On q-divisible Hypergraphs

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Abstract. Let $H(V,E)$ be r-uniform hypergraph. Let $A \subset V$ be a subset of vertices and define $\deg_H(A) = |\{e \in E : A \subset e\}|$.

We say that H is (k,m) divisible if for every k-subset A of $V(H)$, $\deg_H(A) \equiv 0 \pmod{m}$. (We assume that $1 \le k < r$).

Given positive integers $r \ge 2$, $k \ge 1$ and q a prime power, we prove that if H is r-uniform hypergraph and $|E| > (q-1)\binom{|V|}{k}$ then H contains a nontrivial subhypergraph F which is (k,q)-divisible.

Several variations of this result are discussed.

1. Introduction.

Let $H(V,E)$ be r-uniform hypergraph and $f: E(H) \to \mathbf{Z}_m^\star$ be a mapping from the edge set of H to $\mathbf{Z}_m - \{0\}$. Let $A \subset V$ be a set of vertices and define the degree of A by $\deg_H(A) = \sum_{A \subset e} f(e)$, where we assume from now on that $1 \le k < r$.

We say that H is (k,n,m)-divisible if $\deg_H(A) \equiv n \pmod{m}$ for every k-subset A of V. (Of course, with respect to f in the background). Those for $f \equiv 1$ and $r = 2$, H is $(1,1,2)$-divisible means that every vertex v of H has an odd degree.

Some special cases of the following questions were posed (in a slightly different form), by A. Bialostocki in connection with his work on the Zero-sum Ramsey Numbers, and by Y. Roditty to whom I am indebted for telling me the problems **[BD1, BD2, BD3, BCR]**.

Problem 1: Given r, k, n, m and r-uniform hypergraph H, and a mapping $f: E(H) \to \mathbf{Z}_m^\star$. Find a (k,n,m)-divisible subhypergraph F with a maximum number of vertices.

Problem 2: Given r, k, n, m and r-uniform hypergraph H, and a mapping $f: E(H) \to \mathbf{Z}_m^\star$. Find a (k,n,m)-divisible subhypergraph F with a maximum number of edges.

Problem 3: Given r, k, n, m and r-uniform hypergraph H, and a mapping $f: E(H) \to \mathbf{Z}_m^\star$. Find a (k,n,m)-divisible subhypergraph F with a maximum number of edges and such that also $\sum_{e \in F} f(e) \equiv 0 \pmod{m}$.

The tools necessary to deal with such problems were developed few years ago by Alon, Friedland, and Kalai [AFK], where among many interesting results they proved the assertion of the theorem mentioned in the abstract in the case of graphs.

The main tool in [AFK] which we need here is the following extension and variation of Chevalley's theorem [BS].

Theorem A. [AFK] *Let $q = p^d$ be a prime power and n be a positive integer. For $1 \le i \le m$ let $a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)})$ be a vector with integer coordinates. Suppose $m > (q-1)n$ then there exists a subset $\phi \neq I \subset \{1, 2, \dots, m\}$ s.t. the following congruences hold*

$$(\star) \qquad \sum \left\{ a_j^{(i)} : i \in I \right\} \equiv 0 \pmod q \quad \text{for} \quad j = 1, 2, \dots, n.$$

Moreover, if $\sum_{j=1}^{n} a_j^{(i)} \equiv 0 \pmod p$ for $i = 1, 2, \dots, m$ and $m > (q-1)n + \frac{q}{p} - q$ then $(\star)$ holds.

Using this theorem they proved in [AFK] the following result.

Theorem B. [AFK] *Let $q = p^d$ be a prime power and set*

$$h(n,q) = \begin{cases} (q-1)n & p \text{ odd prime} \\ (q-1)n - \frac{q}{2} & q = 2^d \end{cases}$$

then every graph G on n vertices and $h(n,q) + 1$ edges contains a nonempty $(1,0,q)$-divisible subgraph (a subgraph in which the degree of any vertex is divisible by q, and the edge set of this subgraph is not empty).

The last notion we need before stating our results is the notion of a dense set. A set F of positive integers is called dense if the following holds

$$1) \quad 1 \in F, \qquad\qquad 2) \quad n \notin F \Rightarrow n-1 \in F.$$

2. Results on Problem 1.

We first give some initial results concerning Problem 1. We begin with some theorems on graphs. Our first result concerns dense sets.

Theorem 1. *Let G be a connected graph on at least two vertices, and let F be a dense set. There is a subgraph $H \subset G$, $|H| \ge |G| - 1$, such that $\deg_H v \in F$ for every $v \in V(H)$.*

Proof: We prove the assertion of the theorem by induction on n, the number of vertices in a spanning tree T of G.

For $n = 2$ the assertion is true. Let T be a spanning tree on $n+1$ vertices. If already in T, $\deg_T v \in F$ for every $v \in V(T)$ we are done. Otherwise, take an endpoint v and the remotest vertex u from v for which $\deg_T u \notin F$. Certainly u is not an endpoint because in this case $\deg_T u = 1 \in F$. Delete the edge e, adjacent to u on the unique path from u to v. We obtain two subtrees H, K and suppose $u \in V(H)$. Now in H, $\deg_H w \in F$ for every $w \in V(H)$, including u, because $\deg_H u = \deg_T u - 1$ and F is dense. The other vertices in H were left unchanged.

Apply induction on K (or, otherwise, K is a single vertex) and we are done. ∎

Remark: The proof of Theorem 1 implies an $O(E)$ algorithm to find a subgraph $H \subset G$, $|H| \ge |G| - 1$, and $\deg_H v \in F$. F a dense set. Yet a sharper result can be proved if F is the set of odd integers.

Theorem 2. *Let G be a connected graph on at least two vertices, then*

(i) *if $|G| \equiv 1 \pmod 2$ then there exists $H \subset G$, $|H| = |G| - 1$, s.t. $\deg_H v \equiv 1 \pmod 2$ for every $v \in V(H)$,*

(ii) *if $|G| \equiv 0 \pmod 2$ then there exists $H \subset G$, $|H| = |G|$ s.t. $\deg_H v \equiv 1 \pmod 2$ for every $v \in V(H)$.*

Proof: Observe that F, the set of odd positive integers is a dense set. Hence, by Theorem 1 there exists a subgraph $H \subset G$, $|H| \geq |G| - 1$ s.t. $\deg_H v \equiv 1 \pmod 2$ for any $v \in V(H)$.

(i) If $|G| \equiv 1 \pmod 2$ then because of parity consideration there is no subgraph H s.t. $|H| = |G|$ and $\deg_H v \equiv 1 \pmod 2$ for each $v \in V(H)$.

(ii) We prove the assertion by induction on $|G| = 2n$. For $n = 1$ this is true as $G = K_2$. Consider a spanning tree T of G, $|T| = |G| = 2n$. Then of course $e(T) \equiv 1 \pmod 2$, ($e(G)$ denotes the number of edges of a graph G).

If $\deg_T v \equiv 1 \pmod 2$ for each $v \in V(T)$ we are done, else let v be a vertex having an even degree in T. Clearly, v is not an endpoint, and let $B_1, B_2, \ldots, B_k$, $k \equiv 0 \pmod 2$ be the branches at v.

Now $\sum_{i=1}^{k} e(B_i) = e(T) \equiv 1 \pmod 2$. Thus, at least one of the branches, say B_1, must contain an even number of edges, hence, $|B_1 \backslash v| \equiv 0 \pmod 2$. Consider the resulting trees, $T_1 = B_1 \backslash v$ and $T_2 = T \backslash T_1$, both contain an even number of vertices and induction applies. ∎

Remark: Once again Theorem 2 can be implemented in $O(E)$ algorithm. Observe here that from Theorem 2 we deduce that if G is a connected graph then there exists a set of edges $E_1 \subset E$ such that $|E_1| \geq \frac{|G|-1}{2}$ and E_1 induces a $(1,1,2)$-divisible subgraph. It is also easy to conclude that if G is a graph such that $\delta(G) \geq 1$ ($\delta(G)$ = the minimum degree) then G contains $(1,1,2)$-divisible subgraph H such that $|H| \geq \frac{2}{3}|G|$.

What about $(1,1,k)$-divisible graphs for $k \geq 3$?

The following result gives a hint on this question.

Theorem 3. *Let G be a connected graph on $n \geq 2$ vertices. There exists a $(1,1,k)$-divisible subgraph $H \subset G$ such that $|H| \geq \frac{2n}{k+1}$.*

Proof: We prove the assertion of the theorem by induction on the number of the vertices of a spanning tree T of G.

For $n = 2$ this is true. It is also true for any star $K_{1,m}$ as one can easily check. Suppose T is not a star, then there is an edge $e = (u, v)$ whose endvertices u, v are not endpoints of the spanning tree T. Hence, $T \backslash e$ results in two nontrivial subtrees K and H, both of them contain at least two vertices. Apply the induction hypothesis on K and H to obtain $K' \subset K$ and $H' \subset H$, both K' and H' are $(1,1,k)$-divisible and also $|K'| \geq \frac{2|K|}{k+1}$, $|H'| \geq \frac{2|H|}{k+1}$, hence, $K' \cup H'$ which is $(1,1,k)$-divisible satisfies $|K' \cup H'| \geq \frac{2(|K|+|H|)}{k+1} = \frac{2|T|}{k+1}$ as needed. ∎

Theorem 3 is probably not best possible and we conjecture that the following stronger estimate holds.

Conjecture 1. *Let G be a connected graph, then G contains a $(1,1,k)$-divisible subgraph H such that* $|H| \geq \frac{2(|G|-1)}{k}$.

We conclude the discussion concerning Problem 1 with the following question.

Question 1. *Let $k \geq 3$ be a given integer and let G be a k-connected graph. Is it true that G contains a $(1,1,k)$-divisible subgraph H s.t. $|H| \geq |G| - k$.*

3. Problem 2 and Problem 3.

Recall Theorem A and Theorem B from the introduction and the following well-known result of Pyber [PYB].

Theorem C. [PYB] *Let $k \geq 3$ be an integer and let G be a graph on n vertices and at least $c_k n \log n$ edges, where $c_k > 0$ a constant depends only on k, then G contains a k-regular subgraph ($c_k = 32k^2$ is a valid choice).*

Recall also the following extension of Theorem A to a nonprime-power moduli m.

Theorem D. [AFK, BASM] *Let m be a non-prime power integer, and let n be a positive integer. For $1 \leq i \leq t$ let $a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)})$ be a vector with integer coordinates. Suppose $t \geq (cm \log m)n$, then there exists a non empty subset $I \subset \{1, 2, \dots, t\}$ such that the following set of congruences hold.*

$$\sum \left\{a_j^{(i)} : i \in I\right\} \equiv 0 \pmod{m}, \quad j = 1, 2, \dots, n.$$

Let k, m, n, r be positive integers, $m, r \geq 2$, $r > k > 0$. Define the function $h(n, m, k, r)$ as follows.

$$h(n,m,k,r) = \begin{cases} (m-1)\binom{n}{k} & m = p^d, p \text{ a prime}, \binom{r}{k} \not\equiv 0 \pmod{p} \\ (m-1)\binom{n}{k} - m + \frac{m}{p} & m = p^d, p \text{ a prime}, \binom{r}{k} \equiv 0 \pmod{p} \\ (cm \log m)\binom{n}{k} & m \neq p^d, c \text{ is the constant from Theorem D.} \end{cases}$$

Theorem 4. *Let H be r-uniform hypergraph on n vertices and q edges. Let $f : E(H) \to Z_m^k$. If $q \geq h(n, m, k, r) + 1$ then H contains a nontrivial $(k, 0, m)$-divisible hypergraph.*

Proof: . Let $[H]^k$ be the collection of all k-subsets of $V(H)$.
Clearly, $|[H]^k| = \binom{n}{k}$. For $u_i \in [H]^k$ and $e \in E(H)$ define

$$a_{u_i}^{(e)} = \begin{cases} f(e) & u_i \subset e \\ 0 & \text{otherwise.} \end{cases}$$

For $e \in E(H)$ define $a^{(e)} = \left(a^{(e)}_{u_1}, \ldots, a^{(e)}_{u_t}\right)$ where $t = \binom{n}{k}$. Observe that

$$\sum_{i=1}^{t} a^{(e)}_{u_i} = \binom{r}{k} f(e) \equiv 0 \pmod{p} \text{ if } p \mid \binom{r}{k}.$$

Hence, by Theorems A,C,D, according to the relations between m, k, r, there exists a non-empty subset $E' \subset E(H)$ such that

$$\sum \left\{a^{(e)}_{u_j} : e \in E'\right\} \equiv 0 \pmod{m} \text{ for all } 1 \leq j \leq \binom{n}{k}.$$

The induced r-uniform subhypergraph on E' is a non-trivial $(k, 0, m)$-divisible hypergraph. ∎

We are now in a position to give some estimates concerning Problem 2 and Problem 3.

Theorem 5. *Let m, k, r be positive integers such that $m \geq 2, r > k > 0$. Let H be r-uniform hypergraph on n vertices and e edges and let $f: E(H) \to Z^*_m$. Then H contains a $(k, 0, m)$-divisible subhypergraphs having at least* $\max\{0, e - h(n, m, k, r)\}$ *edges.*

Proof: If $e \leq h(n, m, k, r)$ the result is obvious. Suppose $e > h(n, m, k, r)$. Then by Theorem 4, H contains a nontrivial $(k, 0, m)$-divisible subhypergraph H_1. Put $e_1 = |E(H_1)|$ and consider the hypergraph $F = H(V, E \backslash E(H_1))$. If $|E(F)| \leq h(n, m, k, r)$ then $e_1 \geq e - h(n, m, k, r)$ and we are done, otherwise, we may use Theorem 4 once more and F contains a nontrivial $(k, 0, m)$-divisible subhypergraph H_2, where $e_2 = |E(H_2)|$. Clearly, $E(H_1) \cup E(H_2)$ induces a $(k, 0, m)$-divisible subhypergraph on $e_1 + e_2$ edges. This process will terminate only after the deletion of $(k, 0, m)$-divisible subhypergraphs $H_1, H_2, \ldots, H_t$ such that $e_1 + e_2 + \ldots + e_t \geq e - h(n, m, k, r)$ and, clearly, $\cup_{i=1}^{t} E(H_i)$ induces a $(k, 0, m)$-divisible subhypergraph of H. ∎

For the particular case when $r = 2, k = 1, m = 2$ Theorem 5 implies that every graph G contains a $(1, 0, 2)$-divisible subgraph on at least $|E(G)| - |G| + 1$ edges. What about $(1, 1, 2)$-divisible subgraphs?

Theorem 6. *Let G be a connected graph on $|G| \geq$ vertices.*

(i) *If $|G| \equiv 0 \pmod{2}$, G contains a $(1, 1, 2)$-divisible subgraph H, $|E(H)| \geq |E(G)| - |G| + 1$.*
(ii) *If $|G| \equiv 1 \pmod{2}$, G contains a $(1, 1, 2)$-divisible subgraph H, $|E(H)| \geq |E(G)| - 2|G| + 3$.*

Proof: By Theorem 2, G contains a $(1, 1, 2)$-divisible subgraph H s.t. $|H| = |G|$, if $|G| \equiv 0 \pmod{2}$, and $|H| = |G| - 1$ if $|G| \equiv 1 \pmod{2}$.

(i) If $|G| \equiv 0 \pmod{2}$ then delete from $E(G)$ the edges of $E(H)$ to obtain a subgraph $G(V, E(G) \backslash E(H)) := G'$.

By Theorem 5, G' contains a $(1,0,2)$-divisible subgraph F such that $|E(F)| \geq |E(G')| - |G'| + 1$.

Clearly, $E(F) \cup E(H)$ is $(1,1,2)$-divisible subgraph Q in which $|E(Q)| \geq |E(H)| + |E(G')| - |G'| + 1 = |E(G)| - |G| + 1$.

(ii) If $|G| \equiv 1 \pmod 2$ the same argument works with slight changes. Recall that if G is connected then there exists a vertex v s.t. $G \backslash v = H$ is also connected. We may lose $|G| - 1$ edges incident to the vertex $v \in G \backslash H$. Also, $|G'| = |G| - 1$. Taking this in account, the same calculation gives, $|E(Q)| \geq |E(H)| + |E(G')| - |G'| + 1 \geq |E(G)| - 2|G| + 3$.

∎

Remark: Observe that if G is 2-connected and v is a vertex that realizes $\delta(G)$ then $G \backslash v$ is also connected, hence, in this case one can improve Theorem 6(ii) to the bound $|E(G)| - |G| - \delta(G) + 2$. Notice here also that combining Theorem 5 with the ideas of Theorem 6 it is easy to show that if H is r-uniform hypergraph and H contains a subhypergraph F s.t. $V(F) = V(H)$, and F is (k,n,m)-divisible and also $|E(H)| \geq h(n,m,k,r) + 1$ then, in fact, H contains a (k,n,m)-divisible subhypergraph F' s.t.

$$|E(F')| \geq |E(H)| - h(n,m,k,r).$$

So, for example, if G has a 1-factor (perfect matching) then G contains a $(1,1,9)$-divisible subgraph H s.t. $|E(H)| \geq |E(G)| - 8|G|$. This principle can be applied to graphs having k-factor (k-regular spanning subgraphs) as well.

Now we turn our attention to Problem 3 from the introduction. Recall that Problem 3 asked for a (k,n,m)-divisible hypergraph F such that $\sum_{e \in F} f(e) \equiv 0 \pmod m$.

Recall the trivial fact that every collection $a_1, a_2, \dots, a_m$ of m integers contains a subset that adds to $0 \pmod m$. Combining this fact with our results on Problem 2 we obtain:

Theorem 7. *Let m,k,r be positive integers, $m \geq 2$, $r > k > 0$. Let H be r-uniform hypergraph on n vertices and e edges, and let $f: E(H) \to Z_m^*$. Then H contains a $(k,0,m)$-divisible subhypergraph F s.t. $\sum_{e \in E(F)} f(e) \equiv 0 \pmod m$ and $|E(F)| \geq \max\{0, \lfloor \frac{e}{m} - h(n,m,k,r) \rfloor\}$.*

Proof: We may assume $\lfloor \frac{e}{m} - h(n,m,k,r) \rfloor = q \geq 1$, otherwise the assertion is trivial. Take m disjoint subsets of edges $E_1, \dots, E_m \subset E(H)$ s.t. $|E_i| = h(n,m,k,r) + q$. This is possible because $|\cup_{i=1}^m E_i| \leq e$. By Theorem 5, for $1 \leq i \leq m$ the hypergraph induced by E_i contains a $(k,0,m)$ divisible subhypergraph F_i, $|E(F_i)| \geq q$. For each $1 \leq i \leq m$ let $a_i = \sum_{e \in F_i} f(e)$. By the previous remark, in $\{a_1, \dots, a_m\}$ there exists a subset $a_{i_1}, \dots, a_{i_k}$ s.t. $\sum_{j=1}^k a_{i_j} \equiv 0 \pmod m$.

Clearly, $F = \cup_{j=1}^{k} E(F_{i_j})$ is $(k,0,m)$-divisible with the additional property that $\sum_{e\in F} f(e) \equiv 0 \pmod{m}, |E(F)| \geq q$. ∎

The estimate in Theorem 7 can be improved under certain conditions, using the following theorem due to Erdős, Ginzburg and Ziv [EGZ].

Theorem E. [EGZ] *Let $m \geq 2$ be an integer and $a_1, a_2, \dots, a_{(t+1)m-1}$ be a collection of $(t+1)m-1$ integers. Then there exists a subset $I \subset \{1,2,\dots,(t+1)m-1\}$, $|I| = tm$, s.t. $\sum_{i\in I} a_i \equiv 0 \pmod{m}$.*

Theorem 8. *Let m, k, r be positive integers, $m \geq 2, r > k > 0$. Let H be r-uniform hypergraph on n vertices and e edges. Finally, let $f\colon E(H) \to Z_m^*$. Then H contains a $(k,0,m)$-divisible subhypergraph F s.t.*

$$\sum_{e\in F} f(e) \equiv 0 \pmod{m}, |E(F)| \geq \max_{1\leq t} \left\{ tm \left\lfloor \frac{e}{(t+1)m-1} - h(n,m,k,r) \right\rfloor \right\}.$$

Proof: We may assume $\lfloor \frac{e}{(t+1)m-1} - h(n,m,k,r) \rfloor = q \geq 1$ for given $t \geq 1$; otherwise there is nothing to prove.

Take $(t+1)m-1$ disjoint subsets of edges $E_i \subset E(H), 1 \leq i \leq (t+1)m-1$ such that $|E_i| = h(n,m,k,r) + q$. This choice is possible because $|\cup_i E_i| \leq e$.

By Theorem 5 for $1 \leq i \leq (t+1)m-1$, the hypergraph induced by E_i contains a $(k,0,m)$-divisible subhypergraph F_i, $|E(F_i)| \geq q$. As before let $a_i = \sum_{e\in F_i} f(e)$ $1 \leq i \leq (t+1)m-1$.

By Theorem E there exists a subset $I \subset \{1,2,\dots,(t+1)m-1\}$, $|I| = tm$ s.t. $\sum_{i\in I} a_i \equiv 0 \pmod{m}$.

Clearly, $F = \cup_{i\in I} E(F_i)$ is $(k,0,m)$-divisible subhypergraph s.t. $\sum_{e\in F} f(e) \equiv 0 \pmod{m}$ and $|E(F)| \geq tmq$ as claimed. ∎

Final Remarks

We believe that Problems 1-3, stated in the introduction, deserve more consideration and further ideas must be incorporated to deal with the case of (k,n,m)-divisibility when $n \neq 0$.

We close this paper with two more questions.

Question 2. *Let n, m be positive integers. Does there exist a constant $c = c_m$ such that if G is an m-connected graph, then G contains a $(1,n,m)$-divisible subgraph H s.t. $|E(H)| \geq |E(G)| - c|G|$?*

Question 3. *Let G be a connected graph on $n \equiv 1 \pmod{2}$ vertices.*

Is it true that G contains a $(1,1,2)$-divisible subgraph H s.t. $|E(H)| \geq |E(G)| - \frac{3}{2}|G| + c$ (for some $c \geq 0$).

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