A characterization of trees in which no edge is essential to the domination number

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Abstract. For any tree T let $\gamma(T)$ represent the size of a minimum dominating set. Let E_0 represent the collection of trees with the property that, regardless of the choice of edge e belonging to the tree T, $\gamma(T-e) = \gamma(T)$. We present a constructive characterization of E_0.

Introduction.

We begin with some definitions and terminology. A set S of vertices of a graph G is a *dominating set* if every vertex of G is either in S or adjacent to a member of S. Let $\gamma(G)$ be the cardinality of a dominating set of minimum size. We shall say D is a *γ-set of* G if D is a dominating set and $|D| = \gamma(G)$.

As we are interested in the effect of removing edges it will be useful to call an edge *essential* if $\gamma(G-e) > \gamma(G)$ and not essential otherwise, that is, if $\gamma(G-e) = \gamma(G)$. E_0 is the collection of trees T in which no edge is essential. For instance, the paths of order 4 and 7 as well as the graph F_m, $m \geq 2$, in Figure 1 are in E_0.

In [3] Fink, Jacobson, Kinch, and Roberts, introduced the *bondage number* of a graph as being the cardinality of a smallest set of edges whose removal from G results in a graph with domination number greater than $\gamma(G)$. They proved that any tree has bondage number 1 or 2, and pose as an open problem classifying trees of bondage number 2. E_0 is the collection of trees that have bondage number 2.

Other authors have considered similar problems related to adding or deleting edges or vertices in graphs and investigating the effect on various graphical parameters. See [1], [2], and [6]. In [5], Haynes, Lawson, Brigham and Dutton study minimum and maximum degree, maximum clique size and vertex and edge independence under adding or deleting a single edge. In [4] Gunther, Hartnell, and Rall give a constructive characterization of the two classes of trees whose independence number is unchanged under single edge addition or deletion.

The notation P_n will represent a path on n vertices. We shall use the expression *attach a* P_2, say $[x, y]$ to a vertex, say v, in a tree T to refer to the operation of joining the vertices x and v by an edge. Similarly, to *attach a* P_3, say $[x, y, z]$, to v refers to joining x and v by an edge.

An important tree in the work that follows will be called F_m. This tree can be viewed as being formed from m copies of P_4, each of the form $[t_i, y_i, s_i, u_i]$, and a single vertex, say w, which is joined to each y_i (see Figure 1).

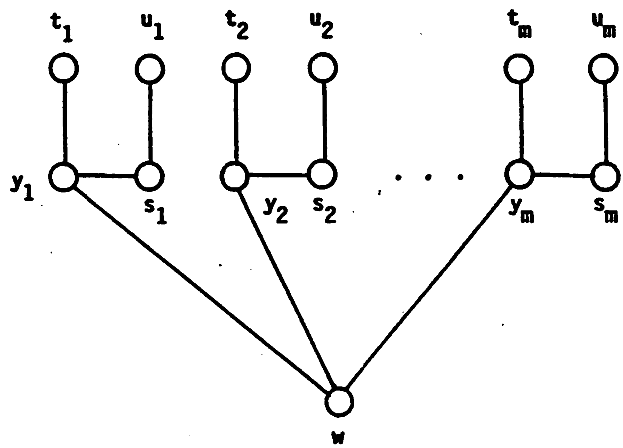


Figure 1

When we say that we *attach* F_m to a vertex v in a tree T we mean that v and w are joined by an edge.

A vertex v of a tree T will be called a *level* vertex of T if $\gamma(T - v) = \gamma(T)$ and a *down* vertex if $\gamma(T - v) < \gamma(T)$.

As we shall demonstrate how to build larger trees in E_0 from existing ones we need to consider the following 4 operations on a tree T:

type (1): Attach a P_2 to T at v where v is a level vertex of T belonging to at least one γ-set of T.

type (2): Attach a P_3 to T at v where v is a down vertex of T.

type (3): Attach F_1 to T at v where v belongs to at least one γ-set of T.

type (4): Attach F_m, $m \geq 2$, to T at v, where v can be any vertex of T.

Finally, let $C = \{T \mid T \text{ is a tree and } T = K_1, T = P_4, T = F_m, \text{ for some } m \geq 2,$ or T can be obtained from P_4 or F_m ($m \geq 2$) by a finite sequence of operations of type (1), (2), (3), or(4)$\}$.

It will be shown that E_0 and C are identical.

Preliminary results.

To understand the structure of a tree from E_0 it is necessary to know something about the effect of vertex removal on the domination number. Although in general the removal of a vertex can result in the domination number increasing, in our first lemma we prove that if a tree T has no essential edges, then each vertex of T is either a level vertex or a down vertex. The first statement of the following result is Corollary 3.1 of [3]. Its proof is included here for completeness.

Lemma 1. *If T is a tree in E_0, then no vertex of T can be adjacent to more than one leaf. Furthermore, for any vertex v of T, $\gamma(T-v) \leq \gamma(T)$.*

Proof: Let T be a tree in E_0. Assume w is a vertex in T with two leaves, say w_1 and w_2, attached. Without loss of generality, we may assume that w belongs to a γ-set of T. Clearly, there is a γ-set of $T - ww_1$, which includes both w and w_1, and hence, $\gamma(T-ww_1) > \gamma(T)$, a contradiction.

If some vertex v of T has the property that $\gamma(T-v) > \gamma(T)$, then v must belong to every γ-set of T. Let D be any γ-set of T. Suppose $N(v) = \{v_1, v_2, \ldots, v_k\}$. In the forest $T - vv_i$ let T_i represent the component which contains v_i, and let S_i represent the component which contains v.

If $\gamma(T_j) \leq |D \cap V(T_j)|$ for every j, $1 \leq j \leq k$, then choose a γ-set M_j of T_j. $\cup_{j=1}^{k} M_j$ is a γ-set for $T-v$ and yet $|\cup_{j=1}^{k} M_j| \leq \sum_{j=1}^{k} |D \cap V(T_j)| = \gamma(T) - 1$, a contradiction. Therefore, fix i with $1 \leq i \leq k$ and so that $\gamma(T_i) > |D \cap V(T_i)|$. (In fact, $\gamma(T_i) = |D \cap V(T_i)| + 1$.) Since T belongs to E_0, $\gamma(T) = \gamma(T - vv_i)$ and it then follows that $\gamma(S_i) < |D \cap V(S_i)|$. That is, there is a γ-set D_1 of S_i with cardinality less than $|D \cap V(S_i)|$. This implies that D_1 does not include v. But if we consider a γ-set, say D_2, of T_i, we then can form a γ-set of T, namely, $D_1 \cup D_2$, which does not contain v, a contradiction. Hence, for every vertex v of a tree T in E_0 it is the case that $\gamma(T-v) \leq \gamma(T)$. ∎

Lemma 2. *If T is any tree in E_0 and v is a vertex of T with a P_2 attached at v, then v is a level vertex of T. Furthermore, there is a γ-set for T which includes v.*

Proof: Let $\gamma(T) = n$ where T is a tree in E_0 and v is a vertex of T with a P_2, say $[x, y]$, attached. Suppose v is a down vertex. Then $\gamma(T-v) = n-1$. Let D be a γ-set for $T - v$. Assume without loss of generality that $x \in D$. But then D is a dominating set for T of size $\gamma(T) - 1$, a contradiction. Therefore, such a vertex v must be a level vertex of T.

Also v belongs to some γ-set of T. This can be seen by taking a γ-set, F, for $T - xy$. We may assume $v \in F$ (since either x or v must be in F). Now $|F| = \gamma(T - xy) = \gamma(T)$ since $T \in E_0$. Therefore, F is a γ-set of T and $v \in F$. ∎

Lemma 3. *If T belongs to E_0 and a vertex u of T has a P_3 attached, then u is a down vertex of the tree T' obtained from T by removing the P_3.*

Proof: Denote the P_3 attached at u by $[v, w, x]$. Let $\gamma(T) = n$. Then $\gamma(T') = n-1$. Let $e = xw$. Since $T \in E_0$, $\gamma(T-e) = n$. Let D be a γ-set for $T-e$. $x \in D$ and we may assume that $v \in D$. Then $|D \cap V(T')| = n-2$. If $u \in D \cap V(T')$ then $D \cap V(T')$ is a dominating set for T' of cardinality $n-2$, a contradiction.

This implies $u \notin D \cap V(T')$. Therefore, $D \cap V(T') \subseteq V(T'-u)$ and $D \cap V(T')$ is a dominating set of $T'-u$. Then $\gamma(T'-u) \leq |D \cap V(T')| = n-2 = \gamma(T')-1$ and so u is a down vertex of T'. ∎

The next four lemmas will be needed in our characterization of trees in E_0. Collectively they show that the class of trees in E_0 is closed under the four operations specified in the Introduction.

Lemma 4. *If T belongs to E_0 and S is the tree obtained from T by a type (1) operation, then S belongs to E_0.*

Proof: Suppose $T \in E_0$ and v is a level vertex of T which belongs to at least one γ-set of T. Let S be the tree obtained from T by attaching a P_2, say $[x, y]$ at v. Since v is a level vertex of T, $\gamma(S) = \gamma(T) + 1$. If $e \in E(T)$ and D is a γ-set for $T - e$, then $D \cup \{x\}$ is a dominating set for $S - e$. This implies that $\gamma(S-e) \leq |D \cup \{x\}| = |D| + 1 = \gamma(T-e) + 1 = \gamma(T) + 1 = \gamma(S)$. Therefore, e is not essential in S.

Let D' be a γ-set for T with $v \in D'$. $|D'| = \gamma(T)$. $D'' = D' \cup \{y\}$ is a dominating set for $S - e$ for $e = vx$ or $e = xy$. Thus, $\gamma(S - e) \leq |D''| = |D'| +1 = \gamma(T) + 1 = \gamma(S)$. Hence, neither $e = vx$ nor $e = xy$ is essential in S. Therefore, S belongs to E_0. ∎

Lemma 5. *If T belongs to E_0 and S is the tree obtained from T by a type (2) operation, then S belongs to E_0.*

Proof: Let T be a tree in E_0 and v be a down vertex of T. Let S be the tree obtained from T by attaching a P_3, say $[x, y, z]$, to T at v. First we note that $\gamma(S) = \gamma(T) + 1$. Let D be a γ-set of $T - v$. Since v is a down vertex of T, $|D| = \gamma(T) - 1$.

If $e = vx$ or $e = xy$, let $D' = D \cup \{v, y\}$. If $e = yz$, let $D' = D \cup \{x, z\}$. In each of these cases D' dominates $S - e$ and so $\gamma(S-e) \leq |D'| = |D| + 2 = \gamma(T) + 1 = \gamma(S)$. On the other hand, if $e \in E(T)$, let D be a γ-set for $T - e$. Now $|D| = \gamma(T-e) = \gamma(T)$ since $T \in E_0$. Let $D' = D \cup \{y\}$. As D' is a dominating set for $S - e$ we have $\gamma(S - e) \leq |D'| = |D| + 1 = \gamma(T) + 1 = \gamma(S)$.

Hence, for all $e \in E(S)$, $\gamma(S - e) = \gamma(S)$, and so S belongs to E_0. ∎

Lemma 6. *If T belongs to E_0 and S is the tree obtained from T by a type (3) operation, then S belongs to E_0.*

Proof: Let T be a tree in E_0 and v be a vertex of T which is in at least one γ-set of T. Let S be the tree obtained from T by attaching F_1, say $[w, t, y, s, u]$ at v, where t and u are of degree one and y is of degree three with neighbors t, s, and w. Observe that $\gamma(S) = \gamma(T) + 2$.

If $e \in E(T)$, then let D be a γ-set of $T - e$ and $D' = D \cup \{y, s\}$. We have $|D'| = |D| + 2 = \gamma(T - e) + 2 = \gamma(T) + 2 = \gamma(S)$. But D' dominates $S - e$ so $\gamma(S - e) = \gamma(S)$ for the case $e \in E(T)$.

If $e \in \{vw, wy, ys\}$, then let D be a γ-set of T which includes v, and let $D' = D \cup \{y, s\}$. Certainly $|D'| = |D| + 2 = \gamma(T) + 2 = \gamma(S)$ and D' dominates $S - e$. If $e = yt$ choose $D' = D \cup \{t, s\}$ and if $e = su$ select $D' = D \cup \{u, y\}$, where again D is a γ-set of T which includes v. As before D' dominates $S - e$ and $|D'| = |D| + 2 = \gamma(S)$.

Therefore, for all $e \in E(S)$, $\gamma(S - e) = \gamma(S)$ and, hence, S belongs to E_0. ∎

Finally, we show that a type (4) operation maintains a tree's membership in E_0.

Lemma 7. *If T belongs to E_0 and S is the tree obtained from T by a type (4) operation, then S belongs to E_0.*

Proof: Let T be a tree in E_0 and v be an arbitrary vertex of T. Let S be the tree obtained from T by attaching F_m, for $m \geq 2$, say $[w, t_1, y_1, s_1, u_1, \ldots, t_m, y_m, s_m, u_m]$ at v. First observe that $\gamma(S) = \gamma(T) + 2m$. Consider any edge $e \in E(S)$.

If $e \in E(T)$, let D be a γ-set for $T - e$ and let $D' = D \cup \{y_1, s_1, y_2, s_2, \ldots, y_m, s_m\}$. Since D' dominates $S - e$ we have $\gamma(S - e) \leq \gamma(T) + 2m = \gamma(S)$.

If $e \in E(S) - E(T)$, let D be a γ-set for T and $D_1 = \{y_1, s_1, y_2, s_2, \ldots, y_m, s_m\}$. If $e = vw$, $e = wy_i$ or $e = y_j s_j$, for some i or j, let $D' = D \cup D_1$. If $e = s_i u_i$, for some i, let $D_2 = (D_1 - \{s_i\}) \cup \{u_i\}$ and let $D' = D \cup D_2$. If $e = y_i t_i$, for some i, let $D_3 = (D_1 - \{y_i\}) \cup \{t_i\}$ and let $D' = D \cup D_3$. In each case D' dominates $S - e$ which implies that $\gamma(S - e) = \gamma(S)$.

Hence, no edge e of S is essential in S, and so S belongs to E_0. ∎

The characterization.

We are now ready to establish our main result.

Theorem 1. *The family E_0 is contained in C.*

Proof: It is straightforward to verify that every tree of order 9 or less which belongs to E_0 is also in C. Assume that if T is a tree in E_0 on fewer than n (where n is an integer and at least 10) vertices, then T belongs to C.

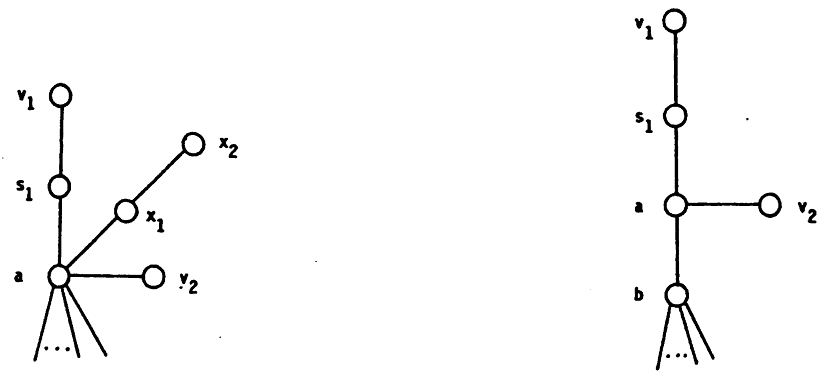
Now let T be a tree of order n where $T \in E_0$. Choose a leaf, say v_1, at the end of a longest path in T. Let s_1 be the unique neighbor of v_1 in T. Since s_1 cannot have 2 leaves as neighbors by Lemma 1, it follows that $\deg_T(s_1) = 2$.

Let v_1, s_1, a and b be the four vertices at the end of this longest path where a is adjacent to s_1 and b. Exactly one of the following two cases must hold.

(I) a has a leaf as a neighbor, or
(II) a is not adjacent to a leaf.

We consider these separately as follows:

(I) Suppose a is adjacent to a leaf v_2.



(a) (b)

Figure 2

(a) In addition assume a has at least one other P_2 (other than $[s_1, v_1]$) attached (see Figure (2(a)).

Let $T' = T - \{v_1, s_1\}$. Thus, $\gamma(T') = \gamma(T) - 1$. First we show that $T' \in E_0$. Let D be a γ-set for T'. Hence, $|D| = \gamma(T) - 1$ and we may assume without loss of generality that $x_1 \in D$ and that $a \in D$.

If $e = ax_1$, then D is a dominating set for $T' - e$ which implies that $\gamma(T' - e) = \gamma(T')$.

If $e = x_1x_2$, then $D' = (D - \{x_1\}) \cup \{x_2\}$ is a dominating set for $T' - e$. Again $\gamma(T' - e) = \gamma(T')$.

If $e \in E(T')$, $e \neq ax_1$, $e \neq x_1x_2$, then let D be a γ-set for $T - e$. Now $|D| = \gamma(T)$, since $T \in E_0$. Assume without loss of generality that $s_1 \in D$, $x_1 \in D$. But then $D' = D - \{s_1\}$ is a dominating set for $T' - e$ implying $\gamma(T' - e) \leq |D'| = \gamma(T) - 1 = \gamma(T')$ which means $\gamma(T' - e) = \gamma(T')$.

Thus, for all $e \in E(T')$, $\gamma(T' - e) = \gamma(T')$ and so $T' \in E_0$.

By induction, $T' \in C$. By Lemma 1 vertex a is either level or down. Since a has a leaf as a neighbor, it follows that a is a level vertex and must also belong to at least one γ-set of T'. Hence, T can be obtained from T' by a single Type (1) operation, thus, implying T belongs to C.

(I)(b) Assume a has no other P_2 (other than $[s_1, v_1]$) attached and, hence, $\deg_T(a) = 3$ (see Figure 2(b)).

(I)(b)(1) Assume b is adjacent to a leaf v_3 (see Figure 3(a)).

Let $T' = T - \{v_1, s_1\}$. Thus, $\gamma(T') = \gamma(T) - 1$.

Let $e \in E(T')$. Since a and b each have a leaf as a neighbor, we can choose a γ-set, say D, for $T - e$ in such a way that at least one of a and b belongs to D. Now $\gamma(T - e) = \gamma(T)$ so $|D| = \gamma(T)$. Note that a is dominated by $D' = D - (D \cap \{s_1, v_1\})$. Therefore, D' is a dominating set for $T' - e$ and it follows that $\gamma(T' - e) = \gamma(T')$.

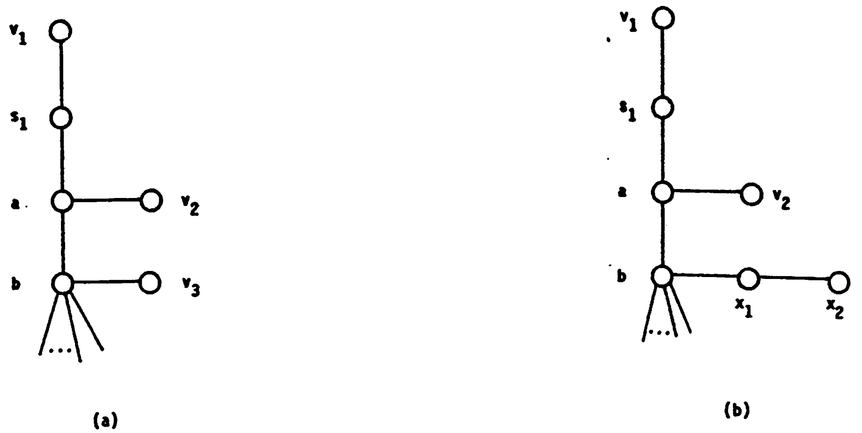


Figure 3

Thus, $T' \in E_0$ and so by the induction hypothesis, $T' \in C$. Since vertex a is adjacent to a leaf and $T' \in E_0$, it follows that a is level and belongs to at least one γ-set of T'. Hence, T can be obtained from T' by a single Type (1) operation and it follows that $T \in C$.

(I)(b)(2) Assume b is not adjacent to a leaf but has a P_2, say $[x_1, x_2]$, attached (see Figure 3(b)).

Let $T' = T - \{v_1, s_1, a, v_2\}$. Hence, $\gamma(T') = \gamma(T) - 2$. If $e \in E(T')$ but $e \neq bx_1$ and $e \neq x_1x_2$, then we can select a γ-set D for $T - e$ in such a way that $x_1 \in D$. But $D' = D - (D \cap \{v_1, s_1, a, v_2\}) = D \cap V(T')$ is a dominating set for $T' - e$. This implies $\gamma(T' - e) = \gamma(T')$.

Suppose $e = bx_1$ or $e = x_1x_2$. Let D be a γ-set for $T - x_1x_2$. Assume without loss of generality that $b \in D$ and $x_2 \in D$. Then $D' = D \cap V(T')$ is a dominating set for $T' - e$. Since $T \in E_0$, $|D| = \gamma(T) \Rightarrow |D'| = \gamma(T) - 2$ and so $\gamma(T' - e) = \gamma(T')$.

Therefore, $T' \in E_0$. By the induction hypothesis $T' \in C$. By Lemma 2, b is a level vertex in T' and belongs to some γ-set for T'.

Thus, we can obtain T from T' by 2 applications of a Type (1) operation. That is, first attach P_2 at b and then attach P_2 at a. Hence, $T \in C$.

(I)(b)(3) b is not adjacent to a leaf of T, b has no P_2 attached but $\deg_T(b) \geq 3$.

Since v_1 was chosen to be the end of a longest path in T, we need only consider the following three subcases.

Subcase (i)

Suppose b has a neighbor, say x, of degree $r + 1$ $(r \geq 1)$, which has r P_2's attached, say $[y_1, z_1], [y_2, z_2], \ldots, [y_r, z_r]$ (see Figure 4).

But this is a forbidden structure in a tree in E_0, for if $1 \le i \le r$ then any γ-set for T must contain one of y_i, z_i. Therefore, if D is a γ-set for T, we may assume that $s_1, a, y_1, y_2, \ldots, y_r \in D$ and $|D \cap \{v_1, s_1, a, v_2, x, y_1, \ldots, y_r, z_1, \ldots, z_r\}| = r + 2$.

However, if $e = y_1 z_1$ then in $T-e$, if D' is a γ-set for $T-e$, $|D' \cap \{v_1, s_1, a, v_2, x, y_1, \ldots, y_r, z_1, \ldots, z_r\}| = r+3$, a contradiction. [If D' could be chosen so that $|D' \cap V(T - \{v_1, s_1, a, v_2, x, y_1, \ldots, y_r, z_1, \ldots, z_r\})| \le \gamma(T) - (r+3)$, then one could construct a dominating set for T of cardinality strictly less than $\gamma(T)$.]

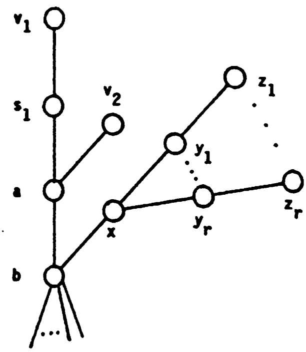


Figure 4

Subcase (ii)

Suppose b has a neighbor, say x, of degree $r+2$ $(r \ge 2)$, which has r P_2's, say $[y_1, z_1], \ldots, [y_r, z_r]$ as well as a leaf, say w, attached (see Figure 5(a)).

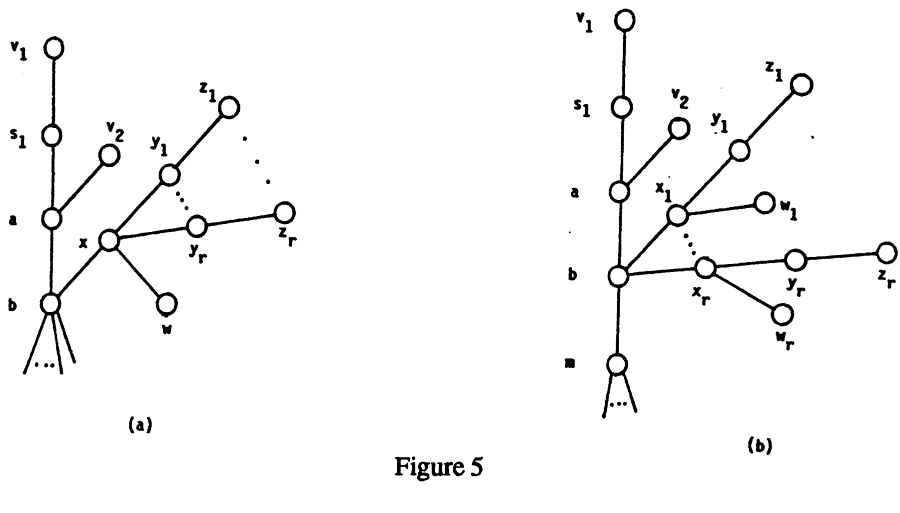


Figure 5

But this is case (I)(a) with the role of a in (I)(a) taken by x here. Hence, within this case we are left with only the following possibility.

Subcase (iii)

Suppose b is of degree $r + 2$ $(r \geq 1)$, and b has r neighbors, other than a of degree 3, say $x_1, \dots, x_r$, where each x_i has a leaf, say w_i, and a P_2, say $[y_i, z_i]$, attached (see Figure 5(b)).

Let $T' = T - \{v_1, s_1, a, v_2, b, x_1, \dots, x_r, y_1, \dots, y_r, w_1, \dots, w_r, z_1, \dots, z_r\}$. Now $\gamma(T') = \gamma(T) - 2(r + 1)$. First we show that $T' \in E_0$. Let $e \in E(T')$ and let D be a γ-set for $T - e$. One may assume that $\{s_1, a, x_1, \dots, x_r, y_1, \dots, y_r\} \subseteq D$, $b \notin D$, and that $|D \cap V(T')| = \gamma(T) - 2(r + 1)$. (Note: we can replace b by m in D, if necessary.) But then $D' = D \cap V(T')$ is a dominating set for $T' - e$ which implies that $\gamma(T' - e) = \gamma(T')$. Therefore, $T' \in E_0$.

Now by the induction hypothesis $T' \in C$. But then T is obtained from T' by a single application of a type (4) operation which implies that $T' \in C$. This completes (I)(b)(3).

(I)(b)(4) $\deg_T(b) = 2$. That is, we have the structure illustrated in Figure 6.

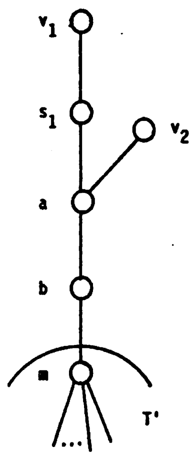


Figure 6

Let $T' = T - \{a, s_1, v_1, v_2, b\}$. Clearly $\gamma(T') = \gamma(T) - 2$. Let $e = ab$. Since $T \in E_0$ there exists a γ-set D for $T - e$ with cardinality $\gamma(T - e) = \gamma(T)$. Since b is a leaf in $T - e$, either $b \in D$ or $m \in D$. Assume without loss of generality that $m \in D$. But D is also a γ-set for T. Let $D' = D \cap V(T')$. Then D' is a γ-set for T' and $m \in D'$. Thus, m belongs to a γ-set for T'.

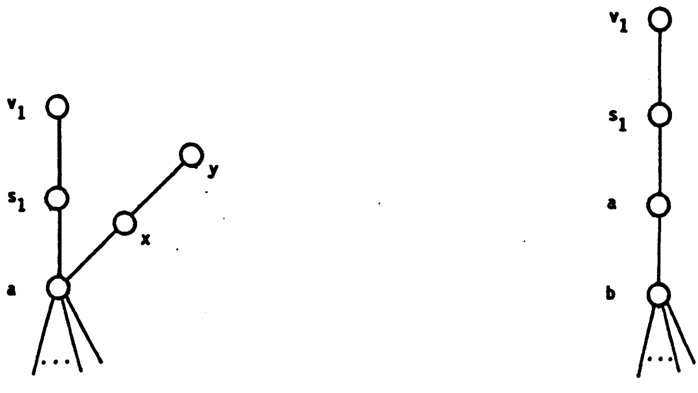
Let $e \in E(T')$ and let D be a γ-set for $T - e$. Since $T \in E_0$, $|D| = \gamma(T - e) = \gamma(T)$. We may assume without loss of generality that $s_1 \in D$, $a \in D$, and

$b \notin D$ (if necessary replace b in D by m). Then $D' = D \cap V(T')$ is a dominating set for $T' - e$ implying that $\gamma(T' - e) = \gamma(T')$. Therefore, $T' \in E_0$.

By the induction hypothesis $T' \in C$ so since T is obtained from T' by a single application of a type (3) operation, $T \in C$. This completes case (I) where a has a leaf as a neighbor.

(II) Suppose a has *no* leaf as a neighbor.

(a) $\deg_T(a) \geq 3$. Then a must have a P_2 say $[x, y]$, attached (see Figure 7(a)).



(a) (b)

Figure 7

By Lemma 2, a is a level vertex of T. Let $T' = T - \{v_1, s_1\}$. Then $\gamma(T') = \gamma(T) - 1$. First we show that $T' \in E_0$. Let D be a γ-set for $T - xy$. $T \in E_0$ so $|D| = \gamma(T)$. We may assume that $s_1 \in D$, $y \in D$ and $a \in D$. Now consider $e \in E(T')$.

If $e = ax$ or $e = xy$, then $D' = D - \{s_1\}$ is a dominating set for $T' - e$ implying $\gamma(T' - e) = \gamma(T')$ so e is not essential in T'. A similar argument holds for any other e belonging to a P_2 attached at a. Suppose e is any other edge in T' (that is, one not on a P_2 attached at a). If M is a γ-set for $T - e$, then $M' = M \cap V(T')$ is a dominating set for $T' - e$. Hence, $\gamma(T' - e) = \gamma(T')$ so e is not essential in T'. Therefore, $T' \in E_0$. By the induction hypothesis, $T' \in C$. By Lemma 2, a belongs to some γ-set for T'. Since T can be obtained from T' by a single application of a type (1) operation, T belongs to C.

(II)(b) $\deg_T(a) = 2$. Then we have the P_3, namely $[a, s_1, v_1]$, attached at b (see Figure 7(b)).

Let $T' = T - \{v_1, s_1, a\}$. Thus, $\gamma(T') = \gamma(T) - 1$. Let $e \in E(T')$ and let D be a γ-set for $T - e$. Since $T \in E_0$, $|D| = \gamma(T)$. We may assume without loss of generality that $s_1 \in D$ and that $a \notin D$ (for if $a \in D$ replace it in D by

b). Thus, T' is dominated by $D' = D - \{s_1\} = D \cap V(T')$, and $|D'| = |D| - 1$, so D' is a dominating set for $T' - e$. Therefore, $\gamma(T' - e) \leq |D'| = |D| - 1 = \gamma(T) - 1 = \gamma(T')$. Hence, $\gamma(T' - e) = \gamma(T')$, for all $e \in E(T')$. Thus, $T' \in E_0$. By the induction hypothesis $T' \in C$. By Lemma 3, b is a down vertex in T' and so since T can be obtained from T' by a single type (2) operation, $T \in C$.

Thus, all cases are handled and so $T \in C$. Therefore, by induction, it now follows that $E_0 \subseteq C$. ∎

We can now state the theorem characterizing trees in E_0.

Theorem 2. *A tree T belongs to E_0 if and only if T belongs to C.*

Proof: By Theorem 1, if T belongs to E_0, then T must belong to C. But K_1 as well as P_4 and F_m, $m \geq 2$, are clearly in E_0. Using Lemmas 4, 5, 6, and 7 it follows that any tree in C is also a member of E_0 . Therefore, $E_0 = C$. ∎

Two related questions.

Having characterized those trees in which no edge is essential to the domination number it is natural to ask which graphs have the property that every edge is essential. It is not surprising that the stars are the only connected graphs satisfying this condition.

Theorem 3. *Let G be a connected graph. $\gamma(G - e) > \gamma(G)$ for every edge e of G if and only if G is a star.*

Proof: Certainly if G is a star, then G has the required property. Conversely, if every edge of G is domination essential then G must be a tree since every connected graph has a spanning tree with the same domination number. Let D be a γ-set of G and let $x \in D$. No neighbor of x, say y, could belong to D or else the edge xy would be inessential. Furthermore, all neighbors of x must be leaves, for if some neighbor y is not a leaf and is adjacent to a vertex $z \neq x$, then the edge yz is inessential. Hence, x must be the center of a star. ∎

We conclude by exhibiting a family of graphs with the property that no matter which set of k edges is deleted, the domination number does not increase. In particular, let H be a graph on $2p$ vertices where $V(H)$ can be partitioned into two sets L and S of equal cardinality such that each vertex of S has degree at least $k + 1$, each vertex of L is a leaf and each vertex of S is adjacent to exactly one vertex of L. Observe that $\gamma(H) = p$ and that necessarily $p \geq k+1$. If $F \subseteq E(H)$ and $|F| \leq k$, then no vertex of S is isolated in the graph $H - F$. Thus, a γ-set D for $H - F$ can be chosen as follows. For each $x \in S$, if F contains the edge xy where $y \in L \cap N(x)$ let $\{y\} = D \cap \{x, y\}$. Otherwise, let $\{x\} = D \cap \{x, y\}$. $|D| = p$ and D dominates $H - F$ so $\gamma(H - F) = \gamma(H)$.

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