On Magic Labelings of Grid Graphs

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Abstract. This paper deals with the problem of labeling the vertices, edges and interior faces of a grid graph in such a way that the label of the face itself and the labels of vertices and edges surrounding that face add up to a value prescribed for that face.

1. Introduction

Various types of labelings of graphs have been intensively studied by combinatorialists for some time. The notion of magic labeling has its origin in classical Chinese mathematics of 13th century (see [2]). Only recently have these labelings been investigated using modern notions of graph theory.

The notions of magic and consecutive labelings of plane graphs were defined by Lih Ko-Wei in [2] where magic labelings of type $(1, 1, 0)$ for wheels, friendship graphs and prisms are given. Magic labelings of type $(1, 1, 1)$ for fans, planar bipyramids and ladders are described in [1].

2. Terminology and Notation

The graphs considered here will be finite. For $n \geq 2$, $m \geq 1$ let G_n^m be the grid graph embedded in the plane (Fig. 1) and let p, q and f be, respectively, the number of vertices, edges and interior faces.

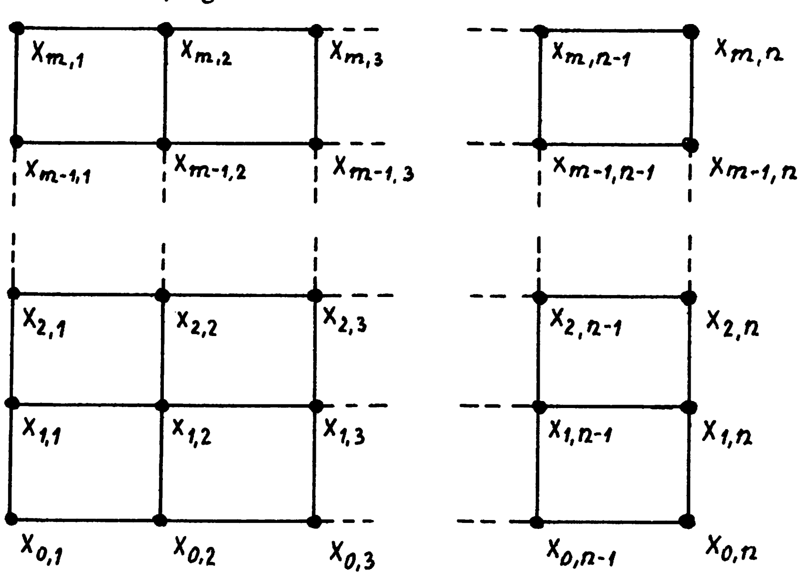

Figure 1

A labeling of type $(1,1,1)$ assigns labels from the set $\{1,2,\ldots,p+q+f\}$ to the vertices, edges and interior faces of a grid graph G_n^m in such a way that each vertex, edge and interior face receives exactly one label and each number is used exactly once as a label. If we label only vertices, only edges or only faces, we call such a labeling a vertex labeling, an edge labeling or a face labeling, respectively.

The weight of a face under a labeling is the sum of the label of the face itself and the labels of vertices and edges surrounding that face. A labeling is said to be magic if all interior faces have the same weight. A labeling is said to be consecutive if the weights of all interior faces constitute a set of consecutive integers. Two labelings g and g' are said to be complementary if the sum of the g-weight and g'-weight of each interior face is a constant.

We shall use the expressions $a = \frac{(-1)^i+1}{2}$, $b = \frac{(-1)^{i+1}+1}{2}$, $c = \frac{(-1)^j+1}{2}$, $d = \frac{(-1)^{j+1}+1}{2}$ (for $i \in I$ and $j \in J$) to simplify later notations. Set $I = \{1,2,\ldots,n\}$ and set $J = \{0,1,2,\ldots,m\}$. The expression $\lfloor r \rfloor$ will denote the greatest integer less than or equal to r.

3. The Main Results

Define the labeling g_1 of a grid graph G_n^m as follows.

$$\begin{aligned} g_1(x_{j,i}) =& ac\left(\frac{m+1}{2}(2n-i+2) - \left\lfloor\frac{m+1}{2}\right\rfloor - \frac{j}{2}\right) \\ &+ ad\left(\frac{i}{2}(m+1) - \frac{j+1}{2} + 1\right) \\ &+ bc\left(\frac{m+1}{2}(i-1) + \frac{j}{2} + 1\right) \\ &+ bd\left(\frac{m+1}{2}(2n-i+1) - \left\lfloor\frac{m+1}{2}\right\rfloor + \frac{j+1}{2}\right) \end{aligned}$$

for $i \in I$ and $j \in J$.

Theorem 1. *The vertex labeling g_1 of G_n^m is magic if $n \geq 2$ and $m \geq 1$.*

Proof: First we shall show that the vertex labeling g_1 uses each integer $1,2,\ldots,p$. We will consider only the case where m and n are even. The other three cases are dealt with similarly.

If i is odd and j is even then $g_1(x_{j,i})$ is equal successively to $(1,2,\ldots,\frac{m}{2}+1)$, $(m+2, m+3,\ldots,\frac{3m}{2}+2),\ldots,((mn-2m+n)/2, (mn-2m+n)/2+1,\ldots,(mn-m+n)/2)$, if i is even and j is odd then $g_1(x_{j,i})$ is equal successively to $(\frac{m}{2}+2, \frac{m}{2}+3,\ldots,m+1), (\frac{3m}{2}+3, \frac{3m}{2}+4,\ldots,2m+2),\ldots,((mn-m+n)/2+1, (mn-m+n)/2+2,\ldots,(mn+n)/2)$, if i and j are even then $g_1(x_{j,i})$ successively attains values $((mn+n)/2+1, (mn+n)/2+2,\ldots,(mn+m+n)/2+1), ((mn+2m+n)/2+2, (mn+2m+n)/2+3,\ldots,(mn+3m+$

$n)/2+2),\ldots,(mn-m+n, mn-m+n+1,\ldots,mn+n-\frac{m}{2})$ and finally if i and j are odd then $g_1(x_{j,i})$ is equal successively to $((mn+m+n)/2+2,(mn+m+n)/2+3,\ldots,(mn+2m+n)/2+1),((mn+3m+n)/2+3,(mn+3m+n)/2+4,\ldots,(mn+4m+n)/2+2),\ldots,((2mn-m+2n)/2+1,(2mn-m+2n)/2+2,\ldots,mn+n)$. It is not difficult to check that for every $i \in I-\{n\}$ and $j \in J-\{m\}$ the weight of each interior face $w_{j,i}=g_1(x_{j,i})+g_1(x_{j,i+1})+g_1(x_{j+1,i})+g_1(x_{j+1,i+1})$ is $2p+3$ if m is even and $2p+2$ if m is odd. This proves that all interior faces of G_n^m under the vertex labeling g_1 have the same weight.

Define the edge labeling g_2 of G_n^m as follows.

$$
\begin{aligned}
g_2(x_{j,i}x_{j,i+1}) =&(ac+bd)\left(\frac{m+1}{2}(n-i)-\frac{j-1}{2}\right)\\
&+(ad+bc)\left(\frac{m+1}{2}(n+i-1)-\frac{j-1}{2}\right)
\end{aligned}
$$

if m is even and n is odd,

$$
\begin{aligned}
=&ac\left(\frac{m+1}{2}(n+i)-\frac{m+j}{2}\right)\\
&+ad\left(\frac{m+1}{2}(n-i)-\frac{j-1}{2}\right)\\
&+bc\left(\frac{m+1}{2}(n-i+1)-\frac{m+j}{2}\right)\\
&+bd\left(\frac{m+1}{2}(n+i-1)-\frac{j-1}{2}\right)
\end{aligned}
$$

if m and n are even,

$$
\begin{aligned}
=&c\left(\frac{m+1}{2}(n+i-1)-\frac{j}{2}\right)\\
&+d\left(\frac{m+1}{2}(n-i)-\frac{j-1}{2}\right)
\end{aligned}
$$

if m is odd; $i \in I-\{n\}$ and $j \in J$.

$$
\begin{aligned}
g_2(x_{j,i}x_{j+1,i}) =&a\left((m+1)(n-1)+\left\lfloor\frac{n+1}{2}\right\rfloor m+\frac{i}{2}+\left\lfloor\frac{n}{2}\right\rfloor j\right)\\
&+b\left((m+1)(n-1)+\frac{i+1}{2}+\left\lfloor\frac{n+1}{2}\right\rfloor j\right);
\end{aligned}
$$

$i \in I$ and $j \in J-\{m\}$.

Theorem 2. *The edge labeling g_2 of G_n^m is consecutive if $n \geq 2, m \geq 1$ and $n+m \neq 3$.*

Proof: We now restrict ourselves to the case where m and n are even. The other three cases are dealt with similarly. We can proceed analogously as in the proof of Theorem 1 that if i is odd and j is even then $g_2(x_{j,i}x_{j,i+1})$ successively attain values $(1, 2, \ldots, \frac{m}{2}+1), (m+2, m+3, \ldots, \frac{3m}{2}+2), \ldots, ((mn-2m+n)/2, (mn-2m+n)/2+1, \ldots, (mn-m+n)/2)$, if i is even and j is odd then $g_2(x_{j,i}x_{j,i+1})$ attain values $(\frac{m}{2}+2, \frac{m}{2}+3, \ldots, m+1), (\frac{3m}{2}+3, \frac{3m}{2}+4, \ldots, 2m+2), \ldots, ((mn-3m+n)/2, (mn-3m+n)/2+1, \ldots, (mn-2m+n)/2-1)$, if i and j are odd then $g_2(x_{j,i}x_{j,i+1})$ is equal successively to $((mn-m+n)/2+1, (mn-m+n)/2+2, \ldots, (mn+n)/2), ((mn+m+n)/2+2, (mn+m+n)/2+3, \ldots, (mn+2m+n)/2+1), \ldots, ((2mn-3m+2n)/2, (2mn-3m+2n)/2+1, \ldots, mn-m+n-1)$ and if i and j are even then $g_2(x_{j,i}x_{j,i+1})$ successively assume values $((mn+n)/2+1, (mn+n)/2+2, \ldots, (mn+m+n)/2+1), ((mn+2m+n)/2+2, (mn+2m+n)/2+3, \ldots, (mn+3m+n)/2+2), \ldots, (mn-2m+n-1, mn-2m+n, \ldots, (2mn-3m+2n)/2-1)$. If i is odd then $g_2(x_{j,i}x_{j+1,i})$ successively attain values $(mn-m+n, mn-m+n+1, \ldots, mn-m+\frac{3n}{2}-1), mn-m+\frac{3n}{2}, mn-m+\frac{3n}{2}+1, \ldots, mn-m+2n-1), \ldots, (\frac{n}{2}(3m+1)-m, \frac{n}{2}(3m+1)-m+1, \ldots, \frac{3}{2}mn-m+n-1)$ and finally if i is even then $g_2(x_{j,i}x_{j+1,i})$ successively assume values $(\frac{3}{2}mn-m+n, \frac{3}{2}mn-m+n+1, \ldots, \frac{3n}{2}(m+1)-m-1), (\frac{3n}{2}(m+1)-m, \frac{3n}{2}(m+1)-m+1, \ldots, \frac{3}{2}mn-m+2n-1), \ldots, (2mn-m+\frac{n}{2}, 2mn-m+\frac{n}{2}+1, \ldots, 2mn-m+n-1)$.

This proves that the edge labeling g_2 uses each integer $1, 2, \ldots, q$.

It remains to show that the weights of all interior faces constitute a set of consecutive integers. The weights of all 4-sided faces of G_n^m constitute the set $\{u_{j,i}: u_{j,i} = g_2(x_{j,i}x_{j,i+1}) + g_2(x_{j,i}x_{j,i+1}) + g_2(x_{j,i}x_{j+1,i}) + g_2(x_{j,i+1}x_{j+1,i+1}), i \in I-\{n\}$ and $j \in J-\{m\}\}$.

We distinguish four cases.

Case 1. If m is even and n is odd then it is simple to verify that the previous set of weights of 4-sided faces consists of the consecutive integers $\{\frac{7mn}{2}+3n-2m, \frac{7mn}{2}+3n-2m+1, \ldots, \frac{9mn}{2}+3n-3m-1\}$.

Case 2. If m and n are even then the set consists of the consecutive integers $\{\frac{7mn-5m}{2}+3n, \frac{7mn-5m}{2}+3n+1, \ldots, \frac{9mn-7m}{2}+3n-1\}$.

Case 3. If m is odd and n is even then the set consists of the consecutive integers $\{\frac{7mn-5m-1}{2}+3n, \frac{7mn-5m-1}{2}+3n+1, \ldots, \frac{9mn-7m-3}{2}+3n\}$.

Case 4. If m and n are odd then the set consists of the consecutive integers $\{\frac{7mn-1}{2}+3n-2m, \frac{7mn-1}{2}+3n-2m+1, \ldots, \frac{9mn-3}{2}+3n-3m\}$.

Thus the edge labeling g_2 is consecutive and the proof is completed.

Theorem 3. *For $n \geq 2$, $m \geq 1$ and $n + m \neq 3$ the grid graph G_n^m has a magic labeling of type* $(1,1,1)$.

Proof: Label the vertices and the edges of G_n^m by g_1 and $p + g_2$, respectively. From the previous theorems it easily follows that in the resulting labeling of type $(1,1,0)$ the weights of interior faces constitute a set of consecutive integers.

Hence, if g_3 is the complementary face labeling with values in the set $\{p + q + 1, \dots, p + q + f\}$ then the labelings g_1, $p + g_2$ and g_3 combine to a magic labeling of type $(1,1,1)$.

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References

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2. Lih Ko-Wei, *On magic and consecutive labelings of plane graphs*, Utilitas Math. 24 (1983), 165–197.