, if $j = q$ then $U_v = \phi$, hence $j \leq q-1$. By the choice of j and the hypothesis of the theorem, $n_j > a_j$ so that $|X \cap V(H_{jj'})| \geq 2$ for at least one j'. Say $v_{jj'x'} \in X \cap V(H_{jj'})$. Then $(X - \{v, v_{jj'x'}\}) \cup \{v_{jbx'}\}$ dominates G_R.

Case 2: $v_{jx1} \notin X_j$ for all $x = 1, \ldots, a_j$.
Then $s_{jx} \in X_j$ for at least one $x = 1, \ldots, a_j$, $x \neq \ell$, since $v_{j\ell 1}$ is dominated and $s_{j'\ell} \notin X$ for all $j' \geq j$. Let D consist of s_{jx} together with the $n_j - 1$ vertices of M_j as described above. (Note that D is a dominating set of $\widetilde{H}_j$ of the form D_2 of Lemma 4(iii).) Since $|X_j| > n_j$ there exists a vertex $v \in X_j - D$.

Subcase 2(a): $v = s_{jb}$ for some $b = 1, \ldots, a_j$, $b \neq x, \ell$.
Note that D dominates $\widetilde{H}_j - S_j$ and that $\langle S_j \rangle = \overline{K}_{a_j}$, hence $U_v \cap V(\widetilde{H}_j) = \phi$. We now proceed exactly as in Subcase 1(a) to show that X is not 2-minimal.

Subcase 2(b): $v = v_{jbc}$ for some $b = 1, \ldots, a_j$ and some $c = 2, \ldots, n_j$.
Here U_v consists of vertices s_{yb}, $y = j, \ldots, q$ and as in Subcase 1(b), X is not 2-minimal.

We therefore conclude that X is not d-minimal, where $d = \max\{2, a_{i-1} + 1\}$. ∎

Corollary 7. *Let* $i \geq 2$ *and suppose* $m_{a_j} > m_{a_{j+1}} + a_{j-1}$ *for all* $2 \leq a_j \in I - \{a_q\}$. *Then* $\Gamma_i(G_R) \leq m_i$ *for all* $i = 2, \ldots, k$.

Proof: By definition, $\Gamma_{j+1}(G_R) \leq \Gamma_j(G_R)$ for all $j \geq 1$. By Theorem 2,

$$\Gamma_{a_{j-1}+1}(G_R) \leq m_{a_j} \text{ for } a_j \geq 2.$$

Hence

$$\Gamma_{a_j}(G_R) \leq \Gamma_{a_j-1}(G_R) \leq \cdots \leq \Gamma_{a_{j-1}+1}(G_R) \leq m_{a_j} = m_{a_j-1} = \cdots = m_{a_{j-1}+1}$$

so that for each $i = a_{j-1} + 1, \ldots, a_j$ and each j such that $a_j \geq 2$,

$$\Gamma_i(G_R) \leq m_i.$$

It follows that $\Gamma_i(G_R) \leq m_i$ for all $i = 2, \ldots, k$. ∎

Observe that it now follows from Corollaries 3 and 7 that $\Gamma_i(G) = m_i$ for all $i = 2, \ldots, k$.

6. The Upper Domination Number of G_R

It is evident from Lemma 4(iv)-(v) that G_R has minimal dominating sets of larger cardinality than the set D_1^* defined in Theorem 1 and we therefore still need to determine $\Gamma_1(G)$. We begin by remarking that if $a_q = 1$, then $q = 1$ and $m_1 > m_2 = \cdots = m_k = k$, while $n_1 = m_1 - m_2 + 1$ by definition. In this case G_R consists of K_{1,n_1} together with $k - 1$ isolated vertices and it is obvious that

$$\begin{aligned}\Gamma_1(G_R) &= k - 1 + n_1 \\ &= k - 1 + m_1 - m_2 + 1 \\ &= m_1.\end{aligned}$$

In what follows we therefore assume that $a_q \geq 2$. We now find an expression for m_1 in terms of the n_j's, $j = 1, \ldots, q$.

Proposition 3. *If* $a_q \geq 2$, *then*

$$m_1 = k - 2 + \sum_{j=1}^{q} n_j.$$

Proof: Note that

$$n_1 = \begin{cases} m_1 - m_2 - a_q + 3 & \text{if } a_1 = 1 \\ m_{a_1} - m_{a_1+1} + a_1 & \text{if } a \geq 2. \end{cases}$$

Now,

$$k-2+\sum_{j=1}^{q} n_j = k-2+n_1+(m_{a_2}-m_{a_3}+a_2-a_1)+\dots$$

$$+(m_{a_q}-m_{a_q+1}+a_q-a_{q-1})$$

$$= k-2+n_1+m_{a_2}-a_1-m_{a_{q+1}}+a_q \quad (\text{since } m_{a_i+1}=m_{a_{i+1}})$$

$$= n_1+m_{a_2}-a_1+a_q-2 \quad (\text{since } m_{a_{q+1}}=k).$$

If $a_1 = 1$, then $m_2 = m_{a_2}$ so that by the definition of n_1,

$$k-2+\sum_{j=1}^{q} n_j = m_1.$$

If $a_1 \geq 2$, then $m_2 = m_{a_1}$ and $m_{a_1+1} = m_{a_2}$ so that it follows from the definition of n_1 and the fact that $m_1 = m_2 + a_q - 2$ in this case that,

$$k-2+\sum_{j=1}^{q} n_j = m_1.$$

∎

We now find a minimal dominating set of G_R with m_1 vertices. (Recall that $L = V(\overline{K}_{k-a_q})$ ($L = \phi$) if $k > a_q$ ($k = a_q$).)

Proposition 4. *The set*

$$\widehat{D}_1 = L \cup (S_q - \{s_{q1}\}) \cup (\cup_{j=1}^{q} V(H_{j1})) - \{v_{q11}\}$$

is a minimal dominating set of G_R containing m_1 vertices.

Proof: It is clear that $\widehat{D}_1$ dominates G_R and that

$$|\widehat{D}_1| = k-a_q+a_q-1+\sum_{j=1}^{q} n_j - 1 = k-2+\sum_{j=1}^{q} n_j = m_1 \quad \text{(by Proposition 3).}$$

Also each vertex in $\widehat{D}_1$ is an isolated vertex of $\langle \widehat{D}_1 \rangle = \overline{K}_{m_1}$ which means that $\widehat{D}_1$ is a minimal dominating set of G_R. ∎

Corollary 8. $\Gamma_1(G_R) \geq m_1$. ∎

Theorem 3. *If X is a minimal dominating set of G_R, then $|X| \leq m_1$.*

Proof: Suppose, to the contrary, that X with $|X| > m_1$ is a minimal dominating set of G_R. If there exists an integer i satisfying the conditions of Corollary 4, let

i' be the largest such integer; otherwise, let $i' = 0$. If $i' \geq 1$ then, by Corollary 4, $X \cap (V(G_{i'}) - S_{i'}) = \phi$. Let $X_j = X \cap V(\widetilde{H}_j)$ and $X_j^* = X_j - S_j$ for each $j \in Q$. Note that $|X_j^*| \geq n_j - 1$ for each $j > i'$ and by Corollary 4, $|X_j^*| = 0$ for each $j \leq i'$. Let $R_j \subseteq X_j^*$ be any set of $n_j - 1$ vertices of M_j (as defined in Theorem 2) such that R_j consists of exactly one vertex of each of $n_j - 1$ copies of K_{a_j}, $j > i'$, and let $R_j = \phi$ for $j \leq i'$.

If we let $r = |X - \cup_{j=1}^{q} R_j|$, then

$$
\begin{aligned}
r &= \sum_{j=1}^{q}(|X_j| - |R_j|)\\
&= \sum_{j=1}^{q}|X_j| - \sum_{j=1}^{q}|R_j|\\
&\geq |X| - \sum_{j=1}^{q}(n_j - 1) \qquad (6)\\
&\geq m_1 + 1 + q - \sum_{j=1}^{q} n_j\\
&= k - 2 + 1 + q \quad \text{(by Proposition 3)}\\
&\geq a_q + q - 1.
\end{aligned}
$$

We now proceed to count the numberr of vertices in $X - \cup_{j=1}^{q} R_j$ in a different way and show that this number does not exceed $a_q + q - 2$, hence finding a contradiction to (6).

For each j such that $[\cup_{\ell=1}^{a_j}\{v_{j\ell 1}\}] \cap (X_j^* - R_j) \neq \phi$, choose a vertex $v \in [\cup_{\ell=1}^{a_j}\{v_{j\ell 1}\}] \cap (X_j^* - R_j)$ and let $F_j = R_j \cup \{v\}$. Then $F_j \subseteq X$ and by the definitions of R_j and F_j, F_j is a dominating set of $\widetilde{H}_j$ of the form D_1 of Lemma 4(ii). If $s_{j\ell'} \in X$ for some $\ell' = 1, \ldots, a_j$, then since $X \cap [\cup_{\ell=1}^{a_j}\{v_{j\ell 1}\}] \neq \phi$, the contrapositive of Lemma 6 (ii) implies that $s_{j'\ell'} \notin X$ for all $j' \neq j$. Let there be exactly t_1 distinct integers j for which F_j is defined, with J the set of these integers j. Let t_2 denote the number of distinct integers ℓ' such that $s_{j\ell'} \in X$ for some $j \in J$. Note that $0 \leq t_1 \leq q$ and $0 \leq t_2 \leq a_q$.

Next, suppose $v = v_{j\ell x} \in X_j^* - R_j$ for some $\ell = 1, \ldots, a_j$ and some $x = 2, \ldots, n_j$, or $v = v_{j\ell x} \in X_j^* - F_j$ if $x = 1$ and F_j is defined. Since $v_{j\ell' x} \in X$ ($x = 1, \ldots, n_j$) for some $\ell' \neq \ell$, it is clear that v is a non-isolate of $\langle X \rangle$ and that U_v consists of vertices of the form $s_{j'\ell}$, $j' = j, \ldots, q$. Consequently, *for ℓ fixed,*

$$X \cap \{s_{i\ell} | a_i \geq \ell\} = \phi \text{ and } X \cap V(H_{j\ell}) = \{v\}. \qquad (7)$$

Furthermore, if $v' = v_{\eta \ell y}$, $\eta \neq j$ has the same property as v, suppose without losing generality that $j < \eta$ and let $s_{\eta' \ell} \in U_{v'}$. But then $j < \eta'$ and v is adjacent to $s_{\eta'\ell}$ (by the construction of G_R), contradicting the fact that $s_{\eta'\ell} \in U_{v'}$. Hence if there are t_3 such vertices v, then $t_3 \leq a_q$. It is also important to note that if $v_{j\ell x}$ is such a vertex, then for some $\ell' \neq \ell$, $v_{j\ell' x} \in R_j$ or $v_{j\ell' x} \in F_j$ if F_j is defined. By letting $R_j' = (R_j - \{v_{j\ell' x}\}) \cup \{v_{j\ell x}\}$ if $x \neq 1$ $(F_j' = (F_j - \{v_{j\ell' x}\}) \cup \{v_{j\ell x}\}$ if $x = 1)$ and interchanging the roles of $v_{j\ell x}$ and $v_{j\ell' x}$, we see that *for ℓ fixed,*

$$X \cap \{s_{i\ell'} | a_i \geq \ell'\} = \phi, \quad X \cap V(H_{j\ell'}) = \{v_{j\ell' x}\} \tag{8}$$

and no $v_{\eta\ell' y}$ has the same property as $v_{j\ell' x}$. Hence for $\ell \neq \ell'$, at most one of $v_{j\ell x}$ and $v_{j\ell' x}$ has the required property.

Suppose further that X contains t_4 vertices $s_{j\ell}$ where $j \notin J$ and $s_{j'\ell} \notin X$ for all $j' \neq j$, and t_5 vertices $s_{j\ell}$ where $j \notin J$ and $s_{j'\ell} \in X$ for some $j' \notin J \cup \{j\}$. Suppose in the latter case that there are t_6 distinct values of ℓ such that $s_{j\ell} \in X$.

Note that $r = |X - \bigcup_{j=1}^{q} R_j| = \sum_{i=1}^{5} t_i$.

It is also immediately obvious that

$$0 \leq t_1 + t_4 + t_5 \leq q \tag{9}$$

and we claim that

$$0 \leq t_2 + t_3 + t_4 + t_6 \leq \begin{cases} a_q - 1 & \text{if } t_3 > 0 \\ a_q & \text{otherwise.} \end{cases} \tag{10}$$

Firstly, it is obvious that $t_2 + t_4 + t_6 \leq a_q$; hence (10) holds if $t_3 = 0$. If $t_3 > 0$, it follows from the first parts of (7) and (8) that $t_2 + t_3 + t_4 + t_6 \leq a_q - 1$ and so (10) also holds in this case.

We now prove that $\sum_{i=1}^{5} t_i \leq a_q + q - 2$. If $t_1 = q$, then $\bigcup_{j=1}^{q} F_j \subseteq X$ dominates G_q and hence $t_2 = \cdots = t_6 = 0$ by the minimality of X. Thus $\sum_{i=1}^{5} t_i = q \leq a_q + q - 2$ if $a_q \geq 2$. Suppose $t_1 = q - 1$. Since $t_5 \geq 2$ if $t_5 \neq 0$, it follows from (9) that $t_5 = 0$ (and hence $t_6 = 0$). But if $t_1 = q - 1$, then $t_2 + t_3 + t_4 + t_6 \leq a_q - 1$ even if $t_3 = 0$, for suppose $t_3 = 0$ and $t_2 + t_4 + t_6 = a_q$. By Corollary 4, $X \cap (V(G_1) - S_1) = \phi$. Hence F_j is defined for each $j = 2, \ldots, q$ (since $t_1 = q - 1$). But $s_{qa_q} \in X$ if $t_2 + t_4 + t_6 = a_q$ and $N[s_{qa_q}] \subseteq N[F_q]$ so that $X - \{s_{qa_q}\}$ dominates G_R, contradicting the minimality of X. Since $t_5 = t_6 = 0$, it now follows that $t_1 + t_2 + t_3 + t_4 \leq q - 1 + a_q - 1$, i.e., $r \leq a_q + q - 2$.

Finally, suppose $t_1 \leq q - 2$. If $t_5 = 0$ then by (10), $r \leq a_q + q - 2$. If $t_5 \neq 0$ (and thus $t_6 \neq 0$) and $t_3 \neq 0$, then (9) and (10) imply that

$$\sum_{i=1}^{5} t_i \leq q + t_2 + t_3 \leq q + a_q - 1 - t_4 - t_6 \leq a_q + q - 2.$$

Thus, suppose $t_3 = 0$ and $t_5, t_6 \neq 0$. If $t_1 + t_5 \leq q - 1$, then $r \leq a_q + q - 2$ follows from (9) and (10). If $t_1 + t_5 = q$ then $t_4 = 0$. We prove that $t_2 + t_6 \leq a_q - 1$.

If, to the contrary, $t_2 + t_6 = a_q$, then for each integer $\ell = 1, \ldots, a_q$ there exists an integer j such that $s_{j\ell} \in X$. In particular, $s_{qa_q} \in X$. If F_q is defined, then $N[s_{qa_q}] \subseteq N[F_q]$, contradicting the minimality of X. If F_q is not defined, then $q \notin J$, and since $t_1 + t_5 = q$ and $t_4 = 0$, this implies that $s_{ja_q} \in X$ for $j \neq q$ which is impossible by the construction of G_R. Hence $t_2 + t_6 \leq a_q - 1$, implying that $t_2 \leq a_q - 2$ because $t_6 > 0$. Once again

$$r = t_1 + t_2 + t_5 \leq a_q + q - 2.$$

We have now shown for all possible values of t_1 that $r \leq a_q + q - 2$, contradicting (6). We thus conclude that $|X| \leq m_1$, whereby the theorem is proved. ∎

Corollary 9. $\Gamma_1(G_R) = m_1$. ∎

We summarise the preceding results as follows:

Theorem 4. *Consider the sequence* $R : m_1 \geq \cdots \geq m_k \geq m_{k+1} = k$. *Let* $K = \{1, \ldots, k\}$ *and* $I' = \{i \in K - \{1\} | m_i > m_{i+1}\}$. *If* $I' = \phi$ *define* I *as*

$$I = \begin{cases} \phi & \text{if } m_1 = m_2 \\ \{1\} & \text{otherwise.} \end{cases}$$

If $I' \neq \phi$, *let* $a' = \max_{i \in I'}\{i\}$; *in this case we require that* $m_1 \geq m_2 + a' - 2$. *Define* I *as*

$$I = \begin{cases} I' & \text{if } m_1 = m_2 + a' - 2 \\ I' \cup \{1\} & \text{otherwise;} \end{cases}$$

say $I = \{a_1, \ldots, a_q\}$ *where* $|I| = q$ *and* $a_1 \leq \cdots \leq a_q = a'$. *If* $m_{a_j} > m_{a_{j+1}} + a_{j-1}$ *for all* $2 \leq a_j \in I - \{a_q\}$, *then* R *is a domination sequence; in fact, if* G_R *is constructed as described above then* $\Gamma_i(G_R) = m_i$ *for all* $i = 1, \ldots, k$ *and* $\gamma(G_R) = k$. ∎

Corollary 10. *Any sequence* R *of the form* $m_1 \geq m_2 \geq m_3 = \cdots = k$ *is a domination sequence.*

Proof: The result follows immediately since R trivially satisfies the conditions of Theorem 5. ∎

7. Concluding Remarks

That the result obtained in Theorem 4 is not the best possible can be seen by considering the sequence R given by

$$m_1 = 6, \quad m_2 = m_3 = m_4 = 5, \quad m_5 = k = 4.$$

Here $I' = \{4\}$ and hence $a_q = 4$, and $m_1 = 6 < m_2 + a_q - 2 = 7$. Therefore R does not satisfy the conditions of Theorem 4. Nevertheless, R is the domination sequence of the path P_{12} (see [1] for the calculation of $\Gamma_i(P_n)$).

Acknowledgement

Research support from the Canadian National Sciences and Engineering Research Council is gratefully acknowledged. This paper was written while Dr. Cockayne enjoyed a visiting position at UNISA in 1987.

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