The Weight Bound and Subgeometric Packings

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1. Introduction.

Suppose that we have a set of v elements and that we select a family F made up of k-element subsets (these are usually called blocks). Let the cardinality of the family F be b. Suppose further that we demand that no pair of elements from the v-set be repeated among the blocks of F and that the number b be as large as possible. Then we say that we have a packing design of type (2,k,v). The cardinality b is usually denoted by D(2,k,v).

To distinguish this situation from other related problems, we often speak of this problem as the classical packing problem. This sort of packing is of interest in its own right, but is also of use in many areas of combinatorics, such as coding theory.

As an illustration, we readily see that D(2,4,9) = 3, since we can take sets 1234, 1567, and 2589, from 9 elements, but no other 4-sets (this is not a very effective packing, since it includes only half of the pairs from 9 elements, but it is the best that we can do). Similarly, we find that D(2,3,6) = 4, since we can take four triples 123, 145, 246, and 356, but no others. In this second case, the left-over pairs are 16, 25, 34, and they form a graph known as the *defect graph* of the packing. In the special case that the defect graph is null, the packing becomes a Balanced Incomplete Block design with $\lambda = 1$.

Of course, the packing problem, like any other problem involving designs, can be phrased exclusively in graph-theoretic terms: we seek to pack as many as possible disjoint copies of the complete graph K_k into the complete graph K_v (cf., for example, the terminology in [1]). From this graph-theoretic point of view, the packing problem is the problem of minimizing the cardinality of the defect graph.

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2. The Classical Counting Bound.

Suppose that the defect graph is the null graph. Then every pair of elements from the v-set occurs in the family of blocks (k-sets). Also, every element occurs a constant number of times, say r, since counting of pairs involving a fixed element x of frequency r_x gives the equation

$$v - 1 = r_x(k - 1).$$

Thus $r_x = (v - 1)/(k - 1) =$ a constant r.

This particular packing can be called a perfect packing, but was introduced in Statistics by Frank Yates in 1935 as a Balanced Incomplete Block Design. By counting the number of elements appearing in the packing array (that is, the BIBD), one immediately has the result

$$bk = rv.$$

Now suppose that the packing is not perfect, that is, the defect graph is not null. Then each element i has a replication number r_i, and the analogue of the Fisher-Yates result bk = rv is the equation

$$bk = \Sigma r_i \quad (i = 1,2,3, ..., v).$$

Now let R be the maximum of all the replication numbers r_i. Then we can write down the Counting Bound Equation

$$bk \leq Rv.$$

It is clear that $R \leq \lfloor (v - 1)/(k - 1) \rfloor$, and so we have

$$bk \leq v \lfloor (v - 1)/(k - 1) \rfloor.$$

Since b must be an integer, we obtain the classical counting bound

$$b \leq \lfloor v \lfloor (v - 1)/(k - 1) \rfloor /k \rfloor.$$

This bound has been in use for some time (cf. [4]), and evolved independently in coding theory [3]. Basically, for small values of k, the exact value of b is usually equal to this counting bound, or differs from it by only one unit (there is the very occasional exception). See, for example, [12] for k = 3 and [1] for k = 4; the results for k = 5 (still incomplete) are the subject of current studies by Mills and Mullin (see also [2] and [15]).

Until recently, the fact that studies were made only of cases where $k \leq 5$ obscured an important phenomenon, which is somewhat analogous to what happens in quantum theory in Physics. To put matters very roughly, packings behave differently according as they are "large" or "small" packings. The dividing line betwen the two situations occurs at the geometry that exists when $v = k^2 - k + 1$ and $k - 1$ is a prime power.

When such a finite geometry exists, the classical counting bound is exact and gives $D(2,k,v) = k^2 - k + 1 = v$. For values of v greater than $k^2 - k + 1$, the counting bound works very well; for values of v less than $k^2 - k + 1$, the counting bound is extremely poor. So the Counting Bound bears an analogy to Newtonian Mechanics; it works well for "large" situations and very badly for "small" situations. Since the dividing line is at the value $v = k^2 - k + 1$, we refer to the "small" packings as *subgeometric packings.*

Just as one illustration of how poor the counting bound can be, let us look at the value of D(2,6,26). The counting bound gives us

$$D(2,6,26) \leq \lfloor 26 \lfloor 25/5 \rfloor /6 \rfloor = 21.$$

The exact value of D(2,6,26) is equal to 13.

Basically, the reason that the counting bound is so poor in "small" situations is that it uses the value R, the maximum possible value for the occurrence of any element in the packing. But it ignores totally the interaction of the elements upon one another. This is not important for "large" packings, but is a very serious drawback for "small " packings. In a "small" packing, the fact that a few elements occur R (or nearly R) times puts a very severe constraint on the other elements and does not permit these other elements to appear with a frequency that is anywhere near R.

In the next section, we shall describe the weight bound which is excellent for the "small" situations where the counting bound fails to be of use; the weight bound does take into account the interactions of the various elements and considers the design as a whole (it does not allow a few elements to take on high frequencies without considering the effect that such high frequencies have on the other elements). On the other hand, the weight bound is of no use for the "large" situations.

3. The Weight Bound.

The weight bound has recently been developed and employed in a series of papers [6] - [16]. We shall here give a brief summary of some of its principal features.

Take a block B in a packing, and define the *weight* of the block to be the quantity

$$w(B) = b - 1 - \Sigma(r_i - 1),$$

where the summation is over all elements of the block B. It is obvious that $w(B) \geq 0$, since w(B) can be interpreted as the number of blocks in the packing that are disjoint from the block B.

If we add together all the quantities w(B), we obtain the weight of the entire design as

$$w(D) = b(b - 1) - \Sigma r_i(r_i - 1),$$

where the summation is now over all v varieties that occur in the design. Since it is a sum of non-negative quantities, the weight of the design must be non-negative itself.

Use of the weight approach is made possible by a result on balance, given in [5]. This is merely a formalization of the fact that one would expect: For a given value of b, the design of maximum weight has all the r_i as nearly equal as possible. Consequently, if this maximum weight should be negative, the value of b under discussion is immediately ruled out. If this maximum weight is zero, it immediately follows that every block in the design must have weight zero, and this restriction frequently eliminates certain possibilities.

We illustrate how the weight method is used with an example.

Consider D(2,6,26) and suppose, if possible, that a packing in 14 blocks could exist. Then the number of elements in the packing array is

$$bk = 14(6) = 84 = 3(26) + 6 = 4(6) + 20(3).$$

So the packing of maximum weight would contain 6 elements of frequency 4 and 20 elements of frequency 3. Its weight would be

$$w(D) = 14(13) - 6(4)(3) - 20(3)(2) < 0.$$

Consequently, no packing in 14 blocks is possible; so we next consider the possibility of a packing in 13 blocks.

For b = 13, we have

$$bk = 13(6) = 78 = 26(3).$$

So the packing of maximum weight has 26 elements of frequency 3, and has weight

$$w(D) = 13(12) - 26(3)(2) = 0.$$

Thus we see that there could be a packing in 13 blocks, and such a packing is easily achieved by taking the dual of the triple system with parameters (13,26,6,3,1). Hence D(2,6,26) = 13. Indeed, since there are exactly two non-isomorphic triple systems on 13 elements, we have the result that there are exactly two non-isomorphic packing designs of type (2,6,26).

This result is just a special case of a more general result (cf. [8]): If (v,b,r,k,1) is a legitimate set of parameters for a Balanced Incomplete Block Design, then the packing number D(2,r,b) = v if and only if the Balanced Incomplete Block Design exists.

4. Conclusion.

We conclude with a couple of other examples. Consider the case of D(2,7,38) where the counting bound works out to be 32.

We easily find that a packing design of maximum weight in 20 blocks has negative weight. On the other hand, if we try b = 19, we find that the packing design of maximum weight has weight zero and comprises 19 elements of frequency 4 and 19 elements of frequency 3.

As often happens, this information tells us how to construct the packing design; the design is the dual of a pairwise balanced design on 19 elements which appear in 19 blocks of length 4 and 19 blocks of length 3. One such design can be constructed by cycling, modulo 19, on the two initial blocks (1,2,4,9) and (1,5,11).

Our final example is a consideration of D(2,9,69), where the classical counting bound is 61. The weight bound is 46; indeed, the theorem quoted in the last section shows that the packing number is 46 if and only if a Balanced Incomplete Block Design with parameters (46,69,9,6,1) exists. Unfortunately, this is one of the family of Wallis Designs (cf. [7]), a family about which relatively little is known.

However, if we take the finite geometry PG(2,8) on 73 points and delete 4 points, no three collinear, together with all lines that contain these 4 specific points, we are left with a packing on 69 points that contains

$$73 - 9 - 8 - 7 - 6 = 43$$

blocks. Consequently, we can make the statement that

$$43 \leq D(2,9,69) \leq 46.$$

REFERENCES

[1] A.E. Brouwer, *Optimal Packings of K_4's into a K_n*, J. of Combinatorial Theory, Series A , **26(3)** (1979), 278-297.

[2] B. Gardner, *Some Small Packing Numbers*, Ars Combinatoria **31** (1991), 255-258.

[3] S.M. Johnson, *A New Upper Bound for Error-correcting Codes*, IRE Transactions on Information Theory **IT-8** (1962), 203-207.

[4] J. Schönheim, *On Maximal Systems of k-tuples*, Studia Scientiarum Math. Hungarica **1** (1966), 363-368.

[5] R.G. Stanton, *Two Lemmas on Balance*, Bulletin of the ICA **2** (1991), 87-88.

[6] R.G. Stanton, *Some Interesting Sextuple Coverings*, J. of Combinatorial Mathematics and Combinatorial Computing **9** (1991), 149-153.

[7] R.G. Stanton, *On Wallis BIBDs*, Congressus Numerantium **83** (1991), 193-200.

[8] R.G. Stanton, *Some General Considerations on Packings*, Utilitas Mathematica **40** (1991), 219-230.

[9] R.G. Stanton, *Packings in Enneads*, to appear, Australasian J. of Combinatorics **6** (1992).

[10] R.G. Stanton and J.L. Allston, *The Weight Method for Small Packings*, to appear, Congressus Numerantium **86** (1992).

[11] R.G. Stanton and R.W. Buskens, *Excess Graphs and Bicoverings*, Australasian J. of Combinatorics **1** (1990), 207-210.

[12] R.G. Stanton and M.J. Rogers, *Packings and Coverings by Triples*, Ars Combinatoria **13** (1982), 61-69.

[13] R.G. Stanton, D.M.F. Stone, and E.A. Ruet d'Auteuil, *Some Sextuple Coverings with Normal Points*, Utilitas Mathematica **39** (1991), 119-128.

[14] R.G. Stanton, D.M.F. Stone, and E.A. Ruet d'Auteuil, *Packings in Septuples*, J. of Combinatorial Mathematics and Combinatorial Computing **10** (1991), 97-104.

[15] R.G. Stanton, D.M.F. Stone, and E.A. Ruet d'Auteuil, *On Small Packing Numbers*, Bulletin of the ICA **3** (1991), 57-64.

[16] R.G. Stanton, D.M.F. Stone, and E.A. Ruet d'Auteuil, *A Survey of the Small Sextuple Covers*, Bulletin of the ICA **2** (1991), 59-66.