On Tricovers of Pairs by Quintuples: $v \equiv 0 \pmod 4$

A.M. Assaf
Central Michigan University

W.H. Mills
Institute for Defense Analyses

R.C. Mullin
University of Waterloo

Abstract

A tricover of pairs by quintuples of a v-set V is a family of 5-subsets of V (called blocks) with the property that every pair of distinct elements from V occurs in at least three blocks. If no other such tricover has fewer blocks, the tricover is said to be minimum, and the number of blocks in a minimum tricover is the covering number $C_3(v,5,2)$, or simply $C_3(v)$. It is well known that $C_3(v) \geq \lceil v\lceil 3(v-1)/4\rceil/5\rceil = B_3(v)$, where $\lceil x\rceil$ is the least integer not less than x. It is shown here that if $v \equiv 0 \pmod 4$ and $v \geq 8$, then $C_3(v) = B_3(v)$.

1 Introduction

Let V be a finite set of cardinality v. A (k,t)-*cover of index* λ is a family of k-subsets of V (called blocks) with the property that every t-subset of V occurs in at least λ of the blocks. The covering number $C_\lambda(v,k,t)$ is defined to be the number of blocks in a minimum (as opposed to minimal) (k,t)-cover of index λ of V.

For $v > k > t > 0$, let

$$B_\lambda(v,k,t) = \lceil v\lceil(v-1)\ldots\lceil(v-t+1)\lambda/(k-t+1)\rceil\ldots/(k-1)\rceil/k\rceil.$$

Then the quantity $B_\lambda(v,k,t)$ is a lower bound for $C_\lambda(v,k,t)$ (see [31]). Many researchers have been involved in determining the covering numbers known to date (see bibliography). Our interest here is in the case $k=5$, $t=2$, $\lambda=3$, $v \equiv 0 \pmod 4$. For simplicity, let $C_3(v,5,2)$ be denoted by $C_3(v)$ and $B_3(v,5,2)$ be denoted by $B_3(v)$. We adopt the convention that $C_3(4) = B_3(4) = 3$. Covers with $t=2$ and $\lambda=3$ are called *tricovers of pairs*, or *pair tricovers*. For $k=5$, these are then *tricovers of pairs by quintuples*. It was shown in [2] that if $v \equiv 1 \pmod 4$, then $C_3(v) = B_3(v)+1$ for $v \equiv 9$ or $17 \pmod{20}$, and $C_3(v) = B_3(v)$ otherwise.

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2 Some designs useful in constructing tricovers

In this section, we require several other types of combinatorial configurations. For the definitions of balanced incomplete block design (BIBD) and resolvable balanced incomplete block design RBIBD with parameters (v, k, λ), the reader is referred to [18].

Definitions of pairwise balanced design (PBD), group divisible design (GDD) and transversal design (TD) can be found in [36]. Strictly speaking, the definitions given there are for the index $\lambda = 1$. To extend these to general index λ, the requirement that the pairs which occur in precisely one block of each of these configurations is to be replaced by the requirement that each such pair occur in precisely λ blocks. For the existence of transversal designs, our authority is [5] unless another reference is given. Similarly, for the existence of resolvable balanced incomplete block designs and balanced incomplete block designs, see [18].

For group divisible designs, we use the notation $GDD(g_1^{n_1} g_2^{n_2} \cdots g_s^{n_s}, K, \lambda)$ to represent such a design with n_i groups of size $g_i, i = 1, 2, \ldots, s$, whose block sizes lie in the set K, and whose index is λ.

For future reference, we note that there exists a $TD(6, n)$ for all positive integers n with the exception of $n \in \{2, 3, 4, 6\}$ and the possible exception of $n \in \{10, 14, 18, 22, 26, 30, 34, 42\}$, see [4], [5], [1], [30], [33] and [34]. We also use the notation $PBD[(k, w^*), v]$ to denote a pairwise balanced design on v points which has a unique block of size w and all other blocks of size k.

We also require the notion of an incomplete tricover. An incomplete tricover $IT(v, w)$ is a triple (V, W, F) where V is a v-set, W is a w-subset of V, and F is a family of 5-subsets (blocks) of W, where F contains exactly $B_3(v) - B_3(w)$ blocks, with the property that every pair of $(V \times V) \backslash (W \times W)$ occurs in at least three blocks and no pair of $W \times W$ appears in any block. Clearly if there exists an $IT(v, w)$, and if $C_3(w) = B_3(w)$, then $C_3(v) = B_3(v)$. The importance of incomplete tricovers lies, in part, in the following lemma.

Lemma 2.1 *Let u, v and w be non-negative integers congruent to* $0 \pmod 4$, *with* $v \not\equiv 0 \pmod{20}$ *and* $w \equiv \pm v \pmod 5$. *Suppose that there exists*

(i) an $IT(v, w)$, and

(ii) a $GDD((v-w)^a u^1, \{5\}, 3)$, and suppose that

(iii) $C_3(u + w) = B_3(u + w)$.

Then $C_3(a(v - w) + u + w) = B_3(a(v - w) + u + w)$.

Proof. Let D denote the GDD in (ii). Let $G_1, G_2, \ldots, G_a$ be the groups of size $v-w$ of D, and G^* be the group of size u. Let W be a set of w points disjoint from the point-set of D. For $i = 1, 2, \ldots, a$, replace the group G_i by the blocks of an incomplete tricover $(G_i \cup W, W, F_i)$, and replace G^* by the blocks of a minimum tricover of $G^* \cup W$. The result is a tricover of $(\cup_{i=1}^{a} G_i) \cup G^* \cup W$ with $B_3(a(v-w)+u+w)$ blocks.

Indeed, the number of blocks in the $GDD((v-w)^a u^1, \{5\}, 3)$ is

$$\frac{3a(a-1)(v-w)^2 + 6au(v-w)}{20}.$$

We note that if $x \equiv 0 \pmod 4$ then $B_3(x) = \lceil 3x^2/20 \rceil$. Moreover $3v^2 \equiv 3w^2 \pmod{20}$. It follows that $IT(v,w)$ has

$$\lceil 3v^2/20 \rceil - \lceil 3w^2/20 \rceil = (3v^2 - 3w^2)/20$$

blocks. Thus our construction has a total of

$$\frac{3a(a-1)(v-w)^2 + 6au(v-w)}{20} + \frac{a(3v^2-3w^2)}{20} + B_3(u+w)$$

$$= \frac{3a(v-w)(av-aw+2u+2w)}{20} + \lceil \frac{3(u+w)^2}{20} \rceil$$

blocks. In particular $3a(v-w)(av-aw+2u+2w)/20$ is an integer. Therefore

$$\begin{aligned} B_3(a(v-w)+u+w) &= \lceil \tfrac{3(a(v-w)+u+w)^2}{20} \rceil \\ &= \lceil \tfrac{3a(v-w)(a(v-w)+2u+2w)+3(u+w)^2}{20} \rceil \\ &= \tfrac{3a(v-w)(av-aw+2u+2w)}{20} + \lceil \tfrac{3(u+w)^2}{20} \rceil, \end{aligned}$$

which is the number of blocks in our construction. □

For future use, we discuss the following notation. Suppose $\{g_1, g_2, \ldots, g_k\}$ is a subset of $\mathbb{Z}_{2t}$, the cyclic group of order $2t$, and S is a set of indeterminates where $s = |S|$ divides $2t$. Then there is a bijection between the members of S and the cosets of the subgroup G consisting of multiples of s in $\mathbb{Z}_{2t}$. Suppose that these cosets are $C_1, C_2, \ldots, C_s$, and let the corresponding members of S be $y_1, y_2, \ldots, y_s$. Then the notation

$$< g_1, g_2, \ldots, g_k > \cup S \pmod{2t}$$

denotes the collection of blocks of size $k+1$ of the form

$$\{g_1 + h,\ g_2 + h,\ \ldots, g_k + h\} \cup \{y_j\}$$

where $h \in C_j$ and h ranges through Z_{2t}. Note that each y_j is contained in $\frac{2t}{s}$ blocks. Further, the notation

$$< g_1, g_2, \ldots, g_k > \cup \text{ S (half orbit)}$$

denotes the collection of blocks

$$\{g_1 + h,\ g_2 + h,\ \ldots, g_k + h\} \cup \{y_j\}$$

where $h \in C_j$ and h assumes values in $\{0, 1, 2, \ldots, t-1\}$. Also

$$< g_1, g_2, \ldots, g_k > +i,\ i \in S$$

denotes the set $< g_1 + i,\ g_2 + i, \ldots, g_k + i >$, $i \in S$, and $\{\infty_i\}_a^b$ denotes the set $\{\infty_a, \infty_{a+1}, \ldots, \infty_b\}$.

3 Tricovers of orders $v \equiv 0,\ 4 \pmod{20}$

The following construction combines other designs to construct tricovers.

Theorem 3.1 *If $v \geq 8$ and $v \equiv 0$ or $4 \pmod{20}$, then $C_3(v) = B_3(v)$.*

Proof. Let v be any integer as described in the hypothesis, and let D_1 be a (5,2) cover of index 1 on $v-2$ points which contains $B_1(v-2, 5, 2)$ blocks (such is shown to exist in [16] and [26]). Without loss of generality, take the point-set of the design $V = \{1, 2, \ldots, v-2\}$. Also let D_2 be a $BIBD(v+1, 5, 1)$ on the point-set $W = V \cup \{v-1,\ v,\ v+1\}$, and again without loss of generality, let $\{1, 2, 3, v, v+1\}$ be a block of this design. In this block, change $v+1$ to $v-1$, and in all other blocks of D_2 change $v+1$ to v. Again let D_3 be a $BIBD(v+1, 5, 1)$ on the point-set W, and without loss of generality, let $\{1, 2, 3, v-1, v+1\}$ be a block of this design. In this block change $v+1$ to v, and in all other blocks of D_3 change $v+1$ to $v-1$. Then the blocks of D_1, together with the modified blocks of D_2 and D_3, form a tricover. Here D_1 contributes $\lceil (v^2 - 2v)/20 \rceil$ blocks and each of D_2 and D_3 contribute $(v^2 + v)/20$ blocks for a total of $\lceil 3v^2/20 \rceil = B_3(v)$ blocks. □

4 Tricovers of order $v \equiv 8 \pmod{20}$

The following lemma is useful for the case $v \equiv 8 \pmod{20}$.

Lemma 4.1 *If there exists a $PBD[\{5, 9^*\}, 20s+9]$, then $C_3(20s+8) = B_3(20s+8)$.*

Proof. Let D_1 be a $BIBD(20s+5, 5, 1)$ defined on the set $V = \{1, 2, \ldots, 20s+5\}$. Let P be a partition of V which includes the triple $\{20s+1, 20s+2, 20s+3\}$, the pair $\{20s+4, 20s+5\}$, and which is such that all other parts of the partition are of size 4. Adjoin a new point $20s+6$ to each part of P, then adjoin these blocks to D_1. Let the resulting collection of blocks be denoted by D_2. Let D_3 be a $PBD[\{5,9^*\},20s+9]$ defined on the set $W = \{1,2,\ldots,20s+9\}$, and let the points of the set $S = \{20s+1, 20s+2, \ldots, 20s+9\}$ be those which occur in the block B of size 9, and let D_4 be the set of blocks obtained by deleting B from D_3. Let E_1 be the collection of blocks obtained from D_4 by replacing $20s+9$ by $20s+8$, and let E_2 be the collection of blocks obtained from D_4 by replacing $20s+9$ by $20s+7$. Let $Y = W\backslash\{20s+9\}$, $Z = S\backslash\{20s+9\}$, and $T = Y\backslash Z$. Then the blocks of E_1 and E_2 together contain every pair of $T \times \{20s+7, 20s+8\}$ exactly three times and every other pair of $(Y \times Y)\backslash(Z \times Z)$ exactly twice. Denote this collection of blocks by E_3.

Consider the following set F of eight blocks:

$$\begin{array}{ccccc}
20s+1 & 20s+2 & 20s+3 & 20s+6 & 20s+7 \\
20s+1 & 20s+2 & 20s+4 & 20s+5 & 20s+7 \\
20s+3 & 20s+4 & 20s+5 & 20s+6 & 20s+7 \\
20s+1 & 20s+2 & 20s+3 & 20s+6 & 20s+8 \\
20s+1 & 20s+2 & 20s+4 & 20s+5 & 20s+8 \\
20s+3 & 20s+4 & 20s+5 & 20s+6 & 20s+8 \\
20s+1 & 20s+2 & 20s+6 & 20s+7 & 20s+8 \\
20s+3 & 20s+4 & 20s+5 & 20s+7 & 20s+8
\end{array}$$

These blocks contain every pair of $Z \times Z$ at least twice, and every pair including either $20s+7$ or $20s+8$, with the exception of the pair $\{20s+7, 20s+8\}$ occurs at least three times. Also each of the pairs $\{20s+1, 20s+6\}$, $\{20s+2, 20s+6\}$ and $\{20s+3, 20s+6\}$ occurs at least three times.

Let H denote the total collection of the blocks of D_2, E_3 and F together. Delete the block

$$20s+1 \quad 20s+2 \quad 20s+3 \quad 20s+6,$$

and replace the block

$$20s+4 \quad 20s+5 \quad 20s+6$$

by the block

$$20s+4 \quad 20s+5 \quad 20s+6 \quad 20s+7 \quad 20s+8.$$

The result is a tricover of $\{1,2,\ldots,20s+8\}$ which contains $B_3(20s+8)$ blocks. Indeed, the $BIBD(20s+5,5,1)$ contributes $20s^2+9s+1$ blocks to D_2, and the partition contributes another $5s+2$ blocks, to yield a total of $20s^2+14s+3$ blocks (including one block each of sizes 3 and 4) in D_2. Further, the $PBD[\{5,9^*\}, 20s+9]$ contributes $20s^2+17s$ blocks to each of E_1 and E_2, and since there are eight blocks in F, the total number of blocks in $D_2 \cup E_1 \cup E_2 \cup F$ is $60s^2+48s+11$. However, at the end, a block of size 4 is deleted, and the block of size 3 is replaced by a block of size 5, for a total of $60s^2+48s+10 = B_3(20s+8)$ blocks. □

Corollary 4.1.1 *If $v = 20s + 8$, where $s \geq 0$, and $v \notin \{28, 48\}$, then $C_3(v) = B_3(v)$.*

Proof. The required $PBDs$ are constructed in [9]. □

Lemma 4.2 *There exist incomplete tricovers $IT(32,8)$ and $IT(48,12)$.*

Proof. For the construction of an $IT(32,8)$, consider the following set of 144 blocks defined on the set $\mathbb{Z}_{24}$:

$$\begin{array}{llllllllll} 0 & 1 & 2 & 6 & (\text{mod } 24) & 7 & 12 & 15 & 22 & (\text{mod } 24) \\ 3 & 4 & 11 & 16 & (\text{mod } 24) & 9 & 13 & 19 & 21 & (\text{mod } 24) \\ 5 & 8 & 18 & 20 & (\text{mod } 24) & 10 & 14 & 17 & 23 & (\text{mod } 24) \end{array}$$

In these blocks, each pair of distinct residues occurs in at least three blocks. Moreover each translate of the initial set of blocks is a *resolution class*, that is, a set of blocks which contains each residue exactly once. Such a tricover is said to be *resolvable*.

Let $W = \{x_1, x_2, \ldots, x_8\}$, and $V = \mathbb{Z}_{24} \cup W$. Partititon the set of resolution classes above into eight parts, each containing three resolution classes. Adjoin x_i to all blocks of the ith part, $i = 1, 2, \ldots, 8$. The resulting set of 144 blocks is an $IT(32,8)$.

For the construction of an $IT(48,12)$, we again construct a resolvable tricover with block-size 4 on a set of thirty-six points, which has 36 resolution classes. Let the point set be $S = \{(i,j) : i \in GF(9),\ j \in \mathbb{Z}_4\}$. Now consider the following set of 9 blocks:

$$\begin{array}{lllll} (1,0) & (w^2,0) & (w^4,0) & (w^6,0) & \text{mod}(-,4) \\ (w,0) & (w^3,0) & (w^5,0) & (w^7,0) & \text{mod}(-,4) \\ (0,0) & (0,1) & (0,2) & (0,3) & \end{array}$$

where w is a primitive element in $GF(9)$. This constitutes a resolution class, and the set of translates D_2 of this class through $(GF(9), -)$ yields nine resolution classes. Let T be a resolvable transversal design $TD(4,9)$ with the four groups $G_j = \{(i,j) : i \in GF(9)\}$, for $j = 0,1,2,3$. Take each of the blocks of T three times to obtain a set of 27 resolution classes. Adjoin these blocks to those of the 9 resolution classes above.

The points of $W = \{x_1, x_2, \ldots, x_{12}\}$ can be added to these 36 resolution classes of 9 blocks each as in the above case to obtain an $IT(48,12)$. □

Lemma 4.3 *If $v \in \{28, 48\}$, the $C_3(v) = B_3(v)$.*

Proof. For $v = 28$, proceed as follows. In [24], a (5,2)-cover D_1 on 27 points, with index $\lambda = 1$ which has 38 blocks including a minimum sub-cover of seven points, is constructed. If we assume that the point-set is $\{1, 2, \ldots, 27\}$, then we may assume without loss of generality that the blocks of the sub-cover are:

B_1 :	1 2 3 4 5
B_2 :	1 2 3 6 7
B_3 :	4 5 6 7

We delete the block B_3 from the blocks of D_1, and call the resulting set of blocks D_2.

Further, in [28] a (5,2)-cover D_3 on 29 points, with index $\lambda = 2$ (i.e., a bicover) which has 82 blocks, including a minimum sub-bicover of seven points, is constructed. If the point-set of D_3 is $W = \{1, 2, \ldots, 29\}$, then without loss of generality, the blocks of the sub-bicover are

B_4 :	8	9	10	11	12
B_5 :	8	9	10	11	13
B_6 :	8	9	10	11	29
B_7 :	8	9	12	13	29
B_8 :	10	11	12	13	29

Further, this design is constructed by adjoining each point in the sub-bicover to all the blocks of a corresponding 2-resolution class of a $BIBD(22, 4, 2)$, say D_4, on $X = W \backslash \{8, 9, 10, 11, 12, 13, 29\}$, (a 2-resolution class is simply a set of blocks of D_4 which contain every element of the $BIBD$ exactly twice).

Examining the 2-resolvable $BIBD(22, 4, 2)$ generated on the point-set $Z_3 \times Z_7$ in Lemma 3.6 of [28], taking $\beta = 2$, we see that the 2-resolution class corresponding to $\alpha = 0$ contains the following blocks:

E_1 :	(1,2)	(1,5)	(0,1)	(0,6)
E_2 :	(2,3)	(2,4)	(0,1)	(0,6)
E_3 :	(2,1)	(2,6)	(1,3)	(1,4)
E_4 :	(2,1)	(2,6)	(0,2)	(0,5)
E_5 :	(1,1)	(1,6)	(2,2)	(2,5)
E_6 :	(0,3)	(0,4)	(2,2)	(2,5)

Since we are free in the construction of D_3 to label the above set with members in X in any way we choose which is consistent with the choice of elements in the sub-bicover, let us do so in such a way to form the blocks:

B_9 :	4	5	14	28
B_{10} :	15	16	14	28
B_{11} :	6	7	17	18
B_{12} :	6	7	19	20
B_{13} :	1	2	3	21
B_{14} :	22	23	3	21

where each element of X in the set of blocks corresponds to the member of $Z_3 \times Z_7$ in the corresponding position in the list of blocks $\{E_i : i = 1, 2, \ldots, 6\}$.

Without loss of generality, we may assume that it is this 2-resolution class to which the element 29 is adjoined.

Replace blocks $B_9, B_{10}, \ldots, B_{14}$ by the blocks

B_{15} :	4	5	6	14	28
B_{16} :	7	14	15	16	28
B_{17} :	4	6	7	17	18
B_{18} :	5	6	7	19	20
B_{19} :	2	3	14	21	28
B_{20} :	1	3	21	22	23

Now replace the element 29 by the element 28 in all of the unmodified blocks of D_3.

Note that the pairs in the set of elements $\{8, 9, 10, 11\}$ occur at least once in D_2, and three times in D_3. Moreover, since none of these elements occurs in the modified blocks of D_3 prior to the replacement of 29 by 28, each pair of $\{8, 9, 10, 11\} \times \{28\}$ occurs at least four times in D_3, so from the point of view of tricovers, the block B_6 : 8 9 10 11 28 can be deleted. The resulting set of 118 blocks is then a tricover of 28 points, so $C_3(28) = B_3(28)$.

For $v = 48$, since there exists an $IT(48, 12)$ as shown in Lemma 4.2, it is sufficient to show that $C_3(12) = B_3(12)$. Let the point-set for such a tricover be $\mathbb{Z}_{11} \cup \{\infty\}$.

Then the set of blocks

$$\begin{array}{ccccc l} 0 & 1 & 3 & 5 & 10 & (\text{mod } 11) \\ 0 & 1 & 3 & 8 & \infty & (\text{mod } 11) \end{array}$$

yield a tricover with $B_3(12)$ blocks. □

Theorem 4.4 *If $v = 20s + 8$, where $s \geq 0$, then $C_3(v) = B_3(v)$.*

The result follows from Corollary 4.1.1 and Lemma 4.3. □

5 Tricovers for Orders $v \equiv 12 (\text{mod } 20)$

The strategy in this section is to determine $C_3(v)$ for several small values of v, and to use these recursively to complete the spectrum.

Lemma 5.1 *Let m and t be positive integers such that $0 \leq t \leq 5m$. If $C_3(4t) = B_3(4t)$, then $C_3(100m + 4t) = C_3(100m + 4t)$.*

Proof. Hanani [13, Theorem 3.11], has shown that there exists a $GDD((5m)^6, \{6\}, 3)$ for $m \geq 1$. By deleting all but t of the points which occur in one of the groups, and inflating each point by a factor of 4 (using $GDD(4^i, \{5\}, 1)$, where $i = 5, 6$, which can be obtained from a $BIBD(21, 5, 1)$ and a $BIBD(25, 5, 1)$), a $GDD((20m)^5(4t)^1, \{5\}, 3)$ is obtained. By replacing each group G of size $20m$ by the blocks of a tricover of the points of G with $B_3(20m)$ blocks (such exists by Lemma 3.1), and the group G' of size $4t$ by the by the

$B_3(4t)$ blocks of a minimum tricover of the points of G' (such exists by hypothesis), a tricover of $100m + 4t$ points with $B_3(100m + 4t)$ blocks is obtained. Indeed, the $GDD((20m)^5(4t)^1, \{5\}, 3)$ contributes $1200m^2 + 120mt$ blocks, and since $B_3(20m) = 60m^2$, the tricovers substituted on the five groups of size $20m$ contribute another $300m^2$ blocks, and the tricover on the group of size $4t$ contributes another $B_3(4t) = \lceil 12t^2/5 \rceil$ blocks, for a total of $B_3(100m + 4t) = 1500m^2 + 120mt + \lceil 12t^2/5 \rceil$ blocks. Hence, under the hypothesis, $C_3(100m + 4t) = B_3(100m + 4t)$. □

Lemma 5.2 *There exists an $IT(32,8)$, $IT(52,8)$, $IT(72,8)$ and $IT(92,12)$. Further $C_3(12) = B_3(12)$. Therefore, $C_3(v) = B_3(v)$ for $v \in \{12, 32, 52, 72, 92\}$.*

Proof. For $v = 12$, as mentioned earlier, the blocks

$$\begin{array}{cccccl} 0 & 1 & 3 & 5 & 10 & (\text{mod } 11) \\ 0 & 1 & 3 & 8 & \infty & (\text{mod } 11) \end{array}$$

form a tricover of $Z_{11} \cup \{\infty\}$ which has $22 = B_3(12)$ blocks.

For $v = 32$, an $IT(32,8)$ was exhibited in Lemma 4.2. Since $C_3(8) = B_3(8)$ then $C_3(32) = C_3(32)$.

For $v = 52$, we construct an $IT(52,8)$ as follows. Let $W = \{\infty_1, \infty_2, \ldots, \infty_8\}$ and $V = Z_{44} \cup W$. Then the 396 blocks

$$\begin{array}{ll} < 0,9,22,31 > \cup\{\infty_7, \infty_8\} & (\text{half orbit}) \\ < 0,5,22,27 > \cup\{\infty_5, \infty_6\} & (\text{half orbit}) \\ < 0,1,19,30 > \cup\{\infty_i\}_1^4 & (\text{mod } 44) \\ < 0,3,6,21 > \cup\{\infty_7, \infty_8\} & (\text{mod } 44) \\ < 0,5,12,25, > \cup\{\infty_5, \infty_6\} & (\text{mod } 44) \\ < 0,7,16,27 > \cup\{\infty_3, \infty_4\} & (\text{mod } 44) \\ < 0,7,19,30 > \cup\{\infty_1, \infty_2\} & (\text{mod } 44) \\ < 0,4,10,18,20 > & (\text{mod } 44) \\ < 0,5,13,17,28, > & (\text{mod } 44) \\ < 0,1,2,4,10 > & (\text{mod } 44) \end{array}$$

form an $IT(52,8)$. Again, since $C_3(8) = B_3(8)$, then $C_3(52) = B_3(52)$.

For $v = 72$, we proceed as follows. Let $X = \{1, 2, \ldots, 64\}$ and $W = \{x_1, x_2, \ldots, x_8\}$, and $V = X \cup W$. Let D_1 be a resolvable $BIBD(64,4,1)$ design on X. Such a design has 336 blocks which fall into 21 resolution classes which can be combined in sets of three to produce 3-resolution classes $C_1, C_2, \ldots, C_7$. To each block of C_i, adjoin x_i, $i = 1, 2, \ldots, 7$. Now adjoin to these blocks the 416 blocks of a $BIBD(65,5,2)$ on the set $X \cup \{x_8\}$. Let P be a partition of X into 4-sets. To each of these 4-sets, adjoin x_8, then adjoin these 16 blocks to those previously obtained for a total of 768 blocks. The result is an $IT(72,8)$.

For $v = 92$, we construct an $IT(92,12)$ as follows. Let $W = \{\infty_1, \infty_2, \ldots, \infty_{12}\}$, and $V = Z_{80} \cup W$. Then the 1248 blocks

$< 0, 16, 32, 48, 64 > +i,$	$i \in Z_{16}$; 3 times
$< 0, 11, 33, 50 > \cup \{\infty_i\}_1^4$	(mod 80)
$< 0, 11, 33, 50 > \cup \{\infty_i\}_5^8$	(mod 80)
$< 0, 11, 33, 50 > \cup \{\infty_i\}_9^{12}$	(mod 80)
$< 0, 3, 13, 68 > \cup \{\infty_{11}, \infty_{12}\}$	(mod 80)
$< 0, 3, 13, 68 > \cup \{\infty_9, \infty_{10}\}$	(mod 80)
$< 0, 3, 13, 68 > \cup \{\infty_7, \infty_8\}$	(mod 80)
$< 0, 5, 31, 40 > \cup \{\infty_5, \infty_6\}$	(mod 80)
$< 0, 5, 31, 40 > \cup \{\infty_3, \infty_4\}$	(mod 80)
$< 0, 5, 31, 40 > \cup \{\infty_1, \infty_2\}$	(mod 80)
$< 0, 1, 20, 24, 38 >$	(mod 80), 3 times
$< 0, 2, 8, 29, 36 >$	(mod 80), 3 times

form an $IT(92,12)$. □

Lemma 5.3 *If $v = 20s + 12$, where $s \geq 0$ and $v < 500$, then $C_3(v) = B_3(v)$.*

Proof. The proof is given by covering various intervals. For $0 < v < 100$, see Lemma 5.2. For $v = 112$, apply Lemma 5.1 with $m = 1$ and $t = 3$.

The group divisible designs required for the application of Lemma 2.1 below are easily obtained by inflating truncated transversal designs by a factor of 4, and duplicating blocks, unless otherwise indicated.

For $132 \leq v \leq 152$, use Lemma 2.1 and the fact that there exists an $IT(32,8)$. In this case, we use the $TD_3(6,6)$ created by Hanani [13, page 279], and inflate by a factor of 4 using ingredients of index $\lambda = 1$ to obtain a $GDD(24^5 4^1, \{5\}, 3)$ and a $GDD(24^6, \{5\}, 3)$.

For $172 \leq v \leq 192$, we first construct an $IT(36,4)$. A $(5,2)$-cover of index 1 on 36 points with 65 blocks was constructed by Rolf Rees and is exhibited in [27, Lemma 5.4]. This cover has one block of size 4. If the block of size 4 is deleted, and three copies of the remaining blocks are taken, the result is an $IT(36,4)$. We use a $TD_3(6,8)$ to construct $GDD(32^5 u^1, \{5\}, 3)$'s with $u = 8$ and 28, and apply Lemma 2.1.

For $212 \leq v \leq 232$, apply Lemma 5.1 with $m = 2$ and $t = 3$ and 8.

For $v = 252$, proceed as follows. Take a $BIBD(66,6,1)$, say D, and let x be a distinguished point of D, which lies on a distinguished block B. By deleting x, we obtain a set of thirteen disjoint blocks of size 5, which will act as groups. Delete two more points of B to obtain a $GDD(5^{12}3^1, \{5,6\}, 1)$ and inflate the result by a factor of 4 to obtain a $GDD(20^{12}12^1, \{5\}, 1)$, and triplicate the blocks to produce a $GDD(20^{12}12^1, \{5\}, 3)$ which has 8784 blocks. Now replace each group G of size 20 by the blocks of a tricover of G with $C_3(20) = 60$ blocks for a total of 9504 blocks, and replace the group G^* of size 12 by the blocks of a tricover of G^* with $C_3(12) = 22$ blocks. The resulting set of 9526 blocks form the required tricover.

For $v = 272$ and 292, the construction is similar to the preceding. Let D be a resolvable $BIBD(65,5,1)$, and let C be a distinguished class of the 16 resolution classes of D. Let $C_1, C_2, \ldots, C_s$ be s other resolution classes of D, and let $X = \{x_1, x_2, \ldots, x_s\}$ where $s = 3$ or 8, and no x_i is in point set of D. Adjoin x_i to each block in class C_i, $i = 1, 2, \ldots, s$, and take the blocks of C together with X as the groups of a $GDD(5^{13}s^1, \{5,6\}, 1)$ and proceed as in the case $v = 252$.

For $v = 312$, apply Lemma 5.1 with $m = 3$ and $t = 3$.

For $332 \leq v \leq 392$, apply Lemma 2.1 with $a = 5, v = 72, w = 8$, and $u \in \{4, 24, 44, 64\}$.

For $412 \leq v \leq 492$, apply Lemma 2.1 with $a = 5, v = 92, w = 12$, and $u \in \{0, 20, 40, 60, 80\}$. □

The main result of this section is given in the following theorem.

Theorem 5.4 *If $v = 20s + 12$, where $v \geq 0$, then $C_3(v) = B_3(v)$.*

Proof. The cases of $v \leq 500$ were handled in Lemma 5.3. All other cases follow by applying Lemma 5.1, noting that $C_3(v) = B_3(v)$ for $v \in \{12, 32, 52, 72, 92\}$. □

6 Tricovers of orders $v \equiv 16 (\mathrm{mod}\ 20)$

A minimum (5,2) cover D of index $\lambda = 1$ on $w = 20s + 18$ points with point-set $W = \{1, 2, \ldots, w\}$ is said to be *utilitarian* if it contains a subset of points $Z = \{1, 2, \ldots, 11, 12, 13, w-2, w-1, w\}$ and a subset of blocks of the form

$$\begin{array}{lllll} 1\,2\,3 & w-2 & w & & 1\,2\,3\ \ 10\ \ a \\ 4\,5\,6 & w-1 & w & & 4\,5\,6\ \ 11\ \ b \\ 7\,8\,9 & w-1 & w-2 & & \end{array}$$

where no pair from $\{w-2, w-1, w\}$ occurs in any other block, a may be identical with b, but neither a nor b lies in Z, and the pairs $\{10, a\}$, $\{11, b\}$ and $\{12, 13\}$ each occur in at least two blocks of D.

Lemma 6.1 *If there exists a utilitarian cover of $w = 20s + 18$ points, where $s \neq 0$, then $C_3(v) = B_3(v)$ where $v = w - 2$.*

Proof. Let D_1 be a $BIBD(20s+15, 5, 2)$ on the point-set $V^* = \{1, 2, \ldots, v-1\}$. Without loss of generality, we may assume that D_1 contains the block

$$B_1: \ 7\ 8\ 9\ 12\ 13.$$

Let D_2 be a utilitarian cover of order $20s + 18$ which contains the distinguished blocks

$$\begin{array}{llll} B_2: & 1\,2\,3 & v+1 & v+2 \\ B_3: & 4\,5\,6 & v & v+2 \\ B_4: & 7\,8\,9 & v & v+1 \end{array}$$

$$\begin{array}{llll} B_5: & 1\,2\,3 & 10 & a \\ B_6: & 4\,5\,6 & 11 & b \end{array}$$

Wherever $v+1$ and $v+2$ occur outside of these distinguished blocks, replace each symbol by v. Further replace $v+1$ by v and $v+2$ by a in block B_2, $v+2$ by b in block B_3 and $v+1$ by 13 in B_4. Further replace 13 by v in block B_1, a by v in block B_5 and b by v in block B_6. Then the modified blocks of D_1 and D_2 together form a tricover of $V = \{1, 2, \ldots, v\}$ which contains $B_3(v)$ blocks. Indeed, the $BIBD$ denoted by D_1 contains $40s^2 + 58s + 21$ blocks, and the utilitarian design D_2 contains $20s^2 + 38s + 18$ blocks, which yields a total of $B_3(20s + 16) = 60s^2 + 96s + 39$ blocks. □

Lemma 6.2 *Suppose that there exists a* $PBD[\{5, 9^*\}, 20s+17]$ *where* $s > 1$. *Then there exists a utilitarian design on* $w = 20s + 18$ *points.*

Proof. Let D_1 be such a PBD. Let $W^* = \{1, 2, \ldots, w-1\}$ be the point-set of D_1 and without loss of generality, assume that the block of size 9 is

$$B_1: \quad 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22.$$

Since $s > 1$, there is a block disjoint from B_1. Without loss of generality, let it be

$$B_2: \quad 7\,8\,9 \quad w-2 \quad w-1.$$

Now D_1 contains $20s^2 + 33s + 11$ blocks, and B_1 or B_2 meet $70s - 10$ blocks, so, since $s \geq 2$, there exist two blocks which are disjoint from B_1 and B_2. Without loss of generality let these be

$$\begin{array}{llll} B_3: & 1\,2\,3 & 10 & a \\ B_4: & 4\,5\,6 & 11 & b \end{array}$$

where a and b need not be distinct.

We partition $W^* \backslash B_1$ into $5s + 2$ sets of size 4, assuring that $\{1, 2, 3, w-2\}$ and $\{4, 5, 6, w-1\}$ are parts of the partition, and that 12 and 13 occur together in the same part of the partition. Also if $a \neq b$, let $\{10, 11, a, b\}$ be a part of the partition, and if $a = b$ let $\{7, 10, 11, a\}$ be a part of the partition. Add a new symbol w to each part of the partition, and adjoin these blocks and the blocks of a minimum cover of $B_1 \cup \{w\}$ to the blocks of D_1. Then delete the block B_1. The result is a utilitarian cover of w points. Indeed, there are $20s^2 + 33s + 11$ blocks in D_1, including one of size 9 which is deleted. When taken with the $5s+2$ blocks generated by the partition and the 6 blocks of the minimum cover a total of $B_1(20s + 18) = 20s^2 + 38s + 18$ blocks is obtained. □

Corollary 6.2.1 *If $w = 20s + 18$, where $s \geq 2$, then there exists a utilitarian cover of order w.*

Proof. The required *PBDs* are shown to exist in [9].

Theorem 6.3 *Let $v = 20s + 16$, where $s \geq 0$. Then $C_3(v) = B_3(v)$.*

Proof. In view of the preceding lemmas, it remains only to show that the theorem is true for $v \in \{16, 36\}$.

For $v = 16$, let the point-set be $(Z_3 \times Z_5) \cup \{\infty\}$.

Then the required blocks are constructed as follows:

On $(\{0\} \times Z_5) \cup \{\infty\}$ construct the four blocks of a (5,2) minimum bicover of six points.

Then take the following 35 blocks:

$$
\begin{array}{ll}
<(0,0),\ (1,2),\ (1,4),\ (2,0), \infty > & \mathrm{mod}(-,5) \\
<(0,0),\ (1,1),\ (2,3),\ (2,4), \infty > & \mathrm{mod}(-,5) \\
<(0,0),\ (0,1),\ (1,0),\ (2,0),\ (2,3) > & \mathrm{mod}(-,5) \\
<(0,0),\ (0,2),\ (1,3),\ (1,4),\ (2,1) > & \mathrm{mod}(-,5) \\
<(0,0),\ (1,0),\ (1,1),\ (1,2),\ (2,1) > & \mathrm{mod}(-,5) \\
<(0,0),\ (1,3),\ (2,0),\ (2,2),\ (2,3) > & \mathrm{mod}(-,5) \\
<(0,0),\ (1,0),\ (1,3),\ (2,1),\ (2,2) > & \mathrm{mod}(-,5)
\end{array}
$$

This set of 39 blocks forms the required tricover.

For $v = 36$, it is shown in [27, pp. 211-12] that $C_1(36) = 65$. Taking three copies of such a cover shows that $C_3(36) = 195$. □

7 Conclusion

We have shown that if $v = 4s$, where $s > 1$, then the tricovering number

$$C_3(v) = \lceil \frac{v}{5} \lceil \frac{3(v-1)}{4} \rceil \rceil.$$

References

[1] R.J.R. Abel, *Four mutually orthogonal Latin squares of orders 28 and 52*, J. Combin. Theory Ser. A 58 (1991), 306-309.

[2] A.M. Assaf, W.H. Mills and R.C. Mullin, *On tricovers of pairs by quintuples: $v \equiv 1 (\mathrm{mod}\ 4)$*, Ars Combinatoria, to appear.

[3] A. Assaf and L.P. Singh, *Packing designs with block size 5 and index 2: the case v odd*, preprint.

[4] T. Beth, D. Jungnickel, H. Lenz, *Design Theory*, Bibliographisches Institut Mannheim, Wien, Zürich 1985.

[5] A.E. Brouwer, *Mutually orthogonal Latin squares*, Math. Cent. Report ZN81, August 1978.

[6] ______, *the number of mutually orthogonal Latin squares - a table up to order 10,000*, Math. Cent. Report ZW123, June 1979.

[7] ______, *Four MOLS of order 10 with a hole of order 2*, J. Statist. Planning and Inference 10 (1984), 203-205.

[8] A.E. Brouwer and G.H.J. van Rees, *More mutually orthogonal Latin squares*, Discrete Math. 39 (1982), 263-281.

[9] A. Hamel, W.H. Mills, R.C. Mullin, Rolf Rees and D.R. Stinson, *The spectrum of $PBD(\{5, k^*\}, v)$ for $k = 9, 13$*, Ars Combinatoria, submitted.

[10] M.K. Fort, Jr. and G.A. Hedlund, *Minimal coverings of pairs by triples*, Pacific J. Math. 8 (1958), 709-719.

[11] B.I. Gardner, *Results on coverings of pairs with special reference to coverings by quintuples*, Congr. Numer. 5 (1971), 169-178.

[12] G. Haggard, *On the function $N(3,2,\lambda,v)$*, Congr. Numer. 6 (1972), 243-250.

[13] H. Hanani, *Balanced incomplete block designs and related designs*, Discrete Math. 11 (1975), 255-369.

[14] H. Hanani, D.K. Ray-Chaudhuri, and R.M. Wilson, *On resolvable designs*, Discrete Math. 3 (1972), 343-357.

[15] A. Hartman, W.H. Mills, and R.C. Mullin, *Covering triples by quadruples: An asymptotic solution*, J. Combin. Theory Ser. A 41 (1986), 117-138.

[16] E.R. Lamken, W.H. Mills, R.C. Mullin, and S.A. Vanstone, *Coverings of pairs by quintuples*, J. Combin. Theory Ser. A 44 (1987), 49-68.

[17] E.R. Lamken, W.H. Mills, and R.M. Wilson, *Four pairwise balanced designs*, Designs, Codes and Cryptography 1 (1991), 63-68.

[18] R. Mathon and A. Rosa, *Tables of parameters of BIBD's with* $r \leq 41$ *including existence, enumeration and resolvability results, an update*, Ars Combin. 30 (1990), 65-96.

[19] W.H. Mills, *On the covering of pairs by quadruples I*, J. Combin. Theory Ser. A 13 (1972), 55-78.

[20] ______, *On the covering of pairs by quadruples II*, J. Combin. Theory Ser. A 15 (1973), 138-166.

[21] ______, *Covering problems*, Congr. Numer. 8 (1973), 23-52.

[22] ______, *On the covering of triples by quadruples*, Congr. Numer. 10 (1974) 563-581.

[23] ______, *A covering of pairs by quintuples*, Ars Combin. 18 (1984), 21-31.

[24] W.H. Mills and R.C. Mullin, *Covering pairs by quintuples: The case* v *congruent to* 3 (mod 4), J. Combin. Theory Ser. A 49 (1988), 308-322.

[25] R.C. Mullin, *On covering pairs by quintuples: the cases* $v \equiv 3$ *or* 11 *modulo* 20, J. Combin. Math. Combin. Comput. 2 (1987), 133-146.

[26] ______, *On the determination of the covering numbers* $C(2,5,v)$, J. Combin. Math. Combin. Comput. 4 (1988), 123-132.

[27] R.C. Mullin and J.D. Horton, *Bicovers of pairs by quintuples:* v *even*, Ars Combin. 26 (1988), 197-228.

[28] R.C. Mullin, J.D. Horton, and W.H. Mills, *On bicovers of pairs by quintuples:* v *odd,* $v \not\equiv 3$ (mod 10), Ars Combin. 31 (1991), 3-19.

[29] R.C. Mullin, E. Nemeth, R.G. Stanton, and W.D. Wallis, *Isomorphism properties of some small covers*, Utilitas Math. 29 (1986), 269-283.

[30] R. Roth and M. Peters, *Four pairwise orthogonal Latin squares of order 24*, J. Combin. Theory Ser. A 44 (1987), 152-155.

[31] J. Schönheim, *On coverings*, Pacific J. Math. 14 (1964), 1405-1411.

[32] R.G. Stanton and M.J. Rogers, *Packings and coverings by triples*, Ars Combin. 13 (1982), 61-69.

[33] D.T. Todorov, *Four mutually orthogonal Latin squares of order 20*, Ars Combin. 27 (1989), 63-65.

[34] ______, *Four mutually orthogonal Latin squares of orders 38 and 44*, Private Communication.

[35] R.M. Wilson, *Concerning the number of mutually orthogonal Latin squares*, Discrete Math. 9 (1974), 181-198.

[36] ______, *Constructions and uses of pairwise balanced designs*, in Combinatorics, Part I; Math. Centre Tracts 55, Amsterdam (1975), 18-41.