Long Paths and the Cycle Space of a Graph

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Abstract. If each pair of vertices in a graph G is connected by a long path, then the cycle space of G has a basis consisting of long cycles. We propose a conjecture regarding the above relationship. A few results supporting the conjecture are given.

Introduction

Our notation will follow that of Bondy and Murty [2]. The *length* of a path or cycle is the cardinality of its edge set. For vertices x and y, an (x, y)*-path* is a path from x to y. For k a positive integer, an $(x, y\colon k)$*-path* is an (x, y)-path of length at least k. A graph G is *k-path-connected* if for every pair of distinct vertices u, v in G there is a $(u, v\colon k)$-path in G. If P is a path, the segment of P from x to y is denoted $P[x, y]$. The distance from x to y in G, $d_G(x, y)$, is the length of a shortest (x, y)-path.

The *cycle space*, $Z(G)$, of a graph G is the vector space of edge sets of eulerian subgraphs of G. The subspace of the cycle space generated by the set of cycles of length at least k will be denoted $Z_k(G)$. A graph G is *k-generated* if $Z(G) = Z_k(G)$. A 2-connected graph is a *k-generator* if it is both k-generated and $(k-1)$-path-connected. In [5] it is proven that any 2-connected graph which contains a k-generator must be a k-generator.

For graphs G and H, the *join* $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$.

Bondy [3] made a conjecture which we state as:

Conjecture 1. *Let G be a 3-connected graph with minimum degree at least d and with at least $2d$ vertices. Then every cycle of G can be written as the symmetric difference of an odd number of cycles, each of whose lengths are at least $2d - 1$.*

This was partially proven in [5] as:

Theorem 1. *Let G be a 3-connected graph with minimum degree at least d. Suppose further that G is non-hamiltonian or G has at least $4d - 5$ vertices. Then G is a $(2d - 1)$-generator.*

When producing [5], no attempt at proving that a representation as a sum of an odd number of cycles was possible. We quickly sketch here how one might take this proven result and extend it.

A *theta-graph* is a 2-connected graph with two vertices of degree three and the remaining vertices of degree two. It is easily established that if H is a 2-connected graph, but H is not a cycle, then H has a theta-subgraph. In particular,

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any 3-connected graph G has a theta-subgraph H. If Bondy's conjecture is true, then each of the three cycles in H is representable as the sum of an odd number of cycles of length at least $2d-1$, and then the sum of all three of these collections contains every edge of H exactly twice, and is thus the empty subgraph.

On the other hand, if the empty subgraph is the sum of an odd number of such cycles, then to prove Bondy's conjecture, one needs only determine whether or not each cycle can be written as a sum of cycles of length at least $2d-1$, without regard to the parity of the set, since one can always add in an odd set of cycles that sums to the empty graph.

We are now left with determining whether or not a given graph admits an odd set of long cycles summing to the empty graph. We thus conjecture that:

Conjecture 2. *Let G be a 3-connected graph with minimum degree at least d, and with at least $2d$ vertices. Then, there is a an odd set of cycles in G, each of which has length at least $2d-1$, and whose symmetric difference is 0.*

Partial Proof: If G is a non-hamiltonian 2-connected graph with minimum degree at least d, and thus with at least $2d$ vertices, then there is an odd set of cycles whose symmetric difference is 0. A basic step in the proof given in [5], is to take a longest cycle, and to find a $(u,v{:}\,k)$-path P in $G-V(C)$ with

(i) $u',v' \in V(C)$, $uu',vv' \in E(G)$, $u' \neq v'$, and
(ii) $k+|N(u)\cap V(C)| \geq d$.

Now let $x_0 = v'$, and let $\{x_1,x_2,\ldots,x_m\}$ be the vertices of $N(u)\cap V(C)-\{x_0\}$ labelled cyclically around C from x_0. Set

$$C_i = C[x_{i+1},x_i]x_iux_{i+1} \quad \text{for} \quad i=1,2,\ldots,m-1,$$
$$C_0 = C[x_1,x_0]x_0vPux_1 \quad \text{and} \quad C_m = C[x_0,x_m]x_muPvx_0.$$

Since, C is a longest cycle, it is not difficult to check that each C_i has length at least $2d$. If m is even, then $\sum C_i = 0$, while if m is odd, $C+\sum C_i = 0$. In each case, we have an odd set of cycles summing to 0. We note also that, if G is a graph with connectivity 2, mimimum degree d and $2d$ vertices, then G is hamiltonian, but has no odd set of cycles summing to 0. Let $\{u,v\}$ be a cutset of two vertices in G, and let G_1 be one of the components of $G-\{u,v\}$. It is immediate that every odd set of cycles, each cycle of which has length at least $2d-1$, must use an odd number of edges from u to G_1. Therefore, one could not easily strengthen Conjecture 2 to include 2-connected graphs.

There is another possible use of theta-subgraphs in extending the results of [5]. An (a,b,c)*-theta-graph* is a theta-graph in which the three paths P, Q and R connecting the two vertices u and v of degree three are of lengths at least a, b, and c, respectively. A $(d-1,d,d)$-theta-graph H is $(2d-1)$-generated, and if $x \in V(P)-\{u,v\}$ and $y \in V(Q)-\{u,v\}$, then there is an $(x,y{:}\,2d-2)$-path in H. Thus, while H is not a $(2d-1)$-generator, it comes reasonably close.

One would expect that a 3-connected graph which contains a $(d-1, d, d)$-theta-subgraph would be a $(2d-1)$-generator. Then, to extend the result in [5] to graphs on at least $3d$ vertices, one would need only prove the following conjecture, since any hamiltonian graph G has a theta-subgraph with $|V(G)| + 1$ edges. ∎

Conjecture 3. *Let G be a 3-connected graph with minimum degree at least d. Suppose that G contains a theta-subgraph with at least $3d+1$ edges, then G contains a $(d-1, d, d)$-theta-subgraph.* ∎

We note a result, similar to [5], by Hartman [4] for 2-connected graphs.

Theorem 2. *Let G be a 2-connected graph with minimum degree at least d. Then G is $(d+1)$-generated, unless d is odd and $G \cong K_{d+1}$.*

Theorem 2 is also proven in [6]. The proof of [6] can easily be modified to allow for a degree sum condition rather than a minimum degree condition. Hartman [4] also notes that a 2-connected graph with chromatic number k must be $(k-1)$-generated.

The proofs given in [1], [5] and [6] are, in part, based on the fact that the classes of graphs being considered are both $(k+1)$-generated and k-path-connected for some k. In fact, if a class of $(k+1)$-generated graphs is closed under the addition of edges to graphs in the class, then each graph in the class is k-path-connected.

In these notes we consider an aspect of the relationship between the properties k-path-connected and ℓ-generated.

Let $\mathcal{G}_k$ denote the class of k-generated graphs and let $\mathcal{P}_k$ denote the class of k-path-connected graphs. It is immediate from these definitions that $\mathcal{P}_{k+1} \subseteq \mathcal{P}_k$ and $\mathcal{G}_{k+1} \subseteq \mathcal{G}_k$.

Let $f(k) = \max\{m \mid \mathcal{P}_k \subseteq \mathcal{G}_m\}, k \geq 2$.

Conjecture 4. *For $k \geq 2$, $f(k) \geq k - g(k)$, where* $\lim_{k\to\infty} \frac{g(k)}{k} = 0$. ∎

Example 1. $K_{2k} \in \mathcal{P}_{2k-1} - \mathcal{G}_{2k}$, for $k \geq 2$. ∎

Example 2. Let P and Q be disjoint paths of length j, $j \geq 2$, and let G be the graph obtained from $P \cup Q$ by adding the four edges from the ends of P to the ends of Q. Then, $G \in \mathcal{P}_{j+2} - \mathcal{G}_{j+3}$. ∎

One could also describe the graph of Example 2 as a subdivision of K_4, in which the edges of a matching have been replaced by paths of length j. There are only two cycles of length at least $j+3$, but the cycle space has dimension 3.

Example 3. Let D denote the graph of the dodecahedron. Then $D \in \mathcal{P}_{18} - \mathcal{P}_{19}$ and $D \in \mathcal{G}_{17} - \mathcal{G}_{18}$. Therefore, $f(18) \leq 17$. ∎

A simple application of the Kozyrev-Grinberg test (see, for example, [2, §9.7] or [7]) shows that $D - v$ has no Hamilton cycle for any vertex v of D. Thus D has no 19-cycles and $Z_{18}(D) \neq Z(D)$. It is easy to represent each 5-cycle as the sum of a 17-cycle and a Hamilton cycle. Also, for any two vertices, u and v, there is a $(u, v\colon 19)$-path if $d(u,v) \neq 2$, and a $(u, v\colon 18)$-path if $d(u,v) = 2$.

In order to justify Conjecture 4 somewhat, we now look at some general properties and then complete the proof that $\mathcal{P}_k \subseteq \mathcal{G}_k$, for a few small values of k. It may be that a slightly stronger statement than Conjecture 4 is possible. Perhaps some simple degree, connectivity, forbidden subgraph, or embedding condition can be found which is such that $G \in \mathcal{P}_k - \mathcal{G}_{k+1}$ only if G satisfies that condition. Examples 1, 2 and 3 indicate that such conditions might be difficult to determine, unless the condition itself implies $G \in \mathcal{P}_k \cap \mathcal{G}_{k+1}$. Even if one could find a suitable side condition, the result would not be quite as appealing as the simplicity of Conjecture 4 as stated above. We might also note that Conjecture 4 already seems sufficiently difficult. Lemma 4 shows exactly what can be achieved for the class $\mathcal{P}_3$.

Lemma 1 demonstrates that there is some relationship between the two families of classes, $\{\mathcal{P}_k\}$ and $\{\mathcal{G}_\ell\}$.

Lemma 1. *$\mathcal{P}_{(k-1)^2+1} \subseteq \mathcal{G}_{k+1}$ and any 2-edge-connected graph in $\mathcal{G}_{2k-1}$ is in $\mathcal{P}_k$.*

Proof: First, suppose that G is a 2-edge-connected graph in $\mathcal{G}_{2k-1}$. Let x and y be vertices of G. If x and y are in different 2-connected components of G, let R be any (x, y)-path, and let y' be the first vertex of $R - x$ which is a cut-vertex of G. For any $(x, y'\colon k)$-path S, the path $S \cup R[y', y]$ will be an $(x, y\colon k)$-path. Therefore, we may restrict our attention to 2-connected graphs.

Let C be any cycle of length at least $2k - 1$ in G. There are two paths P and Q from x and y, respectively, to C, meeting C at distinct vertices x' and y'. Let C' be the segment of C from x' to y' with length at least k. Then $P \cup C' \cup Q$ is an $(x, y\colon k)$-path. We note that the condition that G be 2-edge-connected is necessary since any tree is in $\mathcal{G}_k$, for any k.

Now suppose that $H \in \mathcal{P}_{(k-1)^2+1}$. Let C be a cycle of length at most k in H. Let x and y be vertices of C. There is an $(x, y\colon (k-1)^2 + 1)$-path P in G. The vertices of C decompose P into segments, at least one of which must have length at least k. Let Q be such a segment, and let C' and C'' be the segments of C between the end vertices of Q. Then

$$C = C'Q + C''Q \in Z_{k+1}(G).$$

∎

Corollary. $f(k) \geq 1 + \sqrt{k}$. ∎

We now prove some lemmas which may be useful for the first few cases of Conjecture 4.

Lemma 2. *If every 2-connected k-path-connected graph is in $\mathcal{G}_m$, then every k-path-connected graph is in $\mathcal{G}_m$.*

Proof: Simply examine the 2-connected components of the graph. Every 2-connected component of a k-path-connected graph is k-path-connected. ∎

Lemma 3. *Let C be any cycle in a 2-connected graph G. Suppose that there is a vertex w at distance at least j from C. Then C is the sum of two cycles of length at least $2j+1$.*

Proof: Let P and Q be paths from w to C, disjoint except at w. Let P_1 and P_2 be the sections of C between the end vertices of P and Q. Then each of $C_1 = P \cup Q \cup P_1$ and $C_2 = P \cup Q \cup P_2$ has length at least $2j+1$. Therefore,

$$C = C_1 + C_2 \in Z_{2j+1}(G).$$

∎

Lemma 4. *If G is a 2-connected graph in $\mathcal{P}_3$, then $G \in \mathcal{G}_4$ or $G \cong K_4$.*

Proof: We may assume that G has at least five vertices. Let C be a cycle of length three in G. By Lemma 3, we may assume that every vertex of G is at distance at most one from C. Let $V(C) = \{x, y, z\}$.

Since G has at least five vertices, we may pick two vertices, v and w, not in C, but adjacent to C. We may also assume that all neighbours of v and w are on C, or we could form a 4-cycle with only one edge in common with C. One could then proceed as in Lemma 3.

Thus, v and w have at least two neighbours each on C. Let us suppose that, say, z has no neighbours not on C. Then, $G = K_2 \vee mK_1$, and there is no (x, y)-path of length at least three.

Thus, without loss of generality, $vx, vy, wy, wz \in E(G)$, and then

$$C = xvywzx + xvyzx + xywzx \in Z_4(G).$$

Since any 3-cycle can be written as the sum of cycles of length at least 4, $G \in \mathcal{G}_4$. ∎

Lemma 5. *If $G \in \mathcal{P}_{2k}$, for $k \geq 2$, then each triangle of G is in $Z_{k+2}(G)$.*

Proof: Let $C = xyzx$ be a triangle in G. Let P be an (x, z)-path of length at least $2k$.

If $y \notin V(P)$, then

$$C = Pzx + Pzyx \in Z_{2k+1}(G).$$

If y is on P, we may assume that y is on the first half of P, and thus that $P[y, z]$ has length at least k. If the length of $P[y, z]$ is k, then

$$C = P[x, y]yzx + Pzx + xyP[y, z]zx \in Z_{k+2}(G).$$

If y is the second vertex on P, then

$$C = zxyP[y, z] + zyP[y, z] \in Z_{2k}(G).$$

In the remaining cases,

$$C = xyP[y, z]zx + P[y, z]zx \in Z_{k+2}(G).$$

∎

Lemma 6. $\mathcal{P}_4 \subseteq \mathcal{G}_4$.

Proof: Direct application of Lemma 5. ∎

Lemma 7. $\mathcal{P}_4 \not\subseteq \mathcal{G}_5$.

Proof: Direct application of Example 2. ∎

Lemma 8. $\mathcal{P}_5 \subseteq \mathcal{P}_4 \subseteq \mathcal{G}_4$.

Proof: Direct application of definitions and Lemma 6. ∎

Lemma 9. $\mathcal{P}_5 \subseteq \mathcal{G}_5$.

Proof: Let $G \in \mathcal{P}_5$. By Lemma 8, $\mathcal{P}_5 \subseteq \mathcal{P}_4 \subseteq \mathcal{G}_4$. Therefore, we need only consider 4-cycles in G. Let $C = wxyzw$ be a 4-cycle in G. We need to show that $C \in Z_5(G)$. We consider two cases, depending on the length of the bridges between opposite vertices of C.

Case (i). Suppose that $d_{G-\{wy\}}(xz) \geq 3$. Let P be a $(w, y{:}\,5)$-path in G. If $\{x, z\} \subseteq V(P)$, then $Q = P[x, z]$ is an $(x, z{:}\,3)$-path. Thus

$$C = Qzyx + Qzwx \in Z_5(G).$$

Hence, we may assume that P meets at most one of x and z. If P meets one of these two vertices, then without loss of generality, assume that $x \in V(P)$ and $z \notin V(P)$. Let $Q = P[w, x]$. If Q is a $(w, x{:}\,4)$-path, then

$$C = Qxw + Qxyzw \in Z_5(G).$$

Therefore, we may assume without loss of generality that each of $Q = P[w, x]$ and $R = P[x, y]$ has length at least two. Thus,

$$C = Qxyzw + wxRyzw + QRyzw \in Z_5(G).$$

We may now assume that neither x nor z is on P. But then

$$C = Pyxw + Pyzw \in Z_7(G) \subseteq Z_5(G).$$

Case (ii). We may thus assume that each of $d_{G-\{x,z\}}(w, y)$ and $d_{G-\{w,y\}}(x, z)$ is at most two. Suppose that there is a vertex $s \in N(x) \cap N(z)$ and a vertex $t \in N(w) \cap N(y)$.

If $s = t$, then

$$C = wsxyzw + xsyzwx + yszwxy + zswxyz \in Z_5(G).$$

If $s \neq t$,then

$$C = xszwtyx + xszytwx \in Z_6(G) \subseteq Z_5(G).$$

Now, suppose that there is a vertex $s \in N(x) \cap N(z)$ and that $y \in N(w)$. Then

$$C = wxszyw + wzsxyw \in Z_5(G).$$

Therefore, we may assume that $x \in N(z), y \in N(w), N(x) \cap N(z) = \{w, y\}$ and $N(y) \cap N(w) = \{x, z\}$. We note that $G[\{w, x, y, z\}] \cong K_4$. Now let P be a $(w, x\colon 5)$-path. We consider the possible intersections of P with C.

If $V(P) \cap V(C) = \{w, x\}$, then

$$C = Pxw + Pxyzw \in Z_6(G) \subseteq Z_5(G).$$

If $|V(P) \cap V(C)| = 3$, then without loss of generality, $V(P) \cap V(C) = \{w, x, y\}$. If $P[w, y]$ has length one, then

$$C = P[y, x]xwzy + P[y, x]xy \in Z_5(G).$$

However, if $P[w, y]$ has length exceeding one, then

$$C = P[w, y]yxzw + P[w, y]yzxw \in Z_5(G).$$

The remaining case is that $|V(P) \cap V(C)| = 4$. The vertices of C occur in the order w, y, z, x or in the order w, z, y, x. Suppose that the order is w, y, z, x. Then $P[w, y]$ and $P[z, x]$ must have length one as in the previous paragraph. Therefore, $P[y, z]$ is a $(y, z\colon 3)$-path avoiding $\{x, w\}$. A similar situation arises if the order of the vertices of C along P is w, z, y, x. In this case, at most one of $P[w, z], P[z, y]$ and $P[y, x]$ can have length greater than one. For example, suppose that $P[w, z]$ and $P[z, y]$ have lengths greater than one. Then

$$C = P[w, z]zyxw + P[z, y]yxwz + P[w, y]yxw \in Z_5(G).$$

The other two possibilities are quite similar.

Therefore, for either order of vertices of C on P, we obtain a path Q between vertices which are consecutive on C, avoiding the remaining vertices of C, and having length at least three. Without loss of generality, let Q be a $(y, z\colon 3)$-path in $G - \{x, w\}$. We may also assume that Q is not a $(y, z\colon 4)$-path. Let $Q = ystz$.

Since $\mathcal{G} \in \mathcal{P}_5$, there is a $(y, z\colon 5)$-path R in G. Since our original choice of P was arbitrary, we may assume that $\{x, w\} \subseteq V(R)$, and that some subpath S of R is a path between vertices which are consecutive on C, avoiding the remaining vertices of C, and having length at least three. Note, also that because S is a subpath of R, we know that S cannot be a (y, z)-path. If $V(Q) \cap V(S) \subseteq V(C)$, then C is the sum of three cycles of lengths at least six. Thus, without loss of generality, we may assume that $s \in V(S)$ and that the neighbours of s along S are not y and t.

Let T be a subpath of S from s to $u \in \{x, z\}$ with $T \cap \{s, t, w, x, y, z\} = \{s, u\}$. If $x = u$, then

$$C = Tstzwx + Tsyzwx + xystzwx \in Z_5(G).$$

If $w = u$, then

$$C = Tsyzxw + Tsyxzw \in Z_5(G).$$

Thus, in all cases, we have shown that $C \in Z_5(G)$. ∎

Some Other Examples

(1) 1. The Petersen graph is in $\mathcal{G}_9 - \mathcal{G}_{10}$ and in $\mathcal{P}_8 - \mathcal{P}_9$.

(2) 2. If G is a connected Cayley graph of an abelian group of order n, then from [1] we know:

" (a)" If n is odd or if G is bipartite, then G is n-generated;

" (b)" If G is an odd prism then $\dim(Z_n(G)) = \dim(Z(G)) - 2$;

" (c)" In all other cases, $\dim(Z_n(G)) = \dim(Z(G)) - 1$.

We can examine cases (b) and (c) to determine $\dim(Z_{n-1}(G))$.

If G is an odd prism, $n = 4k + 2$, for some k. There is a unique cycle of length $n - 1$ avoiding any given vertex of G. It is easy to show that these cycles generate the cycle space of G. Therefore, there is a basis for the cycle space of G consisting of $2k$ cycles of length n and two cycles of length $n - 1$. Thus, $G \in \mathcal{G}_{n-1}$.

For case (c), we have that G is not bipartite, has an even number of vertices and is not an odd prism. In this case $Z(G)$ is generated by the Hamilton cycles together with any odd cycle of G. One would need only calculate the length of a longest odd cycle in G. It may be that G has an $(n-1)$-cycle, and that one could prove this by following [1]. Example 3 does not contradict this, but might constitute a warning.

Recapitulation

$\mathcal{P}_k \subseteq \mathcal{G}_k$, for $2 \leq k \leq 5$ and $f(2) = 3$, $f(k) = k$, for $k = 3, 4, 5$.

$G \in \mathcal{P}_3 - \mathcal{G}_4$ only if G has a 2-connected component which is K_4.

The argument for $k = 5$, is rather more complex than the argument for $k \leq 4$. One would tend to expect the arguments to become even more difficult for evaluating $f(6)$. To establish values of $f(k)$ for $k \geq 10$, would probably necessitate new techniques.

Examples 1 and 2 show that $f(k) \leq k$, for $k \geq 3$, and Lemma 1 verifies that $f(k) \geq 1 + \sqrt{k}$, for $k \geq 2$. Since it may be quite difficult to prove Conjecture 4, we offer a weaker version of that conjecture.

Conjecture 5. *For some constant* m, $0 < m < 1$, $f(k) \geq mk$.

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