Covering Complete Graphs by Cliques

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Abstract. Let k, n be positive integers. Define the number $f(k,n)$ by $f(k,n) = \min\{\max\{|S_i|, i = 1, \dots, k\}\}$, where the minimum is taken over all k-tuples $S_1, \dots, S_k$ of cliques of the complete graph K_n which cover its edge set. Because there exists an $(n, m, 1)$-BIBD if and only if $f(k,n) = m$, for $k = \frac{n(n-1)}{m(m-1)}$ the problem of evaluating $f(k,n)$ can also be considered as a generalization of the problem of existence of balanced incomplete block designs with $\lambda = 1$.

In the paper the values of $f(k,n)$ are determined for small k and some asymptotic properties of f are derived. Among others, it is shown that for all k $\lim_{n\to\infty} \frac{f(k,n)}{n}$ exists.

1. Introduction

One of the natural ways of covering graphs is by means of their cliques. There are a variety of invariants connected with this and probably [1] was the first paper concerning this topic (for recent papers see, for example, [2], [4]. More detailed information can be found in the survey papers [3], [5].) In contrast to the above mentioned papers we will confine ourselves to covering complete graphs by a prescribed number of cliques and will search for the coverings where the order of the largest cliques is as small as possible. This problem is, at the same time, a generalization of the problem of the existence of balanced incomplete block designs.

Let $s = \{G_1, \cdots, G_k\}$ be a collection of k cliques of the complete graph K_n. As usual, it will be said that S covers K_n or that S is a (k,n)-covering if $\bigcup_{1\le i\le k} E(G_i) = E(K_n)$. The order of the largest clique in S will be denoted by $c(S)$, that is, $c(S) = \max\{|V(G_i)|, i = 1, \cdots, k\}$. For k and n natural numbers we define the number $f(k,n)$ by $f(k,n) = \min c(S)$, where the minimum is taken over all (k,n)-coverings.

The numbers $f(k,n)$ can also be thought of as a generalization of some other concepts. For example, it is not difficult to see that there exists a $(n, m, 1)$-BIBD (that is, there exists a decomposition of K_n into subgraphs isomorphic to K_m) if and only if $f(k,n) = m$, where $k = \frac{n(n-1)}{m(m-1)}$. This means that the evaluation of

$f(k,n)$ for all pairs (k,n) is a difficult problem as it includes the problem of the existence of $(n,m,1)$-BIBD's which has been intensively studied but is still far from being completely solved.

In this paper the values $f(k,n)$, for small k, are determined and several asymptotic properties of f are derived. Among others, it is shown that for any k $\lim_{n\to\infty} \frac{f(k,n)}{n}$ exists.

2. Preliminaries

In order to avoid ambiguity and formal inaccuracy throughout the paper we will understand by K_n the complete graph on a fixed set of n vertices. We now introduce a concept which will enable us to define a similarity relation on the family of all coverings with a given number of cliques.

For a collection $S = \{G_1, \cdots, G_k\}$ of cliques of K_n we define a set $\mathcal{A}$ by $\mathcal{A} = \{A_x : x \in V(G)\}$, where $A_x = \{j : x \in V(G_j)\}$. We say that the collection S is of type $\mathcal{A}$. For $A \in \mathcal{A}$ let $V(A) = \bigcap_{i\in A} V(G_i)$. Clearly $\bigcup_{A\in\mathcal{A}} V(A) = V(K_n)$. Two (k,n)-coverings S and T of types $\mathcal{A} = \{Ai : i \in I\}$, $\mathcal{B} = \{B_j : j \in \mathcal{J}\}$, respectively, will be called similar if there exists bijections $f : \{1, \cdots, k\} \to \{1, \cdots, k\}$, $g : I \to J$ such that $x \in A_i$ if and only if $f(x) \in B_{g(i)}$, $x \in \{1, \cdots, k\}$, $i \in I$. Note that the relation "to be similar to" is an equivalence relation.

A (k,n)-covering $S = \{G_1, \cdots, G_k\}$ is called minimal if $f(k,n) = c(S)$ and omission of any vertex from one of the G_i results in a collection of cliques which does not cover K_n. Clearly, any (k,n)-covering S with $c(S) = f(k,n)$ contains a minimal (k,n)-covering. In the following Lemmas several properties of minimal coverings, which we will make us of when determining the values of f, are stated.

Lemma 2.1. *Let $S = \{G_1, \cdots, G_k\}$, $k \geq 3$, be a minimal (k,n)-covering of type $\mathcal{A} = \{A_i : i \in I\}$ and the partial ordering on $\mathcal{A}$ and on $\mathcal{B} = \{V(G_i) : 1 \leq i \leq k\}$ be set inclusion. Then*

P1) *$\mathcal{B}$ is an antichain, (that is, no two elements of $\mathcal{B}$ are comparable),*
P2) *$\bigcup_{j\in A_i} V(G_j) = V(K_n)$ for $i \in I$,*
P3) *$\mathcal{A}$ is an antichain,*
P4) *$A_i \cap A_j \neq 0$ for $i, j \in I$, and*
P5) *$1 < |A_i| < k$ for $i \in I$. Moreover, if $f(k-1,n) > f(k,n)$, then*
P6) *for every j, $1 \leq j \leq k$, there exist $s, t \in I$ such that $s \neq t$ and $j \in A_s \cap A_t$.*

Proof: Property P1 follows immediately from the minimality of S.

Let v be a vertex of K_n, $v \in V(A_i)$. Since S covers K_n each edge vw, $w \in V(K_n) - \{v\}$ must occur in one of the G_j, $j \in A_i$; and this implies P2.

To prove P3 suppose that for some $i, j \in I$, $A_i \subset A_j$. Let $t \in A_j - A_i$ and z be a vertex of $V(A_j)$.

The edge zw, $w \in V(K_n)$ is covered by some G_m, $m \in A_i$. Thus, $S' = \{G'_1, \cdots, G'_k\}$, where $G'_s = G_s$, $s \neq t$, $G'_t = G_t - z$ is also a (k,n)-covering, which contradicts the minimality of S.

Let v and w be vertices from $V(A_i)$, $V(A_j)$, respectively. The edge vw can be covered only by a clique G_t, where $t \in A_i \cap A_j$. Hence $A_i \cap A_j \neq 0$, for any $i, j \in I$ and P4 follows.

Assume $|A_i| = 1$ for some $i \in I$. Then, for $A_i = \{j\}$, $G_j = K_n$ and by minimality of S, this implies $k = 1$, a contradiction. If $|A_i| = k$ for some $i \in I$, then according to P3 $\mathcal{A}$ contains only one set and consequently all the G_i equal K_n. However, this contradicts the minimality of S and P5 is proved.

Since $|V(G_j)| \geq 1$, each of the sets A_y, $y \in V(G_j)$, contains j. There are at least two such sets unless $A_y = A_x$ for all $x, y \in V(G_j)$. But then the clique G_j could be deleted and we would have $f(k-1,n) = f(k,n)$, a contradiction. Now the proof is complete.

Lemma 2.2. *Let n and k be natural numbers, $k \geq 3$, and let $\mathcal{A} = \{A_i : i \in I\}$ be the type of a minimal (k,n)-covering S. Then*

B1) $f(k,n) \geq \left[\frac{m.n}{k}\right]$, *and*

B2) $f(k,n) \geq \left[\frac{n+M}{2}\right]$, *where* $m = \min\{|A_i| : i \in I\}$, *and* $M = \max\{|V(A_i)| : i \in I, |A_i| = 2\}$.

Proof: Denote by $G_1, \cdots, G_k$ the cliques of S. From the definition of m and the fact that S comprises a covering of K_n every vertex of K_n belongs to at least m of the G_i. Thus $\sum_{1 \leq i \leq k} V(G_i) \geq m.n$ and consequently

$$\frac{m.n}{k} \leq \max\{|V(G_i)|, i = 1, \cdots, k\} = c(S) = f(k,n).$$

In order to prove B2 assume without loss of generality that for $A_1 = \{1,2\}$, $|V(A_1)| = M$. In view of P2 we have $V(G_1) \cup V(G_2) = V(K_n)$ and so $n = |V(K_n)| = |V(G_1) \cup V(G_2)| = |V(G_1)| + |V(G_2)| - |V(G_1) \cap V(G_2)| \leq 2.f(k,n) - M$ and B2 follows

To finish this section we give a simple sufficient condition for a collection of k cliques to be a (k,n)-covering.

Lemma 2.3. *Let $S = \{G_1, \cdots, G_k\}$ be a collection of k cliques of K_n of type $\mathcal{A} = \{Ai : i \in I\}$ such that*

$$\bigcup_{1 \leq i \leq k} V(G_i) = V(K_n), \quad \text{and} \quad A_i \cap A_j \neq 0 \quad \text{for} \quad i,j \in I.$$

Then S is a (k,n)-covering.

Proof: The fact that $\bigcup_{1 \leq i \leq k} V(G_i) = K_n$ guarantees that the G_i cover all the vertices of $V(K_n)$ while $A_i \cap A_j \neq 0$ guarantees that S covers all the edges of K_n.

3. A linear programming problem

Consider a (k,n)-covering S of type $\mathcal{A} = \{A_1, \cdots, A_t\}$. We assign to S a linear programming problem $P(S)$ in the following way. Let $\bar{x} = (x_1, \cdots, x_t, y)$ be a $(t+1)$-vector of unknowns and let $\bar{A} = (a_{ij})$ be a $(k\times(t+1))$-matrix with $a_{i,t+1} = -1$ for $i = 1, \cdots, k$, and $a_{ij} = 1$, if $i \in A_j$, otherwise $a_{ij} = 0$ for $1 \leq j \leq t$, $1 \leq i \leq k$. Minimize the objective function $g(\bar{x}) = 0\cdot x_1 + \cdots + 0\cdot x_t + y = y$, where the constraints on the problem are $\bar{A}\bar{x} \leq \bar{0}$, $\bar{x} \geq \bar{0}$ and $x_1 + \cdots + x_t = n$.

It is easy to see that $P(S)$ has a solution. We will denote by $y(S)$ the minimum value of the objective function g and by $\text{int}y(S)$ the minimum value of g for integral vector $\bar{x}$.

Theorem 3.1. *Let S be a (k,n)-covering. Then*

$$f(k,n) \leq y(S) + 2^k,$$

and there exists a (k,n)-covering T with $c(T) = \text{int}y(S)$. Moreover, if S is minimal, then

$$\lceil y(S) \rceil \leq f(k,n) = \text{int}y(S).$$

Proof: Let S be a (k,n)-covering of type $\mathcal{A} = \{A_1, \cdots, A_t\}$ and let $(x_1, \cdots, x_t, y)$ be a feasible integral solution of $P(S)$. Consider a decomposition $B_1, \cdots, B_t$ of $V(K_n)$ such that $|B_j| = x_j, j = 1, \cdots, t$. Then the collection $T = \{H_1, \cdots, H_k\}$ of cliques of K_n given by $V(H_i) = \cup B_j$, where the union is taken over all j with $i \in A_j, i = 1, \cdots, k$ is either of type $\mathcal{A}$ or of type $\mathcal{B}$; $\mathcal{B}$ being a subfamily of $\mathcal{A}$. (Note the latter case happens when some of the x equal 0.) So by Lemma 2.3 T is a (k,n)-covering. Clearly, $c(T) \leq y$ and we get

$$f(k,n) \leq \text{int}y(S). \tag{3.1}$$

Clearly, if $(x_1, \cdots, x_t, y)$ is a minimal feasible integral solution, then $c(T) = y\text{int}y(S)$. Further, if $(x_1, \cdots, x_t, y)$ is a minimal feasible vector of $P(S)$, that is, $y = y(S)$, then $(\lfloor x_1 \rfloor, \cdots, \lfloor x_{t-1} \rfloor, n - \sum_{1\leq i\leq t-1} \lfloor x_1 \rfloor, \lceil y \rceil + t - 1)$ is an integral feasible vector, where $|\mathcal{A}| = t \leq 2^k$, as $\mathcal{A}$ is a system of subsets of $\{1, 2, \cdots, k\}$. Thus $\text{int}y(S) \leq \lceil y \rceil + 2^k - 1 \leq y(S) + 2^k$ and the first inequality follows from (3.1).

Now let S be a minimal (k,n)-covering. Put $x_i' = |V(A_i)|, i = 1, \cdots, t$, and $y' = c(S) = f(k,n)$. It is a routine matter to check that $(x_1', \cdots, x_t', y')$ is an integral feasible vector of $P(S)$. Thus, $\text{int}y(S) \leq f(k,n)$ and together with (3.1) and the obvious fact that $\lceil y(S) \rceil \leq \text{int}y(S)$ we get the second inequality.

Now we establish a relationship between two similar coverings from the viewpoint of the corresponding linear programming problems.

Theorem 3.2. *Let S and T be (k,n)- and (k,m)-coverings, respectively, of the same type. Then $\bar{x}$ is a feasible vector of $P(S)$ if and only if $\frac{m}{n}\bar{x}$ is a feasible vector of $P(T)$. In particular, $\frac{y(S)}{n} = \frac{y(T)}{m}$.*

Proof: Denote by $\mathcal{A} = \{A_1, \cdots, A_t\}$ the type of S, by $\mathcal{B} = \{B_1, \cdots, B_t\}$ the type of T. Because S and T are similar it is possible to rearrange the cliques of T and the sets of $\mathcal{B}$ in such a way that $B_i = A_i$, $i = 1, \cdots, t$. This means that the constraints on $P(S)$ and $P(T)$ differ from each other only in the last condition. Instead of $x_1 + \cdots + x_t = n$ in $P(S)$ we have $x_1 + \cdots + x_t = m$ in $P(T)$. Therefore $(x_1, \cdots, x_t, y)$ is a feasible vector of $P(S)$ if and only if $(\frac{m}{n}x_1, \cdots, \frac{m}{n}x_t, \frac{m}{n}y)$ is a feasible vector of $P(T)$. Consequently, $y(T) = \frac{m}{n}y(S)$, which yields the required assertion.

4. Values of $f(k,n)$ for small k

It is easy to see that $f(1,n) = n$, $n \geq 2$. Assume $S = \{G_1, G_2\}$ is a $(2,n)$-covering. But then $|V(G_i)| < n$, $i = 1,2$, implies that the edge x_1x_2 of K_n, where $x_i \in V(K_n) - V(G_i)$, $i = 1,2$, is not covered. Thus, also in this case, $f(2,n) = n$, $n \geq 2$. The first non-trivial value of $f(k,n)$ is that when $k = 3$.

Theorem 4.1. *Let n be a natural number. Then*

i) $f(3,n) = \lceil \frac{2}{3}n \rceil$ *for* $n \geq 3$,
ii) $f(4,n) = \lceil \frac{3}{5}n \rceil$ *for* $n \geq 13$,
iii) $f(5,n) = \lceil \frac{5}{9}n \rceil$ *for* $n \geq 39$.

Proof: Throughout this proof $S = \{G_1, \cdots, G_k\}$ will be a minimal (k,n)-covering of type $\mathcal{A} = \{A_i : i \in I\}$. Put $a_t = |\{A_i : i \in I, |A_i| = t\}|$, and $b_t = \sum_{|(A_i)|=t} |V(A_i)|$, $t = 1, \cdots, k$. Clearly, $\sum_{1 \leq t \leq k} b_t = n$, and

$$\frac{\sum_{1 \leq t \leq k} t.b_t}{k} \leq f(k,n) \tag{4.1}$$

In all three cases, $k = 3,4,5$, we will first construct a (k,n)-covering in order to obtain an upper bound on $f(k,n)$ and, at the same time, to show that $f(k-1,n) > f(k,n)$ (that is, that the assumptions of P6 in Lemma 2.1 are fulfilled). Then by inspecting the set systems with properties P1–P6 and subsequently solving the corresponding linear programming problems associated with some of them we will show that $f(k,n)$ equals this upper bound.

i) $k = 3$ and $n > 3$.

Let $T = \{G_1, G_2, G_3\}$ be a collection of cliques of K_n of type $\mathcal{B} = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{2,3\}\}$, where $|V(B_i)| = \lceil \frac{n-i+1}{3} \rceil$. From Lemma 2.3 T is a $(3,n)$-covering and $c(T) = \lceil \frac{2}{3}n \rceil$. Therefore, $f(3,n) \leq \lceil \frac{2}{3}n \rceil < f(2,n)$.

Consider a minimal $(3,n)$-covering $S = \{G_1, G_2, G_3\}$ of type $\mathcal{A} = \{A_i : i \in I\}$. Applying P5 we obtain $|A_i| = 2, i \in I$, and according to P3 and P6, $\mathcal{A} = \{\{2,3\},\{1,3\},\{2,3\}\}$. The minimum value of the objective function of $P(S)$ is $\frac{2}{3}n$ and is attained by the vector $(\frac{1}{3}n, \frac{1}{3}n, \frac{1}{3}n, \frac{2}{3}n)$. By Theorem 3.1 $f(k,n) \geq \lceil \frac{2}{3}n \rceil$, which establishes part i).

ii) $k = 4$ and $n \geq 13$.

Let $T = \{G_1, \cdots, G_4\}$ be a collection of cliques of type $\mathcal{B} = \{B_1 = \{1,3\}, B_2 = \{2,3\}, B_3 = \{1,2,4\}, B_4 = \{3,4\}\}$, where $|V(B_i)| = \lceil \frac{n-i}{5} \rceil, i = 1,2,4$, $|V(B_3)| = \lceil \frac{2n-1}{5} \rceil$. Following Lemma 2.3, T is a $(4,n)$-covering and $c(T) = \lceil \frac{3}{5}n \rceil$, which implies $f(4,n) \leq \lceil \frac{3}{5}n \rceil < f(3,n)$. Consider a minimal $(4,n)$-covering $S = \{G_1, G_2, G_3, G_4\}$ of type $\mathcal{A} = \{A_i : i \in I\}$. The proof of the reverse inequality will be broken into two cases. First, let $a_2 < 3$. From P5 it follows that $2 \leq |A_i| \leq 3, i \in I$. By inequality (4.1) and the upper bound already obtained, we have

$$\frac{2b_2 + 3b_3}{4} \leq \left\lceil \frac{3}{6}n \right\rceil .$$

As $3(b_2 + b_3) = 3n$, this implies $3n - 4 \lceil \frac{3}{5}n \rceil \leq b_2$. Thus, for $a_2 < 3$ and $n \geq 4$, according to B2 we have $c(S) = f(4,n) \geq \left\lceil \frac{n + \left\lceil \frac{3n-4\lceil \frac{3}{5}n \rceil}{2} \right\rceil}{2} \right\rceil \geq \lceil \frac{3}{5}n \rceil$ and hence $f(4,n) = \lceil \frac{3}{5}n \rceil$. On the other hand, suppose $a_2 > 3$. From P4 the intersection of any two of the A_i is not empty, and so $\mathcal{A}$ contains exactly three sets A_1, A_2, A_3 of cardinality two. In order to satisfy P6 $\mathcal{A}$ has to contain another set and owing to P3 we must have $\bigcap_{1 \leq i \leq 3} A_i \neq 0$. Thus, S and all the minimal $(4,n)$-coverings are similar to the covering T. The minimum value of the objective function of $P(S)$ is equal to $\frac{3}{5}n$ and is attained by the vector $(\frac{1}{5}n, \frac{1}{5}n, \frac{1}{5}n, \frac{2}{5}n, \frac{3}{5}n)$. Theorem 3.1 finishes the proof of this part.

iii) $k = 5$ and $n \geq 39$.

Consider a collection $T = \{G_1, \cdots, G_5\}$ of cliques of type $\mathcal{B} = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{2,3,4\}, B_4 = \{2,3,5\}, B_5 = \{1,4,5\}\}$, where $|V(B_1)| = \lceil \frac{n-6}{9} \rceil$, $|V(B_2)| = \lceil \frac{n-3}{9} \rceil$, $|V(B_3)| = \lceil \frac{2n-5}{9} \rceil$, $|V(B_4)| = \lceil \frac{2n-3}{9} \rceil$, $|V(B_5)| = n - \sum_{1 \leq i \leq 4} |V(B_i)|$.

By Lemma 2.3 T is a $(5,n)$-covering with $c(T) = \lceil \frac{5}{9}n \rceil$. Therefore $f(5,n) \leq \lceil \frac{5}{9}n \rceil < f(4,n)$. Consider a minimal $(5,n)$-covering $S = \{G_1, G_2, \cdots, G_5\}$ of type $\mathcal{A} = \{Ai : i \in I\}$. In view of P4 we have $a_2 \leq 4$. In order to prove that $f(5,n) \geq \lceil \frac{5}{9}n \rceil$ we will distinguish five cases according to the value of a_2.

a) $a_2 = 0$. Then $m = \min_{i \in I} |A_i| \geq 3$ and by B1 $c(S) = f(5,n) \geq \lceil \frac{3}{5}n \rceil \geq \lceil \frac{5}{9}n \rceil$.

b) $a_2 = 1$. According to P5 we have $2 \leq |A_i| \leq 4$, and hence $b_2 + b_3 + b_4 = n$.

With respect to (4.1) we obtain $\frac{2b_2+3b_3+4b_4}{5} \leq \lceil \frac{5}{9}n \rceil$, or $b_2 \geq 3n-5\lceil \frac{5}{9}n \rceil$. From B2 we get $c(S) = f(5,n) \geq \lceil \frac{n+b_2}{2} \rceil \geq \lceil 2n - \frac{5}{2}\lceil \frac{5}{9}n \rceil \rceil$.

c) $a_2 = 2$. In this case to fulfill propositions P1–P6 it must be that $a_4 = 0$ and S is similar to a covering of type. The minimum value of the objective function of $P(S)$ equals $\frac{5}{9}n$ and is attained by vector $(\frac{1}{9}n, \frac{1}{9}n, \frac{2}{9}n, \frac{2}{9}n, \frac{3}{9}n, \frac{5}{9}n)$.

d) $a_2 = 3$. Let A_1, A_2, A_3 are the sets of $\mathcal{A}$ of cardinality 2, which have (in order to satisfy P4) pairwise nonempty intersections. But now P3 and P6 cannot be satisfied simultaneously.

e) $a_2 = 4$. From P4 the intersection of all the A_i of cardinality two is nonempty and it then follows that S is similar to a covering of type $\mathcal{A} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3,4,5\}\}$. The minimum integer value of the objective function of $P(S)$ is $\lceil \frac{4}{7}n \rceil$ and therefore $c(S) = \lceil \frac{4}{7}n \rceil \geq \lceil \frac{5}{9}n \rceil$. The proof of Theorem 4.1 is complete.

The methods used in the proof of Theorem 4.1 are not suitable for $k > 5$. Therefore, we will not obtain all the values of $f(k,n)$ in the cases $k = 6$ and $k = 7$.

Theorem 4.2. *Let n be a natural number. Then*

i) $\lceil \frac{n}{2} \rceil \leq f(6,n) \leq \lceil \frac{n}{2} \rceil + r_n, n \geq 4$, *where* $r_n = 1$ *for* $n \equiv 2 \pmod 4$, *and otherwise* $r_n = 0$, *and*

ii) $\lceil \frac{3}{7}n \rceil \leq f(7,n) \leq \lceil \frac{3}{7}n \rceil + s_n, n \geq 7$, *where* $s_n = 1$ *for* $n \equiv 2 \pmod 7$, *and otherwise* $s_n = 0$.

Proof: Let $S_1 = \{G_1, \cdots, G_6\}$, $S_2 = \{G_1, \cdots, G_7\}$ be collections of cliques of K_n of types $\mathcal{A} = \{A_1 = \{1,4,5\}, A_2 = \{1,2,6\}, A_3 = \{2,3,5\}, A_4 = \{3,4,6\}\}$, $\mathcal{B} = \{B_1 = \{1,2,6\}, B_2 = \{1,3,5\}, B_3 = \{1,4,7\}, B_4 = \{2,3,7\}, B_5 = \{2,4,5\}, B_6 = \{3,4,6\}, B_7 = \{5,6,7\}\}$ ($\mathcal{B}$ is a Fano plane), respectively. Further, let $\sum_{1\leq i\leq 4} |V(A_i)| = n$, $\sum_{1\leq i\leq 7} |V(B_i)| = n$, $|V(A_i)| - |V(A_j)| \leq 1, 1 \leq i < j \leq 4, 0 \leq |V(B_i)| - |V(B_j)| \leq 1, 1 \leq i < j \leq 7$. Then, according to Lemma 2.3, S_1 is a $(6,n)$-covering with $c(S_1) = \lceil \frac{n}{2} \rceil + r_n$ and S_2 is a $(7,n)$-covering with $c(S_2) = \lceil \frac{3}{7}n \rceil + s_n$; thus giving the upper bounds as stated in i) and ii).

To prove the lower bounds consider a minimal $(6,n)$-covering ($(7,n)$-covering) of type $\mathcal{A} = \{A_j : j \in J\}$. Suppose first that there is in a set of cardinality 2. Then, in accordance with B2

$$f(k,n) \geq \left\lceil \frac{n+M}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil . \tag{4.2}$$

If $|A_j| \geq 3$ for each $j \in J$, then B1 implies $f(6,n) \geq \lceil \frac{m.n}{6} \rceil \geq \lceil \frac{3}{6}n \rceil = \lceil \frac{n}{2} \rceil$, $f(7,n) \geq \lceil \frac{m.n}{7} \rceil \geq \lceil \frac{3}{7}n \rceil$. In view of equation (4.2) we arrive at the desired conclusion.

5. Asymptotic properties of f

In order to study the values of $f(k,n)$ for $n \to \infty (k \to \infty)$ we introduce one more concept.

Let $\mathcal{S} = \{S_i\}_{i=1}^{\infty}$, where S_i is a minimal (k, n_i)-covering. Then the sequence $\mathcal{S}$ will be called k-minimal if $\{n_i\}_{i=1}^{\infty}$ is strictly increasing and the coverings S_i are similar.

For any $k > 1$ there exists a k-minimal sequence (see, e.g. the proof of Theorem 5.2).

For a k-minimal sequence $\mathcal{S} = \{S_i\}$ we put $y(\mathcal{S}) = \frac{y(S_i)}{n_i}$, where $y(S_i)$ is as defined in Section 3. In view of Theorem 3.2, the function $y(\mathcal{S})$ is well defined.

Theorem 5.1. *If $\mathcal{S}$ and $\mathcal{T}$ are two k-minimal sequences, then $y(\mathcal{S}) = y(\mathcal{T})$.*

Proof: Suppose $y(\mathcal{S}) > y(\mathcal{T})$. Then there exists n_0 such that for $n \geq n_0$, $n.y(\mathcal{S}) > n.y(\mathcal{T}) + 2^k$. Let S be a minimal (k,n)-covering in $\mathcal{S}$ with $n \geq n_0$ and let T be a (k,n)-covering, which is similar to the coverings in $\mathcal{T}$. Then by Theorems 3.1 and 3.2 we get

$$f(k,n) \leq y(T) + 2^k = n.y(\mathcal{T}) + w^k < n.y(\mathcal{S}) = y(S) \leq f(k,n),$$

which contradicts our assumption.

The next assertion shows that $\lim_{n\to\infty} \frac{f(k,n)}{n}$ exists for arbitrary k.

Theorem 5.2. *Let $\mathcal{S}$ be a k-minimal sequence. Then*

$$\lim_{n\to\infty} \frac{f(k,n)}{n} = y(\mathcal{S}).$$

Proof: Suppose $\mathcal{S} = \{S_i\}_{i=1}^{\infty}$ is a k-minimal sequence, where S_i covers K_{n_i}. From Theorem 3.1 we obtain $y(S_i) \leq f(k,n) \leq y(S_i) + 2^k$, or $\frac{y(S_i)}{n_i} = y(\mathcal{S}) \leq \frac{f(k,n_i)}{n_i} \leq y(\mathcal{S}) + \frac{2^k}{n_i}$. Thus we have $\lim_{i\to\infty} \frac{f(k,n_i)}{n_i} = y(\mathcal{S})$. With respect to Theorem 5.1 all that remains to be shown is that for any k there exists n_0 and a collection $\{\mathcal{S}_j : j \in J\}$ of k-minimal sequence such that J is finite and for every $n > n_0$ at least one of $\mathcal{S}_i$ contains a (k,n)-covering. Consider a sequence $\mathcal{T} = \{T_n\}_{n=1}^{\infty}$, where T_n is a minimal (k,n)-covering. Decompose $\mathcal{T}$ into subsequences such that two coverings belong to the same subsequence if and only if they are similar. As the type of a (k,n)-covering is a collection of subsets of $\{1,\ldots,k\}$ we have a finite number of subsequences. Omitting those which are finite yields a suitable collection of subsequences and the proof is complete.

The following assertion shows that the sequence $\frac{f(k,n)}{n}$ contains a constant infinite subsequence.

Theorem 5.3. *Let* $\lim_{n\to\infty} \frac{f(k,n)}{n} = t$. *Then for an infinite number of n, $f(k,n) = t.n$.*

Proof: First we prove an auxiliary statement.

Lemma 5.4. *Let S be a (k, n_0)-covering. Then there exists a sequence $\mathcal{T}_s = \{T_i\}_{i=1}^{\infty}$, where T_i is a (k, n_i)-covering and $\{n_i\}_{i=1}^{\infty}$ is strictly increasing, such that $\frac{c(T_i)}{n_i} = \frac{y(S)}{n_0}$ for $i = 1, 2, \ldots$.*

Proof of Lemma: Let $\mathcal{S} = \{S_n\}_{n=n_0}^{\infty}$, where S_n is a (k, n)-covering of the same type as S. If $\bar{x}$ is a minimum feasible vector of $P(S)$, then from the proof of Theorem 3.2, $\frac{n}{n_0}\bar{x}$ is a minimum feasible vector of $P(S_n)$. Since $\bar{x}$ is rational (as all the coefficients in $P(S)$ are), then for infinitely many n, the vector $\frac{n}{n_0}\bar{x}$ is integral. This means that for these values of n, int $y(S_n) = y(S_n) = \frac{n}{n_0}y(S)$ and applying Theorem 3.1 we get the sequence $\{T_i\}$.

Now we can proceed to the proof of Theorem 5.3. Suppose there is n_0 such that $f(k, n_0) < t.n_0$. Let S be a minimal (k, n_0)-covering. Then, of Theorem 3.2, $y(S) \le f(k, n_0) < t.n_0$. For the sequence $\mathcal{S}_s = \{T_i\}_{i=1}^{\infty}, t > \frac{y(S)}{n_0} = \frac{c(T_i)}{n_i} \ge \frac{f(k,n_i)}{n_i}$ and consequently $\lim_{i\to\infty} \frac{f(k,n_i)}{n_i} \le \frac{y(S)}{n_0} < t$ which contradicts $\lim_{i\to\infty} \frac{f(k,n)}{n} = t$. Therefore

$$f(k, n) \ge t.n \tag{5.1}$$

for all $n \ge 2$.

Let $\mathcal{S}$ be a k-minimal sequence. Then, for a (k, n_0)-covering $S \in \mathcal{S}$, $\frac{y(S)}{n_0} = t$. Further, for the sequence $\mathcal{T}_s = \{T_i\}_{i=1}^{\infty}$, $\frac{c(T_i)}{n_i} = \frac{y(S)}{n_0} = t$. So $\frac{f(k,n_i)}{n_i} \le \frac{c(T_i)}{n_i} = t$ which together with (5.1) yields $f(k, n_i) = t.n_i$. The fact that $\{n_i\}_{i=1}^{\infty}$ is increasing infinite sequence establishes the assertion.

We would like to end the paper by confirming the expected fact that for large $k \{\frac{f(k,n)}{n}\}_{n=1}^{\infty}$ tends to 0.

Theorem 5.5. $\lim_{k\to\infty}(\lim_{n\to\infty} \frac{f(k,n)}{n}) = 0$.

Proof: Put $\lim_{n\to\infty} \frac{f(k,n)}{n} = c_k$. Clearly, $f(k_1, n) \ge f(k_2, n)$, for $k_1 < k_2$. Therefore $c_{k_1} \ge c_{k_2}$. In addition, $c_k > 0$, and hence $\lim_{k\to\infty} c_k$ exists. Now we will show that there is a subsequence of $\{c_k\}_{k=2}^{\infty}$ which converges to zero.

Consider a positive integer v, $v \equiv 1 \pmod 6$ and put $k = \frac{v(v-1)}{6}$. Then there exists a $(v, 3, 1)$-BIBD containing k blocks which induces a (k, n)-covering S with $c(S) = 3$. Let $\{S_n\}_{n=v}^{\infty}$ be a sequence of (k, n)-coverings which are similar to S. Following Theorem 3.2 we get $\frac{3}{v} = \frac{y(S)}{v} = \frac{y(S_n)}{n}$ and subsequently (Theorem 3.1) $f(k, n) \le \frac{3}{v}n + 2^k$ in view of Theorem 3.1. Thus $c_k \le \frac{3}{v}$ and immediately $c_k = \frac{c_{v(v-1)}}{6} \to 0$ as $v \to \infty$ and $v \equiv 1 \pmod 6$.

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