A Sufficient Condition on Degree Sums of Independent Triples for Hamiltonian Cycles in 1-Tough Graphs

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Abstract. We prove that if G is a 1-tough graph with $n = |V(G)| \geq 13$ such that the degree sum of any three independent vertices is at least $(3n - 14)/2$, then G is hamiltonian.

Introduction

We consider only finite undirected graphs without loops or multiple edges. For any notation and terminology not defined here we refer the reader to [3].

Let $\omega(G)$ denote the number of components of a graph G. Chvátal [4] defined G to be 1-tough if $\omega(G-S) \leq |S|$ for any subset S of $V(G)$ with $\omega(G-S) > 1$. By $\sigma_k(G)$, or just σ_k we denote $\min\{\sum_{i=1}^{k} d(v_i) \mid \{v_1, \ldots, v_k\}$ is an independent set of vertices in $G\}$ $(k \geq 2)$.

The following is a well-known result due to Jung [5].

Theorem 1 ([5]). *Let G be a 1-tough graph on $n \geq 11$ vertices such that $\sigma_2 \geq n-4$. Then G is hamiltonian.*

The purpose of the present paper is to prove the following generalization of Jung's theorem conjectured by Bauer, Morgana, Schmeichel and Veldman (see [1] and [2]).

Theorem 2. *Let G be a 1-tough graph on $n \geq 13$ vertices such that $\sigma_3 \geq (3n-14)/2$. Then G is hamiltonian.*

We will show that our result is, in a sense, best possible. For an integer s we set $H_s = K_s \vee ((s-1) \cdot K_1 + F)$, where F denotes the graph depicted in Figure 1. The graph H_s, is nonhamiltonian and 1-tough, and it is easy to see that $\sigma_3(H_s) \geq (3 \cdot |V(H_s)| - 15)/2$.

[1] Supported by the DEUTSCHE FORSCHUNGSGEMEINSCHAFT, SFB 303

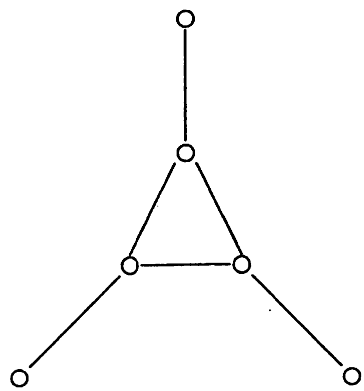

Figure 1

Preliminaries

If C is a cycle in a graph G, we denote by $\overrightarrow{C}$ the cycle C with a given orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We write $uv \in P_C(G)$ if u and v are connected by a path of length at least 2 with all internal vertices in $V(G)\backslash V(C)$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. If $S \subseteq V(C)$, then $S^+ = \{x^+ \mid x \in S\}$ and $S^- = \{x^- \mid x \in S\}$. For $x \in V(G)$, let $N(x)$ be the set of all vertices of G adjacent to x.

Our proof of Theorem 2 heavily relies on the following two lemmas which were established in [1] (the second is implicit in [1, Theorem 9]). As usual, we call a cycle C in a graph G *dominating* if every edge of G has at least one of its vertices on C.

Lemma 1 ([1]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$, and let C be a longest cycle in G. Then C is a dominating cycle. Moreover, if $v \in V(G)\backslash V(C)$ and $A = N(v)$, then $(V(G)\backslash V(C)) \cup A^+$ is independent in G.*

Lemma 2 ([1]). *Let G be a nonhamiltonian 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then G contains a longest cycle C such that* $\max\{d(x) \mid x \in V(G)\backslash V(C)\} \geq \sigma_3/3$.

For the rest of this section, suppose that G is a nonhamiltonian 1-tough graph satisfying the hypothesis of Theorem 2. By Lemma 1 every longest cycle in G is dominating, and by Lemma 2 there exists a longest cycle C in G such that $t := \max\{d(x) \mid x \in V(G)\backslash V(C)\} \geq (3n-14)/6$; let $h \in V(G)\backslash V(C)$ with $d(h) = t$.

We orient C and accordingly enumerate the vertices of $N(h) = \{u_1, \dots, u_t\}$. Since C is a longest cycle, we clearly have $u_i^+ \neq u_{i+1}$ ($i = 1, \dots, t$, indices modulo t).

For $i = 1, \dots, t$ we set $x_i = u_i^+$, $y_i = u_{i+1}^-$ and $L_i = x_i \overrightarrow{C} y_i$. Moreover, we let $X = \{x_1, \dots, x_t\}$, $Y = \{y_1, \dots, y_t\}$ and $Z = X \cap Y$.

Standard arguments yield the following lemma.

Lemma 3. *Let $u_i, u_j \in N(h)$ with $i \neq j$. Then there is no vertex w on $x_i \overrightarrow{C} x_j$ ($y_j \overrightarrow{C} y_i$) such that $x_i w, x_j w^- \in E(G) \cup P_C(G)$ ($y_i w, y_j w^+ \in E(G) \cup P_C(G)$).*

Lemma 4.

(a) $\sum_{i=1}^{t}(|V(L_i)| - 1) = |V(C)| - 2d(h) \leq 3$.
(b) $|Z| \geq d(h) - 3 \geq 2$.

Proof:

(a) Subtract $2d(h) \geq (3n - 14)/3$ from $|V(C)| \leq n - 1$.
(b) By (a), the cardinality of $\{i \in \{1, \dots, t\} \mid |V(L_i)| \geq 2\}$ is at most 3. Thus $|Z| \geq d(h) - 3 \geq 2$ since $d(h) \geq (3n - 14)/6$ and $n \geq 13$. ∎

Lemma 5. *Let $i, j \in \{1, \dots, t\}$, $i \neq j$, such that $x_i y_j \in E(G) \cup P_C(G)$, and suppose that some $z \in Z$ satisfies $d(z) \geq d(h)$. Then*

(a) *$N(z) = N(h)$,*
(b) *z lies on $u_{j+1} \overrightarrow{C} u_i$,*
(c) *$x_j u_{j+1} \notin E(G)$ and $x_j z^+ \notin E(G)$.*

Proof:

(a) By Lemma 1, $X \cup (V(G) \backslash V(C))$ and $Y \cup (V(G) \backslash V(C))$ are independent vertex sets of G, hence $|V(L_i)| \geq 2$ and $|V(L_j)| \geq 2$. By Lemma 4(a), $|V(C)| - 2d(h) \leq 3$, hence $N(z) \subseteq N(h)$ using Lemma 3. Since $d(z) \geq d(h)$, it follows that $N(z) = N(h)$.
(b) This immediately follows from Lemma 3 since $u_{j+1} \in N(z)$ by (a).
(c) By Lemma 3, $x_j u_{j+1} \notin E(G)$ since $x_i y_j \in E(G) \cup P_C(G)$. If $x_j z^+ \in E(G)$, then the cycle $h u_i \overleftarrow{C} z^+ x_j \overrightarrow{C} y_j x_i \overrightarrow{C} u_j z \overleftarrow{C} u_{j+1} h$ contradicts the maximality of C. ∎

Lemma 6. *There are indices $i, j \in \{1, \dots, t\}$, $i \neq j$ such that $x_i y_j \in E(G)$.*

Proof: We show that there is a pair $i \neq j$ of indices such that $x_i y_j \in E(G) \cup P_C(G)$; the assertion then follows from the second statement in Lemma 1.

Assume the contrary of our assertion, and let $p, q \in \{1, \dots, t\}$, $p \neq q$, $v \in V(L_p)$, and $w \in V(L_q)$ such that $vw \in E(G) \cup P_C(G)$. Those vertices must exist since G is 1-tough, and we may assume that w lies on $x_q^+ \overrightarrow{C} y_q^-$. Applying Lemma 4(a), we see that

$$|V(L_p)| + |V(L_q)| \leq |V(C)| - 2d(h) + 2 \leq 5.$$

Thus $|V(L_p)| \leq 2$ and $|V(L_q)| \leq 4$. Now, if a_1 is a vertex on $x_q \overrightarrow{C} w^-$ and a_2 is a vertex on $w^+ \overrightarrow{C} y_q$, then Lemma 3 implies that $a_1 a_2 \notin E(G) \cup P_C(G)$ since $v \in \{x_p, y_p\}$ and $a_1 = x_q$ or $a_2 = y_q$. Moreover, if $V(L_q) = \{x_q, w, w^+, y_q\}$ (without loss of generality), then $v = x_p = y_p$, hence $x_p w^+ \notin E(G) \cup P_C(G)$, since otherwise the cycle $hu_{p+1} \overrightarrow{C} w x_p w^+ \overrightarrow{C} u_p h$ would be a longer cycle than C. But this contradicts the 1-toughness of G because the graph $G - (N(h) \cup \{w\})$ has at least $t + 2$ components. ∎

Lemma 7. *Suppose C and h have been chosen such that $d(h) \geq (3n - 14)/6$ is as small as possible. Then at least one $z \in Z$ satisfies $d(z) \geq d(h)$.*

Proof: First observe that, if some $z \in Z$ satisfies $d(z) \geq (3n - 14)/6$, then $d(z) \geq d(h)$ by the choice of C and h, since $hz^+ \overrightarrow{C} z^- h$ is a longest cycle in G. Thus, taking $\sigma_3 \geq (3n - 14)/2$ into account, we are done if $|Z| \geq 3$. By Lemma 4(a), this holds if h has degree at least 6. If $d(h) < 6$, then $d(h) = 5$ since $d(h) \geq (3n - 14)/6$ and $n \geq 13$. By Lemma 4(b), $|Z| \geq 2$. Hence we may assume $Z = \{z_1, z_2\}$, $d(z_1) \leq 4$ and $d(z_2) \leq 4$. We obtain $(3n - 14)/2 \leq \sigma_3 \leq d(h) + d(z_1) + d(z_2) \leq 13$, hence $n \leq 13$. On the other hand, since $|Z| = 2$ and $d(h) = 5$, $|V(C)| = 2d(h) + 3$. This implies $n \geq |V(C)| + 1 = 2d(h) + 3 + 1 = 14$, a contradiction. ∎

Proof of Theorem 2

The proof is by contradiction. Suppose there exists a nonhamiltonian 1-tough graph on $n \geq 13$ vertices for which $\sigma_3 \geq (3n - 14)/2$. By Lemma 1 every longest cycle in G is dominating, and by Lemma 2 there exists a longest cycle C in G such that some $h \in V(G) \backslash V(C)$ satisfies $d(h) \geq (3n - 14)/6$. Among all longest cycles in G with this property let C be chosen such that $h \in V(G) \backslash V(C)$ with $d(h) = t$ has minimum degree.

We orient C and accordingly enumerate the vertices of $N(h) = \{u_1, \ldots, u_t\}$. For $i = 1, \ldots, t$ we set $x_i = u_i^+$, $y_i = u_{i+1}^-$ and $L_i = x_i \overrightarrow{C} y_i$, where the indices are to be understood modulo t. Moreover, we let $X = \{x_1, \ldots, x_t\}$, $Y = \{y_1, \ldots, y_t\}$ and $Z = X \cap Y$.

By Lemma 7 there exists some $z \in Z$ with $d(z) \geq d(h)$, and by Lemma 6 there are indices $i, j \in \{1, \ldots, t\}$, $i \neq j$, such that i is adjacent to y_j. Assuming that the vertices of $N(h)$ are labeled such that $z = x_1$, we conclude that $i < j$ since, by Lemma 5(b), the vertex $z = x_1$ lies on $u_{j+1} \overrightarrow{C} u_i$. Let i and j be chosen such that i is as large as possible; then Lemma 5(b) implies that none of the vertices of $Y\backslash\{y_j\}$ is adjacent to x_j, hence $N(x_j)\cap((X\cup Y)\backslash\{y_j\}) = \emptyset$. By Lemma 5(c), $x_ju_{j+1} \notin E(G)$ and $x_j\tilde{z}^+ \notin E(G)$ for every $\tilde{z} \in Z$ satisfying $d(\tilde{z}) \geq d(h)$. It follows that if $r \in \{1,\ldots,t\}$, then x_j has at most $|V(L_r)| - 1$ neighbors on $x_r\overrightarrow{C}u_{r+1}$ unless $|V(L_r)| = 1$ and $d(x_r) < (3n-14)/6$. Thus, if $\tilde{Z}$ denotes the set of all vertices in Z having degree less than $(3n-14)/6$, we have

$$d(x_j) \leq |\tilde{Z}| + \sum_{i=1}^{t}(|V(L_i)| - 1)$$

since, by Lemma 1, x_j has no neighbors outside C. Note that $|\tilde{Z}| \leq 2$ since $\sigma_3 \geq (3n-14)/2$ by hypothesis, and that $\sum_{i=1}^{t}(|V(L_i)| - 1) = |V(C)| - 2d(h) \leq n - 1 - 2d(h)$. Thus if a_1, a_2 are distinct vertices of G such that $\{x_j, a_1, a_2\}$ is an independent vertex set of G, then

$$(3n-14)/2 \leq \sigma_3 \leq d(x_j)+d(a_1)+d(a_2) \leq |\tilde{Z}|+(n-1)-2d(h)+d(a_1)+d(a_2). \quad (1)$$

We distinguish three cases.
Case 1: $|\tilde{Z}| = 0$. Setting $a_1 = h$ and $a_2 = x_1$ in (1), we obtain $(3n-14)/2 \leq n-1$ using $d(x_1) = d(h)$. But then $n \leq 12$, contradicting the hypothesis.
Case 2: $|\tilde{Z}| = 1$. Let $\tilde{Z} = \{\tilde{z}_1\}$. Then $d(\tilde{z}_1) \leq d(h) - 1$, and again we arrive at the contradiction $(3n-14)/2 \leq n-1$ by setting $a_1 = h$ and $a_2 = \tilde{z}_1$ in (1).
Case 3: $|\tilde{Z}| = 2$. In this case we choose the two vertices of $\tilde{Z}$ as a_1 and a_2. Then $d(a_1) \leq d(h) - 1$ and $d(a_2) \leq d(h) - 1$, and (1) yields $(3n-14)/2 \leq n-1$. This contradiction completes the proof of Theorem 2. ∎

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