Indecomposable Triple Systems without Repeated Blocks

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Abstract. It is proved in this paper that for any given odd integer $\lambda \geq 1$, there exists an integer $v_0 = v_0(\lambda)$, such that for $v > v_0$, the necessary and sufficient conditions for the existence of an indecomposable triple system $B(3, \lambda; v)$ without repeated blocks are $\lambda(v-1) \equiv 0 \pmod{2}$ and $\lambda v(v-1) \equiv 0 \pmod{6}$.

1. Introduction.

Let V be a finite set containing v elements and λ be a given positive integer, an λ-fold triple system on V, denoted $B(3, \lambda; v)$, is an ordered pair $(V, \mathcal{B})$ where $\mathcal{B}$ is a collection of 3-subsets (called blocks or triples) of V, such that each pair of distinct elements of V is contained in exactly λ triples. A triple system $B(3, \lambda; v)$ is called simple and denoted $NB(3, \lambda; v)$ if it contains no repeated blocks.

Let $(V, \mathcal{B})$ be a $B(3, \lambda; v)$, if there exist $\mathcal{B}_1 \subset \mathcal{B}$ and $1 \leq \lambda_1 < \lambda$ such that $(V, \mathcal{B}_1)$ is a $B(3, \lambda_1; v)$, then $(V, \mathcal{B})$ is called decomposable. Otherwise it is called indecomposable.

In this paper, we consider the existence of indecomposable triple systems without repeated blocks. It is not difficult to show that the following are necessary conditions for the existence of an indecomposable $NB(3, \lambda; v)$:

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{2} \\ \lambda v(v-1) &\equiv 0 \pmod{6} \\ \lambda &\leq v-2. \end{aligned} \tag{1}$$

In the case $\lambda = 1$, a $B(3, 1; v)$ is called a Steiner triple system. Obviously, any Steiner triple system is both simple and indecomposable. It is well known ([5]) that there exists a $B(3, 1; v)$ if and only if

$$v \equiv 1, 3 \pmod{6}. \tag{2}$$

For given $\lambda \geq 2$, it is difficult to determine the existence of an indecomposable $NB(3, \lambda; v)$. The problem is completely solved only for $\lambda = 2, 3$ and 4. A. P. Street ([9]) proved that there exists an indecomposable $NB(3, 2; v)$ if and only if

$$v \equiv 0, 1 \pmod{3}, \quad v > 3 \quad \text{and} \quad v \neq 7 \tag{3}$$

and there exists an indecomposable $NB(3, 3; v)$ if and only if

$$v \equiv 1 \pmod{2}, \quad v > 3. \tag{4}$$

C. J. Colbourn and A. Rosa ([2]) proved that there exists an indecomposable NB$(3,4;v)$ if and only if

$$v \equiv 0,1 \pmod 3, \quad v \geq 10. \tag{5}$$

The present author ([7]) proved that there exists an indecomposable NB$(3,6;v)$ if and only if

$$v \geq 8 \text{ and } v \neq 9 \tag{6}$$

with the following six possible exceptions:

$$v = 10, 11, 12, 13, 15 \text{ and } 16.$$

In general case, we have the following conjecture:

Conjecture ([2], [4]). *Let λ be a fixed positive integer. Then there exists $v_0 = v_0(\lambda)$ such that for $v > v_0$, there exists an indecomposable NB$(3,\lambda;v)$ if and only if $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 6$.*

In this paper, we will prove this conjecture for any odd λ.

2. Recursive constructions.

Let K be a set of positive integers. A pairwise balanced design (PBD) $S(2,K;v)$ is an ordered pair $(V,\mathcal{B})$ where V is a finite set containing v elements, $\mathcal{B}$ is a collection of subsets (called blocks) of V such that for any block $B \in \mathcal{B}$, $|B| \in K$, and each pair of distinct elements of V is contained in exactly one block.

For a given set K of positive integers, let

$$B(K) = \{v \mid \text{ there exists an } S(2,K;v)\}.$$

If $B(K) = K$, then K is called a PBD-closed set.

Let λ be a given positive integer, let

$$INB(\lambda) = \{v \mid \text{ there exists an indecomposable } NB(3,\lambda;v)\}.$$

Lemma 1. *INB(λ) is a PBD-closed set.*

Proof: Let $(V,\mathcal{B})$ be an $S(2,K;v)$ such that $k \in INB(\lambda)$ for each $k \in K$. For any $B \in \mathcal{B}$, $|B| = k$, form an indecomposable NB$(3,\lambda;k)$ and let $\mathcal{A}_B$ denote the collection of its blocks. Let

$$\mathcal{A} = \bigcup_{B \in \mathcal{B}} \mathcal{A}_B.$$

Then $(V,\mathcal{A})$ is an indecomposable NB$(3,\lambda;v)$.

To give further recursive constructions of indecomposable simple triple systems, we introduce the following definitions.

Let $(V,\mathcal{B})$ be a $B(3,\lambda;v)$. Let $V_1 \subset V$, $\mathcal{B}_1 \subset \mathcal{B}$, if $(V_1,\mathcal{B}_1)$ is a $B(3,\lambda;v_1)$, then $(V_1,\mathcal{B}_1)$ is called a subtriple system. The following lemma is obvious.

Lemma 2. *If a triple system contains an indecomposable subtriple system, then it is also indecomposable.*

A transversal design TD$(k,\lambda;n)$ is an ordered triple $(V,\mathcal{G},\mathcal{B})$ where V is a v-set, $v = kn$, $\mathcal{G}$ is a set of n-subsets (called groups) of V, $\mathcal{G}$ partitions V and $\mathcal{B}$ is a collcction of k-subsets (called blocks) such that any block intersects each group in exactly one element, and each pair of elements from distinct groups is contained in exactly λ blocks. When $\lambda = 1$, a TD$(k,\lambda;n)$ is usually denoted TD(k,n) .

A TD$(k,\lambda;n)$ is called simple if it contains no repeated blocks. Two simple TD$(k,\lambda;n)$ on V with same group set $\mathcal{G}$ are called disjoint if they have no common blocks. From t pairwise disjoint simple TD$(k,\lambda;n)$, we can obtain a simple TD$(k,t\lambda;n)$.

Lemma 3. *For any positive integer n, there exist n pairwise disjoint simple TD$(3,n)$.*

Proof: Let $G_1 = Z_n, G_2, G_3$ be three disjoint n-sets and $V = G_1 \cup G_2 \cup G_3$. Form a TD$(3,n)$ on V with G_1, G_2, and G_3 as groups and denote it by $(V,\mathcal{G},\mathcal{B}_0)$. Now let

$$\mathcal{B}_i = \{\{a_1+i,a_2,a_3\}/\{a_1,a_2,a_3\} \in \mathcal{B}_0, (a_1,a_2,a_3) \in G_1 \times G_2 \times G_3\}.$$

Then for each $i \in Z_n, (V,\mathcal{G},\mathcal{B}_i)$ is a TD$(3,n)$. $(V,\mathcal{G},\mathcal{B}_i)$ and $(V,\mathcal{G},\mathcal{B}_j)$ are disjoint if $i \neq j$.

Corollary. *If $1 \le \lambda \le n$, then there exists a simple TD$(3,\lambda;n)$.*

Lemma 4. *If there exists a B$(3,1;v_1)$ and there exists an indecomposable NB$(3,\lambda;v_2)$ containing a subtriple system B$(3,\lambda;v_3)$. Then there exists an indecomposable NB$(3,\lambda;v_1(v_2-v_3)+v_3)$.*

Proof: Let X be a v_1-set and for each $x \in X$, x is a set containing $v_2 - v_3$ points:

$$x = \{x_1, x_2, \dots, x_{v_2-v_3}\}.$$

Let $(X,\mathcal{B})$ be a B$(3,1;v_1)$ on X. For each $B \in \mathcal{B}$, $B = \{x,y,z\}$, form a simple TD$(3,\lambda;v_2-v_3)$ with $\{x_1,x_2,\dots,x_{v_2-v_3}\}$, $\{y_1,y_2,\dots,y_{v_2-v_3}\}$ and $\{z_1,z_2,\dots,z_{v_2-v_3}\}$as groups and denote the block set by $\mathcal{A}_B$. Let V_3 be a v_3-set, $V_3 \cap \{x_1,x_2,\dots,x_{v_2-v_3}\} = \emptyset$ for each $x \in X$. Form a simple B$(3,\lambda;v_3)$ on V_3 and denote the block set by $\mathcal{A}'$. For each $x \in X$, form an indecomposable NB$(3,\lambda;v_2)$ containing $(V_3,\mathcal{A}')$ as a sub B$(3,\lambda;v_3)$, denote the block set by $\mathcal{A}_x \cup \mathcal{A}'$. Let

$$V = V_3 \cup \left\{ \bigcup_{x\in X} \{x_1,\dots,x_{v_2-v_3}\} \right\}$$

$$\mathcal{A} = \mathcal{A}' \cup \left\{ \bigcup_{x\in X} \mathcal{A}_x \right\} \cup \left\{ \bigcup_{B\in\mathcal{B}} \mathcal{A}_B \right\}$$

then $(V,\mathcal{A})$ is an indecomposable NB$(3,\lambda;v_1(v_2-v_3)+v_3)$.

Lemma 5. *If there is an indecomposable NB*$(3, \lambda; v)$, *then there exists an indecomposable NB*$(3, \lambda; 2v + 1)$.

Proof: It is proved ([6]) that any NB$(3, \lambda; v)$ can be embedded in an NB$(3, \lambda; 2v + 1)$. Thus from Lemma 5, if the NB$(3, \lambda; v)$ is indecomposable, then the NB$(3, \lambda; 2v + 1)$ is also indecomposable.

For the same reason, as is proved ([7]) that if $\lambda \equiv 0 \pmod 3$ and $v \geq \lambda + 4$, any NB$(3, \lambda; v)$ can be embedded in an NB$(3, \lambda; 2v + 3)$, we have the following result:

Lemma 6. *Suppose* $\lambda \equiv 0 \pmod 3$ *and* $v \geq \lambda + 4$. *If there exists an indecomposable NB*$(3, \lambda; v)$, *then there exists an indecomposable NB*$(3, \lambda; 2v + 3)$.

3. Proof of the main result.

Let K be a set of positive integers, finite or infinite, let

$$\alpha(K) = gcd\{k - 1 \mid k \in K\}$$
$$\beta(K) = gcd\{k(k - 1) \mid k \in K\}.$$

Lemma 7 ([10]). *For a given set K of positive integers, there exists an integer* v_0 *such that for* $v > v_0$, $v \in B(K)$ *if and only if*

$$\begin{aligned} v - 1 &\equiv 0 \pmod{\alpha(K)} \\ v(v - 1) &\equiv 0 \pmod{\beta(K)}. \end{aligned} \tag{7}$$

Now we are ready to prove our fundamental lemma.

Lemma 8. *For a given* $\lambda \equiv 1 \pmod 2$, *if there is an indecomposable NB*$(3, \lambda; v_0)$, *then there exists a constant* $c = c(v_0, \lambda)$ *such that for* $v > c$, *there exists an indecomposable NB*$(3, \lambda; v)$ *if and only if*

$$\begin{aligned} \lambda(v - 1) &\equiv 0 \pmod 2 \\ \lambda v(v - 1) &\equiv 0 \pmod 6 \\ \lambda &\leq v - 2. \end{aligned} \tag{1}$$

Proof: (I) If $\lambda \equiv 1, 5 \pmod 6$. Then the necessary conditions (1) become

$$v \equiv 1, 3 \pmod 6, \qquad v \geq \lambda + 2. \tag{8}$$

If $v_0 \equiv 1 \pmod 6$, then $2v_0 + 1 \equiv 3 \pmod 6$. By Lemma 5, there is an indecomposable NB$(3, \lambda; 2v_0 + 1)$. Thus, without loss of generality, we may suppose $v_0 \equiv 3 \pmod 6$.

As there is an indecomposable NB$(3,\lambda;v_0)$, we can construct an indecomposable NB$(3,\lambda;2v_0+1)$ containing a sub NB$(3,\lambda;v_0)$. In Lemma 4, let $v_1=3$, $v_2=2v_O+1$ and $v_3=v_0$, then

$$v=v_1(v_2-v_3)+v_3=4v_0+3\in \text{INB}(\lambda),$$

and then

$$\{v_0,2v_0+1,4v_0+3\}\subset \text{INB}(\lambda).$$

Let $K=\text{INB}(\lambda)$. It follows from (8) that

$$\alpha(K)\geq 2,\qquad \beta(K)\geq 6.$$

On the other hand, as $v_0\equiv 3 \pmod 6$, we have

$$(v_0(v_0-1),(2v_0+1)2v_0)=v_0(v_0-1,2(2v_0+1))=2v_0.$$

So we have

$$\alpha(K)\leq(v_0-1,(2v_0+1)-1)=(v_0-1,2)=2$$

and

$$\begin{aligned}\beta(K)&\leq(v_0(v_0-1),(2v_0+1)2v_0,(4v_0+3)(4v_0+2))\\&=(2v_0,16v_0^2+20v_0+6)=(2v_0,6)=6.\end{aligned}$$

Hence $\alpha(K)=2,\beta(K)=6$. From Lemma 7 and Lemma 1, the conclusion then follows.

(II) If $\lambda\equiv 3 \pmod 6$. Then the necessary conditions (1) become

$$v\equiv 1 \pmod 2,\qquad v\geq\lambda+2. \tag{9}$$

If $v_0\equiv 3 \pmod 6$, then $2v_0+1\equiv 1 \pmod 6$. If $v_0\equiv 1 \pmod 6$, then $v_0\geq\lambda+4$, and $2v_0+3\equiv 5 \pmod 6$. By Lemma 6, there exists an indecomposable NB$(3,\lambda;2v_0+3)$. Thus we may suppose $v_0\equiv 5 \pmod 6$. By Lemma 5, we have

$$\{v_0,2v_0+1,4v_0+3\}\subset \text{INB}(\lambda).$$

Let $K=\text{INB}(\lambda)$. It follows from (9) that,

$$\alpha(K)\geq 2,\qquad \beta(K)\geq 2.$$

On the other hand, as $v_0\equiv 5 \pmod 6$, we have

$$(v_0(v_0-1),(2v_0+1)2v_0)=v_0(v_0-1,6)=2v_0.$$

Then

$$\alpha(K) \leq (v_0 - 1, (2v_0 + 1) - 1) = 2$$

and

$$\begin{aligned}\beta(K) &\leq (v_0(v_0 - 1), (2v_0 + 1)2v_0, (4v_0 + 3)(4v_0 + 2)) \\ &= (2v_0, 16v_0^2 + 20v_0 + 6) = (2v_0, 6) = 2.\end{aligned}$$

Hence $\alpha(K) = 2, \beta(K) = 2$. This completes the proof.

It is proved ([3]) that for any positive integer λ with $\lambda \equiv 1 \pmod 2$, there exists an indecomposable NB$(3, \lambda; v)$ for some v. Combining this result with our fundamental lemma, we have proved the main theorem:

Theorem. *Let $\lambda \geq 1$, $\lambda \equiv 1 \pmod 2$. Then there is a constant $v_0 = v_0(\lambda)$ such that for $v > v_0$, there exists an indecomposable NB$(3, \lambda; v)$ if and only if*

$$\begin{aligned}\lambda(v-1) &\equiv 0 \pmod 2 \\ \lambda v(v-1) &\equiv 0 \pmod 6.\end{aligned} \tag{10}$$

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