# Constructing Hadamard Matrices with Orthogonal Pairs[1]

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**Abstract.** A procedure based on the Kronecker product yields $\pm 1$-matrices $X, Y$ of order $8mn$, satisfying $XX^t + YY^t = 8mnI$ and $XY^t = YX^t = 0$, given Hadamard matrices of orders $4m$ and $4n$. This allows the construction of some infinite classes of Hadamard matrices - and in particular orders $8mnp$, for values of $p$ including (for $j \geq 0$) $5, 9^j, 25 \cdot 9^j$, improving the usual Kronecker product construction by at least a factor of 2. A related construction gives Hadamard matrices in orders $4 \cdot 5^i \cdot 9^j$, $0 \leq i \leq 4$. To this end we introduce some disjoint weighing matrices and exploit certain Williamson matrices studied by Turyn and Xia. Some new constructions are given for symmetric and skew weighing matrices, resolving the case of skew $W(N, 16)$ for $N = 30, 34, 38$.

## 1 Introduction

We take the definition of the Kronecker product of matrices $A = (a_{ij})$ and $B$ to be, in block form,

$$A \bigotimes B = (a_{ij}B). \tag{1}$$

The following properties of $\otimes$ will be used without explicit mention throughout the paper:

- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $(A \otimes B)^t = A^t \otimes B^t$
- $(A \otimes B) + \lambda(A \otimes C) = A \otimes (B + \lambda C)$
- $(A \otimes B) + \lambda(C \otimes B) = (A + \lambda C) \otimes B$
- if $A$ is $m \times k$ and $B$ is $n \times 1$ then $A \otimes B$ is $mn \times kl$
- $rank(A \otimes B) = rank(A) rank(B)$,

for all matrices $A, B, C$ and any scalar $\lambda$, whenever the required matrix operations are defined.

As early as 1867, Sylvester [3] noted that the Kronecker product of what are now known as Hadamard matrices is again an Hadamard matrix. An $n \times n$ $\pm 1$-matrix is called an *Hadamard matrix* if it satisfies $HH^t = nI$. These are known to exist only in orders 1, 2 or multiples of 4. Since Seberry [2] has given an asymptotic result which shows Hadamard matrices exist in orders $2^k p$ for any fixed $p$, with $k \geq \lfloor 2 \log_2 (p-3) \rfloor$, the question of existence of Hadamard matrices is concerned primarily with orders $2^k p$ for $p$ odd and $k$ small. Therefore, we have a (loosely

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defined) notion of the comparative value of constructions for Hadamard matrices based on the number of factors of 2 introduced, all other things being equal. While the Kronecker product is of great value in the replication of Hadamard matrices (in fact, is instrumental in the aforementioned result of Seberry), it has this shortcoming, that every nontrivial direct application of it introduces an extra factor of $2^2 = 4$. Here is a way to improve on this by a factor of two (first noted by Agaian [1], whose demonstration of it was considerably more involved):

**Theorem 1.** *Given Hadamard matrices of orders* $4m$ *and* $4n$ *there is an Hadamard matrix of order* $8mn$.

Proof: Write the given matrices in block format as $2 \times 1$ arrays of $2m \times 4m$ (respectively, $2n \times 4n$) submatrices as follows:

$$H = \left(\frac{H_1}{H_2}\right), K = \left(\frac{K_1}{K_2}\right). \tag{2}$$

The required Hadamard matrix is given by

$$U = \frac{1}{2}\left[(H_1 + H_2)^t \bigotimes K_1 + (H_1 - H_2)^t \bigotimes K_2\right]. \tag{3}$$

∎

In the verification of this result, we take note that the matrix $U$ constructed indeed has the right dimensions, and that $\frac{1}{2}(H_1 + H_2)$ and $\frac{1}{2}(H_1 - H_2)$ are $(0, \pm 1)$-matrices which are complementary in the sense that each is zero precisely in those positions in which the other is nonzero. Moreover, $H_1^t H_1 + H_2^t H_2 = H^t H = 4mI$. The rest follows from the properties of the Kronecker product.

In what follows we shall use the more general fact that if a matrix $A$ is partitioned into blocks of rows $A_1, A_2, \ldots A_N$ then $A_1^t A_1 + A_2^t A_2 + \cdots + A_N^t A_N = A^t A$.

## 2 Orthogonal pairs

**Definition 1.** *An* orthogonal pair *(of* $\pm 1$*-matrices) of order* $k$ *is a pair of* $k \times k$ $\pm 1$*-matrices* $(X, Y)$ *satisfying*

$$XY^t = 0, \tag{4}$$

$$XX^t + YY^t = 2kI. \tag{5}$$

It follows immediately that $YX^t = 0$. Thus $(X + Y)(X^t + Y) = (X - Y)(X^t - Y^t) = 2kI$ and so $(X + Y)^t(X + Y) = (X - Y)^t(X - Y) = 2kI$. Comparing terms, we obtain equations like those in the definition, except that the order of multiplication is reversed. Now (4) tells us that the rowspace of $Y$ is contained in the orthogonal complement of the rowspace of $X$. Thus, $rank(X) + rank(Y) \leq k$. But $X + Y$ is nonsingular, and so $rank(X) + rank(Y) \geq k$. Multiplying the second equation by $XX^t$ gives $(XX^t)^2 = 2kXX^t$, and similarly for $YY^t$. Summarizing, we have the following basic information about orthogonal pairs:

**Lemma 2.** *An orthogonal pair $(X,Y)$ of order $k$ satisfies:*

1. $XY^t = YX^t = X^tY = Y^tX = 0$,
2. $XX^t + YY^t = X^tX + Y^tY = 2kI$,
3. *each of* $\begin{pmatrix} \pm X & \pm Y \\ \pm Y & \pm X \end{pmatrix}$ *is an Hadamard matrix,*
4. $rank(X) + rank(Y) = k$,
5. *both $XX^t$ and $YY^t$ have minimal polynomial $x^2 - 2k$.*

Part 3 of this lemma and the following basic theorem for the construction of orthogonal pairs demonstrate the useful connection between orthogonal pairs and Hadamard matrices.

**Theorem 3.** *Given Hadamard matrices of orders $4m$ and $4n$, there is an orthogonal pair of order $4mn$.*

Proof: Let $H$ and $K$ (as in Theorem 1) be partitioned into $m \times 4m$ and $n \times 4n$ submatrices as follows:

$$H = \begin{pmatrix} \underline{H_1} \\ \underline{H_2} \\ \underline{H_3} \\ H_4 \end{pmatrix}, K = \begin{pmatrix} \underline{K_1} \\ \underline{K_2} \\ \underline{K_3} \\ K_4 \end{pmatrix}. \tag{6}$$

Set

$$X = \frac{1}{2}[(H_1 + H_2)^t \bigotimes K_1 + (H_1 - H_2)^t \bigotimes K_2], \tag{7}$$

$$Y = \frac{1}{2}[(H_3 + H_4)^t \bigotimes K_3 + (H_3 - H_4)^t \bigotimes K_4] \tag{8}$$

and observe that the axioms for an orthogonal pair of order $4mn$ are satisfied. ▮

Theorem 1 follows as an easy corollary. At this point we should note that while the order of an orthogonal pair must be even, it is not necessarily divisible by 4. Consider, for example, the pair

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - \\ - & 1 \end{pmatrix}$$

of order 2 (by convention, "$-$" is an abbreviation for "$-1$"). The construction of orthogonal pairs of orders divisible by higher powers of 2 is a trivial matter, considering that if $(X,Y)$ is an orthogonal pair and $H$ is an Hadamard matrix, $(X \otimes H, Y \otimes H)$ is an orthogonal pair.

Here is an intriguing aside, in the light of part 4 of lemma 2: all constructions the author is aware of produce only orthogonal pairs consisting of two matrices of equal rank. This is not necessitated by conditions (4) and (5) alone; we may take, for example, $X = 0$ and $Y$ an Hadamard matrix. Does the additional (combinatorial) condition that $X$ and $Y$ are $\pm 1$-matrices force them to have the same rank?

## 3 Disjoint $W(2p,p)$'s

**Definition 2.** *A square $(0,\pm 1)$-matrix $A$ is a $W(n,w)$, or* weighing matrix of order $n$ with weight $w$, *if $AA^t = wI_n$.*

**Definition 3.** *Matrices $A = (a_{ij})$ and $B = (b_{ij})$ are* disjoint *if $a_{ij}b_{ij} = 0$ for all $i,j$.*

**Theorem 4.** *Let $p$ be a positive integer, $(X,Y)$ an orthogonal pair of order $k$, and $A$ and $B$ disjoint $W(2p,p)$'s. Then $A \otimes X + B \otimes Y$ is an Hadamard matrix of order $2kp$.*

**Corollary 5.** *If there are Hadamard matrices of orders $4m$, $4n$, and two disjoint $W(2p,p)$'s, then there is an Hadamard matrix of order $8mnp$.*

Iterating this result we have:

**Corollary 6.** *If there are Hadamard matrices of orders $4n_1, \ldots, 4n_t$ and a pair of disjoint $W(2p_i,p_i)$'s for each of $p_1, \ldots, p_{t-1}$, then there exists an Hadamard matrix of order $2^{t+1} n_1 \ldots n_t p_1 \ldots p_{t-1}$.*

As for the existence of disjoint $W(2p,p)$'s, there are many such cases. For example, whenever we have an orthogonal design $OD(4t; 2t, 2t)$ we can take $p = 2t$. Weighing matrices $A, B$ obtained in this way will satisfy $AB^t + BA^t = 0$. Conversely, any two such weighing matrices determine the corresponding orthogonal design. These are known for many orders [2]. However, we do not require the strong properties of orthogonal designs. Here is another way to produce an infinite family of these matrices:

**Theorem 7.** *$(X,Y)$ is an orthogonal pair of order $2p$ iff $A = \frac{1}{2}(X+Y)$, $B = \frac{1}{2}(X-Y)$ is a pair of disjoint $W(2p,p)$'s satisfying $AB^t = BA^t$ (Note: we can also write $X = A+B$, $Y = A-B$).*

In light of our comments at the end of the introduction, the case of even $p$ is not as interesting as that of odd $p$, for which we have the following restriction:

**Theorem 8 (Raghavarao, [2]).** *If $n \equiv 2 \mod 4$ and a $W(n,w)$ exists then $w$ is a sum of two squares.*

**Corollary 9.** *Disjoint $W(2p,p)$'s cannot exist for $p \equiv 3 \mod 4$.*

**Corollary 10.** *Orthogonal pairs can only exist in orders $n \equiv 0,2,4 \mod 8$.*

Corollary 9 appears to be the only general restriction on the existence of disjoint $W(2p,p)$'s; while there does not seem to be an orthogonal pair in order 10, and there certainly is no $OD(10;5,5)$, there is a pair of disjoint $W(10,5)$'s:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & - & - & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & 0 & 0 & - & - \\ 0 & 1 & 0 & 0 & - & - & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & - \\ 0 & 0 & 1 & - & 0 & 0 & 1 & 0 & - & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & - \\ 0 & 0 & 1 & 1 & 1 & - & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & - & 1 & 0 & 0 & - & 0 \\ 0 & 1 & 1 & - & 0 & 0 & - & 1 & 0 & 0 \\ 1 & 0 & - & 1 & 0 & 0 & - & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & - & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & 0 \\ 1 & - & 0 & - & 0 & 0 & 0 & 0 & 1 & - \\ 1 & - & 1 & 0 & 0 & 0 & 0 & 0 & - & 1 \end{pmatrix}. \tag{9}$$

A more suggestive way to construct such matrices is to let

$$A = \begin{pmatrix} A_1 & -A_2 \\ A_2^t & A_1^t \end{pmatrix}, B = \begin{pmatrix} B_1 & -B_2 \\ B_2^t & B_1^t \end{pmatrix},$$

where $A_1, A_2, B_1, B_2$ are circulants with first rows $(1,1,-,0,0)$, $(0,1,0,0,1)$, $(0,0,0,1,1)$ and $(1,0,1,-,0)$ respectively.

Turyn [4] considered Williamson matrices $A,B,C,D$ of order $p$ satisfying $AB + CD = AC + BD = AD + BC = 0$, showing them to exist in orders $9^j$, $j \geq 0$. We weaken this slightly to include any quadruple of $\pm 1$-matrices satisfying:

$$\begin{aligned} &i) && AA^t + BB^t + CC^t + DD^t = 4pI \\ &ii) && AB^t - BA^t = AC^t - CA^t = BD^t - DB^t = CD^t - DC^t = 0 \\ &iii) && AD^t + BC^t = 0. \end{aligned} \tag{10}$$

This is slightly weaker than the conditions required for the class "$W_1$" of Williamson matrices in [7], where they are shown to exist in order $25 \cdot 9^j$, $j \geq 0$.

**Theorem 11.** *If $A,B,C,D$ satisfy (10) then*

$$X = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}, Y = \begin{pmatrix} D & -C \\ B & A \end{pmatrix} \tag{11}$$

*is an orthogonal pair of order $2p$.*

Combining this result with theorem 7, we have:

**Corollary 12.** *For $0 \leq i \leq 4$, $j \geq 0$, an Hadamard matrix of order $4 \cdot 5^i \cdot 9^j$ exists, and if Hadamard matrices of orders $4m$ and $4n$ exist, there also exists an Hadamard matrix of order $8mnp$, with $p = 5, 9^j$, or $25 \cdot 9^j$.*

**Corollary 13.** *Matrices satisfying (10) cannot exist in orders $\equiv 3 \mod 4$.*

This last nonexistence result does not seem to be known to either Turyn or Xia. Moreover, if we require the matrices $A, B, C, D$ in [4] (class "$W_2$" in [7]), to be group matrices (as does Turyn), they are regular with row and column sums $a, b, c, d$ respectively. Multiplying the defining relations by the matrix of all ones, we obtain $ab + cd = ac + bd = ad + bc = 0$, $a^2 + b^2 + c^2 + d^2 = 4p$, whence it follows easily that $p$ is a square. Given the evidence at hand it does not seem unreasonable to guess that this must hold in general for matrices satisfying (10).

**Theorem 14.** *If $H$ is an Hadamard matrix of order $2n$, $H_i$, as in Theorem 1, and $A, B$ are disjoint $W(2p, p)$'s partitioned similarly, then*

$$U = \frac{1}{2}\left[(H_1 + H_2)^t \bigotimes A_1 + (H_1 - H_2)^t \bigotimes A_2\right] \tag{12}$$

$$V = \frac{1}{2}\left[(H_1 + H_2)^t \bigotimes B_1 + (H_1 - H_2)^t \bigotimes B_2\right] \tag{13}$$

*is a pair of disjoint $W(2np, np)$'s. Moreover, $UV^t = \pm VU^t$ according as $AB^t = \pm BA^t$.*

## 4 Special types of orthogonal pairs

The defining relations of orthogonal pairs place them somewhere between Hadamard matrices and Williamson matrices. In general, we may consider sets of $q$ $\pm 1$-matrices of order $p$ satisfying $A_1 A_1^t + \cdots + A_q A_q^t = pqI_p$, $A_i A_j^t = A_j A_i^t$. Such matrices lend themselves nicely to similar constructions to ours, utilizing orthogonal designs, and many such constructions have been tried. These are generally based on standard types, such as circulants, symmetric matrices, or those in a "normal form" such as that for Hadamard matrices in which the first row and column have all entries equal to 1.

If $P$ and $Q$ are monomial matrices of the same order as an orthogonal pair $(X, Y)$, it is easy to verify that $(PXQ, PYQ)$ is also an orthogonal pair. Thus we see that we may "normalize" the pair by making all the entries in the first row and column of one of the matrices equal to 1, or those in the first row of one and the first column of the other, but not simultaneously the first row or column of both, for this would contradict (4).

A second special type of orthogonal pair derives its inspiration from the $2 \times 2$ orthogonal pair, $(X, X^t)$, where $X = \begin{pmatrix} 1 & 1 \\ - & - \end{pmatrix}$. We shall say that $X$ is *transpose orthogonal* if $(X, X^t)$ is an orthogonal pair.

**Theorem 15.** *If $H$ is an Hadamard matrix and $X$ is transpose orthogonal then $X \otimes H$ is transpose orthogonal.*

Presently, we shall see another construction for transpose orthogonal matrices, a bit more interesting than this one in that it produces matrice of orders not divisible by 8. However, let us first consider the following variation on the Kronecker product:

**Definition 4.** *Take $A_i, B_j$ to be, respectively, the $i$th row of $A$ and the $j$th column of $B$. Then we write, in block form,*

$$A \oslash B = (B_j A_i). \tag{14}$$

The reader may verify that all the basic properties of $\otimes$ listed in the introduction hold for $\oslash$ except the first two, for which we may substitute:

- $(A \oslash B)(C \oslash D) = AD \otimes BC$
  when these are defined, and
- $(A \oslash B)^t = B^t \oslash A^t$.

**Theorem 16.** *If $H$ is an Hadamard matrix of order $4n$ with $H_i, i = 1, \ldots, 4$ as in Theorem 3 then $X = \frac{1}{2}[(H_1 + H_2)^t \oslash H_3 + (H_1 - H_2) \oslash H_4]$ is transpose orthogonal of order $4n^2$.*

Transpose orthogonal matrices may be applied in the construction of skew weighing matrices as we now illustrate. Let $A, B, C$ be, respectively, the matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & - \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & - \\ 1 & 1 & 0 & 0 & - & 1 \\ 1 & - & 1 & 1 & 0 & 0 \\ - & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & - & 1 & 1 \\ 0 & 0 & - & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{pmatrix}.$$

Then choosing $W$ to be one of

$$\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ B & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & C \\ B & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & C \\ C & 0 & 0 \end{pmatrix},$$

we make $W$ equal to any of a $W(15,4)$, a $W(17,4)$, or a $W(19,4)$ which is, in each case, disjoint from $W^t$.

**Theorem 17.** *If $W, W^t$ are disjoint $W(n, w)$'s and $X$ is transpose orthogonal of order $k$, then $U = W \otimes X \pm W^t \otimes X^t$ is a symmetric (in the case of "+") or* *skew (in the case of "$-$") $W(nk, 2wk)$. In particular, skew $W(N, 16)$ exist for* $N = 30, 34, 38$.

These orders are mentioned specifically because in spite of having small order they are listed as unresolved in [2]. The second one gives some hope that disjoint $W(34, 17)$'s exist, strengthening theorem 4, since in this case $U + I$ is a $W(34, 17)$, which is also listed as unresolved.

A third special type of orthogonal pair occurs when the two matrices are symmetric. We call such a pair a *symmetric orthogonal pair*. An example of a symmetric orthogonal pair is the one of order 2 given in section 2.

**Theorem 18.** *Under the hypotheses of theorem 16,* $X = \frac{1}{2}[(H_1 + H_2)^t \oslash H_1 + (H_1 - H_2) \oslash H_2], Y = \frac{1}{2}[(H_3 + H_4)^t \oslash H_3 + (H_3 - H_4) \oslash H_4]$ *is a symmetric orthogonal pair of order* $4n^2$.

Unfortunately, while this makes it easy to construct symmetric Hadamard matrices of order $8n^2$, there is no way of simultaneously guaranteeing a constant diagonal [6], which would be of interest to graph theorists, since a symmetric Hadamard matrix with constant diagonal corresponds to a special class of graphs [5] (however, these conditions are satisfied by the matrix $H^t \oslash H$ of order $16n^2$). On the other hand, the diagonal is well-behaved in that $tr(X) = tr(Y) = 0$. We also have an analogue of theorem 17:

**Theorem 19.** *If $V, W$ are disjoint symmetric (respectively skew) $W(n, w)$'s and $(X, Y)$ is a symmetric orthogonal pair of order $k$, then $X \otimes V + Y \otimes W$ is a symmetric (respectively skew) $W(nk, wk)$.*

## References


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