Complete k-Caps in $PG(3,q)$ with $k < (q^2+q+4)/2$

G. Faina

Dipartimento di Matematica
Università di Perugia
Via Vanvitelli 06100 Perugia
Italy

Abstract. It is well known that there exist complete k-caps in $PG(3,q)$ with $k \geq (q^2+q+4)/2$ and it is still unknown whether or not complete k-caps of size $k < (q^2+q+4)/2$ and q odd exist. In this paper sufficient conditions for the existence of complete k-caps in $PG(3,q)$, for q odd $q \geq 7$ and $k < (q^2+q+4)/2$, are established and a class of such complete caps is constructed.

1. Introduction and Notation

In $PG(n,q)$, projective space of n dimensions over the field $GF(q)$, consider a set K of size k such that no three points of K are collinear. The set K is called a *k-cap* and when $n = 2$, a *k-arc*. A k-cap in $PG(n,q)$ is *complete* if it is not contained in a $(k+1)$-cap.

The properties of k-caps were first described by B. Segre [6] in 1959, who also indicated a number of interesting open problems; see also Tallini [7] and Hirschfeld [3] for their statistical and coding theoretical connections.

For notation and background material on $PG(n,q)$, elliptic quadrics and k-caps we follow [2] and [4].

We now give some known lower bounds for the number of points on a complete k-cap (see [4, n. 18]):

(1.1) if K is a complete k-cap in $PG(3,q)$, then $k > \sqrt{2q}+1$.

(1.2) in $PG(3,q)$, there exists a complete k-cap with

$$(q^2+q+4)/2 \leq k \leq (q^2+3q+6)/2.$$

(1.3) in $PG(3,q)$ with $q \equiv -1 \pmod 4$, there exists a complete k-cap with

$$k = (q^2+q+4)/2.$$

The problem of determining k-caps with $k < (q^2+q+4)/2$ was first considered by B. Segre [6].

This problem has generated quite a bit of interest and research. See e.g. [4] 18.5. A complete k-cap with $k < (q^2+q+4)/2$ has been constructed by V. Abatangelo (1984) for q even and $q > 255$ [1], but for q odd our problem is still unsolved.

In this paper we give a first example of complete k-caps of $PG(3,q)$, with $q = 7$ and $k < (q^2+q+4)/2$, and a sufficient condition for the existence of complete k-caps of $PG(3,q)$, for $q \geq 7$ odd and $k < (q^2+q+4)/2$ is found.

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2. The Setting

We begin with some relevant definitions which appear for the first time.

Let C be a non singular conic of $PG(2,q)$, with $q \geq 7$ odd. The number q is (C,h)*-constructable* if given any pair of distinct points $\tilde{E}, I$ of $PG(2,q)\backslash C$ such that:

(2.1) $\tilde{E}$ is an external point of C and I is an internal point of C,

(2.2) the polar e of $\tilde{E}$ does not contain I,

(2.3) the line $\tilde{E}I$ through the points $\tilde{E}$ and I is a 0-secant of C, then there exist $h-4$ distinct points, $P_1,\ldots,P_{h-4}$ of $C\backslash\{C_1,C_2\}$, where $C_i = t_i \cap C$, $i = 1,2$, and t_1, t_2 are the 1-secants of C through $\tilde{E}$ such that the set

(2.4) $H_h(C,\tilde{E},I){:=}\{P_1,\ldots,P_{h-4},C_1,C_2,\tilde{E},I\}$ is a complete h-arc of $PG(2,q)$.

It is well known that (see [2, n. 10]) a complete h-arc in $PG(2,q)$, q odd, other than a conic is such that

(2.5) $h \leq q-\sqrt{q}/4+7/4$ and $(h(h-3)/2)+2 > q$.

A triple (E,P,Q) of $PG(3,q)$, with $q \geq 7$ odd, is called *constructable* if the following (2.6)–(2.9) hold:

(2.6) E is a non singular elliptic quadric of $PG(3,q)$,

(2.7) P and Q are two distinct points of $PG(3,q)\backslash E$ such that the line PQ is a 0-secant of E,

(2.8) every secant plane, say σ_i, $i = 1,\ldots,q-1$, of E through PQ intersects E in a non-singular conic, say C_i, such that, if P is an external (internal) point of C_i, then Q is an internal (external) point of C_i,

(2.9) the polar plane of P (of Q), say π_P (π_Q), does not contain $Q(P)$.

It is easy to see that there exist constructable triples (E,P,Q) in every projective space $PG(3,q)$, $q \geq 7$ odd.

Remark. Let $C_P = \{\pi_2 \cap E\}\backslash\{T_1T_2\}$ and let $C_Q = \{\pi_Q \cap E\}\backslash\{T_1T_2\}$. It is not difficult to see that $|C_P| = |C_Q| = q-1$ and that $C_P \cap C_Q$ is the empty set. Thus, if L is the set of lines of type A_iB_j, with $A_i \in C_P$ and $B_j \in C_Q$, then

(2.10) $|L| = (q-1)^2$.

Since τ_i is the polar plane of T_i, $P,Q \in \tau_i$, $i = 1,2$, and $\pi_P \cap \pi_Q \cap E = \{T_1,T_2\}$, from (2.8), we have that

(2.11) $A_iB_j \cap \tau_m \notin \{PQ \cup \pi_P \cup \pi_Q\}$, $m = 1,2$.

Our aim is to demonstrate the follow results.

Theorem 1. *Let (E,P,Q) be a constructable triple of $PG(3,q)$, $q \geq 7$ odd. If q is (C,h)-constructable, then the set K defined as*

$$\bigcup_{i=1}^{q-1} H_h(C_i,P,Q) \cup \{T_1,T_2\}$$

is a complete k-cap of $PG(3,q)$ and $k = (q-1)(h-2)+4$.

Corollary. *If q is (C,h)-constructable with $h < (q^2+5q-8)/(2q-2)$, with $q \geq 7$ odd, then there exists a complete k-cap in $PG(3,q)$ with $k < (q^2+q+4)/2$.*

Theorem 2. *In $PG(3,7)$ there exist complete k-caps with $k < (q^2+q+4)/2$.*

Remark. Since a h-arc, with $h < 6$, is never complete (see [2]) the problem of constructing complete k-caps in $PG(3,5)$ cannot be solved with our method. For $q = 5$, our problem will be dealt in a following paper.

3. Proof of Theorem 1

With our notation, it follows immediately that $PG(3,q) = \bigcup_{i=1}^{q-1} \sigma_i \cup \{\tau_1 \cup \tau_2\}$ and that $\sigma_i \cap K = H_h(C_i,P,Q)$ is a complete h-arc of σ_i, $i = 1,\dots,q-1$.

Thus each point P of σ_i, $i = 1,\dots,q-1$, lies in at least a 2-secant of K. To show that K is a complete k-cap it suffices to prove that every point Z of $\tau_1 \cup \tau_2$ lies in (at least) a 2-secant of K.

The proof shall be given in several steps.

(I) Since P,Q lie in $\tau_i (i=1,2)$, every point Z of PQ lies in at least a 2-secant of K.

(II) If P is an external point of C_i, then $|\pi_P \cap C_i| = 2$ and, by 2.8, $\pi_Q \cap C_i = \phi$. In this case denote by $X_{i,P}$ and by $Y_{i,P}$ the points of $\pi_P \cap C_i$. Thus the lines $PX_{i,P}$ and $PY_{i,P}$ are tangent to C_i. From (2.4), this implies that

$$X_{i,P} \text{ and } Y_{i,P} \text{ lie in } H_h(C_i,P,Q).$$

Since T_i is conained in π_P, $i = 1,2$, it turns out that π_P can intersect exactly $(q-1)/2$ of the conics C_j, $j = 1,\dots,q-1$, since $|\pi_P \cap E| = q+1$, it follows that the non-singular conic $C(P) = \pi_P \cap E$ is contained in K.

(III) With a similar argument we can prove that $C(Q) = \pi_Q \cap E$ is contained in K.

(IV) Since $C(P)((C_Q))$ is a complete arc of π_P (of π_Q), by (II) (by (III)) it follows that every point of $\pi_P \cap \tau_i$ (of $\pi_Q \cap \tau_i$) (i=1,2) lies in (at least) a 2-secant of K.

(V) Let S_i be the set $PQ \cup \{\pi_P \cap \tau_i\} \cup \{\pi_Q \cap \tau_i\}$ $(i = 1,2)$. It is easily to see that $|S_i| = 3q$.

(VI) Let τ_i, $i = 1,2$, be the tangent planes of E through the 0-secant PQ, let $T_i = \tau_i \cap E$ $(i = 1,2)$ let $C_P = \{\pi_P \cap E\} \setminus \{T_1,T_2\}$, and $C_Q = \{\pi_Q \cap E\} \setminus \{T_1,T_2\}$. If A_1,A_2 lie in C_P, B_1,B_2 lie in C_Q and $A_iB_j \neq A_hB_k$, then $A_iB_j \cap \tau_m \neq A_hB_k \cap \tau_m$, with $m,i,j = 1,2$.

To proof this, we observe that we can choose a frame for $PG(3,q)$ such that: $T_1 = (1,0,0,0)$, $T_2 = (0,1,0,0)$, $P = (0,0,0,1)$, $Q = (0,0,1,0)$ and

$U = (1,1,1,1)$, where U is a point of $\pi_Q \cap E \backslash \{T_1, T_2\}$. With respect to this frame, E is represented by

(3.1) $$2axy + bz^2 + 2czt + dt^2 = 0.$$

Thus π_P: $cz + dt = 0$ and π_Q: $bz + ct = 0$.

From (2.8), it follows that $U \notin \pi_P$ and this implies that

(3.2) $$c \neq -d.$$

Since $U \in \pi_Q$, we have that

(3.3) $$c = -b \quad \text{and so} \quad c \neq 0 \quad \text{and} \quad b \neq 0, a \neq 0.$$

Furthermore, $U \in E$ implies that

(3.4) $$d + 2c + 2a + b = 0$$

and, from (3.3), we have that

(3.5) $$d + c + 2a = 0.$$

From (3.3) and (3.5), it follows that E can be represented by

(3.6) $$(1-2v)t^2 - 2zt + 2vxy + z^2 = 0, \quad \text{which} \quad 1 - 2v \neq 0,$$

where v denotes (a/b).

Furthermore, we have that in $PG(3,q)$

(3.7) $$\pi_p\colon ft - z = 0$$

(3.8) $$\pi_Q\colon t - z = 0,$$

where f denotes $1 - 2v$.

It is easy to see that each point of $\pi_Q \cap E$ can be represented by

(3.9) $$A_i = (i, i^{-1}, 1, 1), \quad i = 1, \ldots, q-1$$

and that each point of $\pi_P \cap E$ can be represented by

(3.10) $$B_j = (fj^{-1}, j, f, 1), \quad j = 1, \ldots, q-1.$$

Let A_iB_n and A_jB_m be two distinct lines and suppose that they contain the same point $V = (x', y', z', t')$ of $PG(3, q)$. From (2.11), it follows that $t' \neq 0$ and, if V lies in τ_1, then $y' = 0$. From (3.9) and (3.10), after a straightforward calculation of the cartesian equations of our lines A_iB_n, A_jB_m, we find that the following hold:

$$mi^{-1} - nj^{-1} = 0 \tag{3.11}$$

$$i^{-1} - j^{-1} = n - m, \tag{3.12}$$

and that, necessarily, we have

$$i^{-1} \neq n, j^{-1} \neq m. \tag{3.13}$$

Now, we consider the system of the previous linear equations. If we have that the rank of its matrix

$$M: \begin{pmatrix} m & -n \\ 1 & -1 \end{pmatrix}$$

is < 2, then we obtain that $n = m$. But this implies that $i = j$ and thus $A_iB_n = A_jB_m$ which is a contradiction. It follows that rank of $M = 2$ and that the only solution of our system is given by

$$i^{-1} = n, j^{-1} = m, \tag{3.14}$$

which is a contradiction (see (3.13)).

So our two lines do not contain a same point in τ_1.

A similar argument proves that the our lines do not contain a same point of τ_2. This proves (VI).

(VII) It remains to show that each point of $\tau_i \backslash S_i$ lies in (at least) a 2-secant of K. To show this, we observe that $|\tau_j \backslash S_i| = (q-1)^2, i, j = 1, 2$. Thus, from (2.10), we have that $|L| = |\tau_j \backslash S_i|$. Hence, from (VI) and (2.11), each point of $\tau_j \backslash S_i$ lies in exactly a 2-secant A_mB_n of K and our Theorem 1 is proved.

Our Corollary follows immediately from technical details and its proof will be omitted.

4. Proof of Theorem 2

This follows immediately from Theorem 1 and the next Lemma.

Lemma. *If $q = 7$, then q is $(C, 6)$-constructable for all non-singular conics C of $PG(2,q)$.*

Proof: Let C be any non-singular conic of $PG(2,q)$, $q \geq 7$ odd. Let $\bar{E}$ and I be two points of $PG(2,q)\backslash C$ such that (2.1)–(2.3) hold. Let $A = t_1 \cap C$ and let $B = t_2 \cap C$, were the t_i, $i = 1,2$, are the two tangents to C through $\bar{E}$. Choose a new frame for $PG(2,q)$ as follows:

$$A = (1,0,0),\ B = (0,1,0),\ \bar{E} = (0,0,1) \text{ and } I = (1,1,1).$$

Since q is odd, using homogeneous coordinates, the conic C can be represented by

$$xy - kt^2 = 0$$

where k is a nonsquare of $GF(q)$.

Let $A' = \{IA \cap C\}\backslash\{A\}$ and let $B' = \{IB \cap C\}\backslash\{B\}$. Since the polar of $\bar{E}$ is the 2-secant AB, we have that $I \notin AB$, $A' \neq B$ and $B' \neq A$. We are now ready to prove that

(4.1) if $\bar{E} \notin A'B'$, then q is $(C, 6)$-constructable.

Let $E' = \{A'\bar{E} \cap C\}\backslash\{A'\}$ and let $E'' = \{B'\bar{E} \cap C\}\backslash\{B'\}$. Since $\bar{E} \notin A'B'$, it is easily proved that in $GF(7)$, or $k = 3$ or $k = 5$, and so or $E' = (4,6,1)$ and $E'' = (6,4,1)$ or $E' = (2,6,1)$ and $E'' = (6,2,1)$.

From a result of Lunelli-Sce on complete 6-arcs of $PG(2,7)$ [5], in either case, the 6-arc $\{A, B, \bar{E}, I, E', E''\}$ is a complete arc.

Finally, if $\bar{E} \in A'B'$, then the conic C is given by $xy + t^2 = 0$. Hence, C contains the following two distinct points: $X = (2,3,1)$ and $Y = (4,5,1)$. Thus, from the same result [5], the 6-arc $\{A, B, \bar{E}, I, X, Y\}$ is a complete arc and our Lemma is proved.

From Theorem 1 and from our Lemma, we can to construct complete k-caps of $PG(3,7)$ with $k = 28$. But, if $q = 7$, then $(q^2 + q + 4)/2 = 30$, thus we have proved our Theorem 2.

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