Enumeration of Hamiltonian Cycles in $P_4 \times P_n$ and $P_5 \times P_n$

Y. H. Harris Kwong
SUNY College at Fredonia
Fredonia, NY 14063

Abstract. A scheme for classifying hamiltonian cycles in $P_m \times P_n$ is introduced. We then derive recurrence relations, exact and asymptotic values for the number of hamiltonian cycles in $P_4 \times P_n$ and $P_5 \times P_n$.

1. Introduction

Let P_n denote the path with n vertices, and let $H_m(n)$ be the number of hamiltonian cycles in the cartesian product $P_m \times P_n$. It is easy to verify that $H_1(n) = 0$; $H_2(n) = 1, n \geq 2$; $H_3(2n+1) = 0$ and $H_3(2n) = 2^{n-1}$. The value of $H_4(n)$ was studied recently in [2]:

Theorem 1. *$H_4(n)$ satisfies the recurrence relation*

$$H_4(n) = 2H_4(n-1) + 2H_4(n-2) - 2H_4(n-3) + H_4(n-4)$$

for $n \geq 4$, with initial values $H_4(0) = H_4(1) = 0, H_4(2) = 1$ and $H_4(3) = 2$.

In [2], the authors discovered a complicated necessary and sufficient condition for a cycle to be hamiltonian in $P_4 \times P_n$. After studying a particular pattern within the hamiltonian cycles, they deduced an explicit formula for $H_4(n)$ in terms of binomial coefficients. Meanwhile, the recurrence relation of $H_4(n)$ was also derived.

In this paper, we list four necessary conditions, and apply them to derive the recurrence relation of $H_4(n)$ directly. We then extend the investigation to $H_5(n)$, whose recurrence relation is obtained via its generating function.

Theorem 2. *$H_5(n)$ satisfies the recurrence relation*

$$H_5(n) = 11H_5(n-2) + 2H_5(n-6) \quad \text{for } n \geq 6,$$

with initial values $H_5(0) = H_5(1) = H_5(3) = H_5(5) = 0, H_5(2) = 1$ and $H_5(4) = 14$.

We also evaluate the exact and asymptotic values of $H_4(n)$ and $H_5(n)$.

2. Preliminaries and Notations

Since $P_m \times P_n$ is isomorphic to $P_n \times P_m$, we may consider the vertex-set of $P_m \times P_n$ as $\{0, 1, \cdots, n-1\} \times \{0, 1, \cdots, m-1\}$ so that $P_m \times P_n$ can be represented graphically as a m-by-n grid in the usual cartesian plane. For instance, Figure 1 contains such a representation of $P_5 \times P_{10}$, with one of its hamiltonian cycles drawn in bold lines.

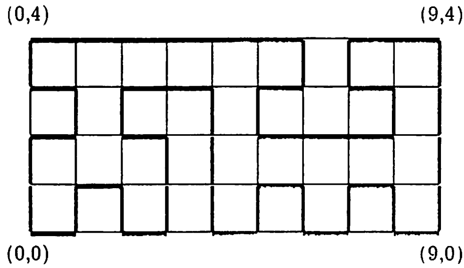

Figure 1. A hamiltonian cycle in $P_5 \times P_{10}$.

It is clear that $H_m(n) = H_n(m)$. Furthermore, we have

Theorem 3. *For* $m, n > 1$, $H_m(n) > 0$ *if and only if* mn *is even.*

Proof: We leave it to the reader to construct a hamiltonian cycle in $P_m \times P_n$ when mn is even. Assume that mn is odd. Define

$$S = \{(i, j) \,|\, i + j \equiv 1 \pmod 2\}.$$

Then $(P_m \times P_n) - S$ consists of $(mn+1)/2$ totally disconnected vertices. Therefore, the number of components in $(P_m \times P_n) - S$ is

$$k((P_m \times P_n) - S) = \frac{mn+1}{2} > \frac{mn-1}{2} = |S|,$$

Thus, $P_m \times P_n$ is not 1-tough, hence it cannot be hamiltonian. [1, p. 219] ∎

To derive a recurrence relation for $H_m(n)$ we have to classify the different shapes a hamiltonian cycle can take. Denote the cell with (i, j) in its upper right corner by C_{ij}. Given any cycle C, we follow [2] by defining $b_{ij} = 1$ if C_{ij} is enclosed within C, and $b_{ij} = 0$ otherwise. These *bit assignments* clearly characterize C, and vice versa. The problem now reduces to finding all bit assignments which induce hamiltonian cycles. For example, Figure 2 displays the bit assignments of the hamiltonian cycle shown in Figure 1.

Note that since every vertex has degree two in any hamiltonian cycle,

$$b_{11} = b_{1,m-1} - b_{n-1,1} = b_{n-1,m-1} = 1.$$

(0,4) (9,4)

1	1	1	1	1	1	0	1	1
0	1	0	0	1	0	0	0	1
1	1	1	0	1	1	1	1	1
1	0	1	0	1	0	1	0	1

(0,0) (9,0)

Figure 2. The bit assignments of a hamiltonian cycle.

To further facilitate our discussion, define the *i-th bit map* of C as

$$x_i = b_{i,m-1}b_{i,m-2}\cdots b_{i2}b_{i1}$$

in binary expansion. Then we can refer to C by its *signature*, which is defined as the 2^{m-1}-ary expansion $x_1x_2\cdots x_{n-1}$. For example, the hamiltonian cycle in Figure 1 has a signature of $BEB8FA3AF$, where we adopt the standard convention of using letters A through F for the hexadecimal digits 10 through 15.

Two cells C_{ij} and C_{hk} are said to be *adjacent* if either (i) $i = h$ and $|j - k| = 1$ or (ii) $|i - h| = 1$ and $j = k$. We call a cell with bit assignment 1 an *1-cell*, and *0-cell* otherwise. We also regard all cells in the x-y plane outside the grid as 0-cells. The following necessary conditions are easy to verify:

(BC) *Boundary Condition:* No adjacent 0-cells can be found on the boundary.

(IC) *Interior Condition:* The configurations, shown in Figure 3, of four cells sharing a common vertex are not allowed:

0	0
0	0

1	1
1	1

0	1
1	0

1	0
0	1

Figure 3. Four forbidden configurations.

(CC) *Connectedness Condition:* There is exactly one contiguous block of adjacent 1-cells.

(EC) *Exterior Condition:* There is exactly one contiguous block of adjacent 0-cells. In other words, any contiguous block of adjacent 0-cells cannot be enclosed entirely by 1-cells: it must have an "outlet" to the exterior.

Note that conditions (CC) and (EC) are equivalent to saying that 1-cells form a simply-connected region. However, we found that it is easier to apply (CC) and (EC) if we leave them as two separate conditions.

3. Value of $H_4(n)$

Proof of Theorem 1: For brevity, denote $H_4(n)$ by $H(n)$. Assume $n \geq 2$. Since $b_{11} = b_{13} = 1$ and $b_{12} \in \{0,1\}$, we have $x_1 \in \{5,7\}$. Let $h_5(n)$ and $h_7(n)$ be the number of hamiltonian cycles in $P_4 \times P_n$ with $x_1 = 5$ and $x_1 = 7$, respectively. Then,

$$H(n) = h_5(n) + h_7(n) \quad \text{for } n \geq 2. \tag{3.1}$$

First consider $x_1 = 5$. It follows from (CC) that $b_{21} = b_{23} = 1$, so $x_2 \in \{5,7\}$. Thus, $x_2 x_3 \cdots x_n$ is the signature of a hamiltonian cycle in $P_4 \times P_{n-1}$. Conversely, given any hamiltonian cycle $x_2 x_3 \cdots x_n$ in $P_4 \times P_{n-1}$, it is clear that $5 x_2 x_3 \cdots x_n$ is hamiltonian in $P_4 \times P_n$. Thus, $h_5(n) = H(n-1)$ for $n \geq 2$. Together with (3.1), we get

$$h_7(n) = H(n) - H(n-1) \quad \text{for } n \geq 2. \tag{3.2}$$

Now assume that $n \geq 3$ and $x_1 = 7$. Due to (IC), there are either one or two 1-cells in the second column, so $x_2 \in \{1,2,4,5\}$. If $x_1 x_2 \in \{71,74\}$, then (CC) and (BC) imply that $b_{31} = 1$ and $b_{33} = 1$, respectively. Thus, $x_3 \in \{5,7\}$ and there are $2H(n-2)$ such hamiltonian cycles. If $x_1 x_2 = 72$, (BC) and (IC) imply that $x_3 = 7$; we have $h_7(n-2)$ hamiltonian cycles in this category.

We are left with the case of $x_1 x_2 = 75$, in which the 0-cell C_{22} needs an outlet to the exterior. Although it is not difficult to derive the number of such hamiltonian cycles directly, the situation is more complex for larger m. Hence, we shall use an alternate approach. Define a *twin cycle* in $P_4 \times P_n$ as a 2-factor $G = 5 x_2 \cdots x_{n-1}$ with two components (that is, a spanning subgraph consisting of exactly two disjoint cycles) such that C_{11} and C_{13} are enclosed in diferent cycles. Topologically, the 1-cells form two disjoint simply-connected regions.

Let $g(n)$ be the number of twin cycles in $P_4 \times P_n$. For instance, $g(1) = 0$ and $g(2) = 1$. Clearly, there are $g(n-1)$ hamiltonian cycles in $P_4 \times P_n$ with $x_1 x_2 = 75$. Thus far, we have obtained

$$h_7(n) = 2H(n-2) + h_7(n-2) + g(n-1) \quad \text{for } n \geq 3. \tag{3.3}$$

In any twin cycle $x_1 x_2 \cdots x_{n-1}$, C_{12} is a 0-cell, which needs an outlet to the exterior. Suppose the first outlet for C_{12} is located at the k-th column. If $k = 2$, then $x_2 \in \{1,4\}$. Similar to the discussion of $x_1 x_2 = 71$, we have $H(n-2)$ such twin cycles. If, however, $k > 2$, then $b_{21} = b_{23} = 1$. From the definition of a twin cycle, G always has $b_{22} = 0$. Therefore, $x_2 = 5$ and there are $g(n-1)$ twin cycles in this case. Hence, $g(n) = g(n-1) + 2H(n-2)$ for $n \geq 3$. It follows immediately that

$$g(n) = 1 + 2\sum_{k=1}^{n-2} H(k) \quad \text{for } n \geq 3. \tag{3.4}$$

Combining with (3.2) and (3.4), we can rewrite (3.3) as

$$H(n) - H(n-1) = 1 + H(n-2) - H(n-3) + 2\sum_{k=1}^{n-2} H(k)$$

for $n \geq 3$. We conclude that for $n \geq 4$,

$$H(n) = 2H(n-1) + 2H(n-2) - 2H(n-3) + H(n-4).$$

▮

To obtain an exact formula of $H_4(n)$, we first have to determine the zeros of the characteristic polynomial $F(x) = x^4 - 2x^3 - 2x^2 + 2x - 1$. Let

$$\mu = \sqrt[3]{\frac{-29 + 3\sqrt{39}}{2}}, \quad \nu = \sqrt[3]{\frac{-29 - 3\sqrt{39}}{2}}, \quad K = \sqrt{\frac{2(\mu + \nu) + 7}{3}},$$

$$G = 4 - (1 + K)^2 \left(1 - \frac{2}{K}\right) \quad \text{and} \quad H = 4 - (1 - K)^2 \left(1 + \frac{2}{k}\right).$$

The zeros of $F(x)$ are, according to Ferrari's formula (see, for example, [3]):

$$\begin{aligned}
\alpha_1 &= ((1 + K) + \sqrt{G})/2 \approx 2.5386, \\
\alpha_2 &= ((1 + K) - \sqrt{G})/2 \approx -1.2762, \\
\alpha_3 &= ((1 - K) + \sqrt{H})/2 \approx 0.3688 + 0.4155i \\
\alpha_4 &= ((1 - K) - \sqrt{H})/2 \approx 0.3688 - 0.4155i
\end{aligned}$$

Since the zeros of $F(x)$ are distinct, we obtain

Theorem 4. *If α_i are the zeros of $F(x) = x^4 - 2x^3 - 2x^2 + 2x - 1$, then*

$$H_4(n) = \sum_{i=1}^{4} \frac{\alpha_i}{F'(\alpha_i)} \alpha_i^n.$$

Proof: It is a routine exercise to show that

$$\sum_{n=0}^{\infty} H_4(n) x^n = \frac{x^2}{1 - 2x - 2x^2 + 2x^3 - x^4} = \sum_{i=1}^{4} \frac{A_i}{1 - \alpha_i x}$$

for some constants A_i, $1 \leq i \leq 4$. Therefore, $\alpha_i^{-2} = A_i \prod_{j \neq i}(1 - \alpha_j \alpha_i^{-1})$, or equivalently, $\alpha_i = A_i \prod_{j \neq i}(\alpha_i - \alpha_j) = A_i F'(\alpha_i)$ for $1 \leq i \leq 4$. ▮

Since $|\alpha_3| = |\alpha_4| < 1$, we also have

Corollary 5. *If* α_1 *and* α_2 *are the real zeros of* $F(x) = x^4 - 2x^3 - 2x^2 + 2x - 1$, *then asymptotically,*

$$H_4(n) \sim \frac{\alpha_1}{F'(\alpha_1)}\alpha_1^n + \frac{\alpha_2}{F'(\alpha_2)}\alpha_2^n \approx 0.1363(2.5386)^n + 0.1162(-1.2762)^n.$$

4. Value of $H_5(n)$

Proof of Theorem 2: Again, for the sake of brevity, denote $H_5(n)$ by $H(n)$. From (BC), b_{12} and b_{13} cannot be both zero. Thus, $x_1 \in \{F, B, D\}$. Define $h_F(n)$, $h_B(n)$ and $h_D(n)$ as the number of hamiltonian cycles in $P_5 \times P_n$ with $x_1 = F, B$ and D, respectively. Clearly, $h_B(n) = h_D(n)$. Therefore,

$$H(n) = h_F(n) + 2h_B(n) \quad \text{for } n \geq 2. \tag{4.1}$$

Because of Theorem 3, $h_F(n) = h_B(n) = 0$ if n is zero or odd. For positive even n, the initial values are $h_F(2) = 1$, $h_F(4) = 8$, $h_B(2) = 0$ and $h_B(4) = 3$.

Type 1: $x_1 = F$.

It follows from (IC) that there are at most two 1-cells in the second column. In fact, $x_2 \in \{8, 1, 4, 2, A, 5, 9\}$. Symmetry of the configurations allows us to group these seven choices of x_2 into four cases.

Case 1.1. $x_1x_2 = F8$. (Symmetric to $x_1x_2 = F1$.)

(BC) and (CC) imply that $b_{31} = 1$ and $b_{34} = 1$, respectively. This in turn implies that $x_3 \in \{F, B, D\}$. Thus, $x_3x_4 \cdots x_{n-1}$ is a hamiltonian cycle in $P_5 \times P_{n-2}$. Conversely, given any hamiltonian cycle $x_3x_4 \cdots x_{n-1}$ from $P_5 \times P_{n-2}$, the cycle $F8x_3x_4 \cdots x_{n-1}$ is hamiltonian in $P_5 \times P_n$. Therefore, there are $H(n-2)$ hamiltonian cycles in Case 1.1.

Case 1.2. $x_1x_2 = F4$. (Symmetric to $x_1x_2 = F2$.)

(BC) implies that $b_{31} = b_{34} = 1$. Now (IC) implies that $b_{33} = 1$. However, $b_{32} \in \{0, 1\}$; so Case 1.2 contributes $h_D(n-2) + h_F(n-2)$ to $h_F(n)$.

Case 1.3. $x_1x_2 = FA$. (Symmetric to $x_1x_2 = F5$.)

While (EC) implies that $b_{33} = 0$, (BC) implies that $b_{31} = 1$. Then it follows from (IC) that $b_{32} = 1$. Depending on the value of b_{34}, we have two subcases.

Subcase 1.3.1. $b_{34} = 0$. (That is, $x_3 = 3$.)

The 0-cell C_{13} has an outlet at C_{34}. Thus, (EC) is satisfied. (BC) implies that $b_{44} = 1$. If we change b_{34} from 0 to 1, we get a hamiltonian cycle $Bx_4x_5 \cdots x_{n-1}$ in $P_5 \times P_{n-2}$. Conversely, starting with any hamiltonian cycle $Bx_4x_5 \cdots x_{n-1}$ from $P_5 \times P_{n-2}$, we obtain a hamiltonian cycle $FA3x_4x_5 \cdots x_{n-1}$ in $P_5 \times P_n$. Thus, Subcase 1.3.1 accounts for $h_B(n-2)$ hamiltonian cycles.

Subcase 1.3.2. $b_{34} = 1$. (That is, $x_3 = B$.)

Certainly, replacing b_{33} by 1 leads to a hamiltonian cycle $Fx_4x_5 \cdots x_{n-1}$ in $P_5 \times P_{n-2}$. The problem is, the converse does not hold. Take, for example, the hamiltonian cycle $Fx_4 \cdots x_{n-1}$ with $b_{43} = 1$. Coupling with $x_1x_2 = FA$ and changing x_3 from F to B will seal both C_{23} and C_{33} (now both 0-cells) from the exterior, contradicting (EC).

Define a *twin cycle* in $P_5 \times P_n$ as a 2-factor $Bx_2 \cdots x_{n-1}$ with two components such that $\{C_{14}\}$ and $\{C_{11}, C_{12}\}$ are enclosed by different cycles. Let $g(n)$ be the number of twin cycles in $P_5 \times P_n$. Then the contribution from Subcase 1.3.2 is precisely $g(n-2)$.

Case 1.4. $x_1x_2 = F9$.

From (IC) and (EC), we have $b_{32}b_{33} \in \{01, 10\}$. Without loss of generality, we may assume $b_{32}b_{33} = 01$. Now (IC) implies that $b_{34} = 1$. The subcases of $b_{31} = 0$ and $b_{31} = 1$ are similar to Subcases 1.3.1 and 1.3.2, respectively. Hence, there are $2[h_B(n-2) + g(n-2)]$ hamiltonian cycles in Case 1.4.

Summary of $x_1 = F$. We conclude that for $n \geq 4$,

$$h_F(n) = 2H(n-2) + 2[h_B(n-2) + h_F(n-2)] + 4[h_B(n-2) + g(n-2)],$$

which can be simplified to, with the aid of (4.1),

$$h_F(n) = 4H(n-2) + 2h_B(n-2) + 4g(n-2). \tag{4.2}$$

Type 2: $x_1 = B$. (Symmetric to $x_1 = D$.)

(CC) implies that $b_{24} = 1$. Since $b_{21} \in \{0, 1\}$, we have $x_2 \in \{9, A, E\}$.

Case 2.1. $x_1x_2 = B9$.

Since (CC) implies that $b_{31} = b_{34} = 1$, $x_3 \in \{B, D, F\}$. Similar to Case 1.1, there are $H(n-2)$ such hamiltonian cycles.

Case 2.2. $x_1x_2 = BA$.

Because of (CC) and (BC), we have $b_{31} = b_{34} = 1$. Now as a consequence of (IC), $b_{32} = 1$. Thus, $b_{33} \in \{0, 1\}$. Similar to Case 1.2, Case 2.2 contributes $h_B(n-2) + h_F(n-2)$ hamiltonian cycles to the evaluation of $h_B(n)$.

Case 2.3. $x_1x_2 = BE$.

Routine argument leads to $b_{31} = b_{32} = 1$ and $b_{33} = 0$, while b_{34} can be either 0 or 1. The two configurations are similar to those found in Subcases 1.3.1 and 1.3.2. Hence, Case 2.3 covers $h_B(n-2) + g(n-2)$ hamiltonian cycles.

Summary of $x_1 = B$. We have proved that, for $n \geq 4$,

$$h_B(n) = 2H(n-2) + g(n-2). \tag{4.3}$$

Recurrence relation for $H(n)$. Concluding from (4.1)–(4.3), we get

$$H(n) = 8H(n-2) + 2h_B(n-2) + 6g(n-2) \quad \text{for } n \geq 4. \tag{4.4}$$

It now remains to find a recurrence relation of $g(n)$ for $n \geq 4$. Note that $g(2) = 1, g(3) = 0, g(4) = 6$; and in general, $g(n) < h_F(n)$ for even n. The six configurations counted by $g(4)$ are displayed in Figure 4.

1	0	1
0	0	1
1	0	1
1	1	1

1	0	1
0	0	1
1	1	1
1	0	1

1	1	1
0	0	0
1	1	1
1	0	1

1	1	1
0	0	0
1	0	1
1	1	1

1	1	1
0	0	1
1	0	0
1	1	1

1	1	1
0	0	1
1	0	1
1	0	1

Figure 4. Six cases counted by $g(4)$.

Since C_{13} needs an outlet to the exterior, $b_{23} = 0$. There are two cases.

Case A: $b_{24} = 0$.

Since C_{13} has already found its outlet at C_{24}, (EC) is satisfied. Now (CC) implies that $b_{21}b_{22} \in \{10, 01\}$. Numbers of such hamiltonian cycles are $H(n-2)$ and $h_B(n-2) + h_F(n-2)$, respectively.

Case B: $b_{24} = 1$.

Since (IC) forbids $b_{21}b_{22} = 11$, we have three subcases.

Subcase B1. $b_{21}b_{22} = 01$. If we replace b_{13} by 1, the two disjoint cycles become connected to form a hamiltonian cycle with $x_1x_2 = FA$. As in Case 1.3, the contribution is $h_B(n-2) + g(n-2)$.

Subcase B2. $b_{21}b_{22} = 10$. Again, replacing b_{13} by 1 leads to $x_1x_2 = F9$ studied in Case 1.4. Therefore, the contribution is $2[h_B(n-2) + g(n-2)]$.

Subcase B3. $b_{21}b_{22} = 00$. Since C_{13} has an outlet at C_{21}, $x_3 \in \{B, D, F\}$. There are $H(n-2)$ such twin cycles.

Recurrence relation for $g(n)$. We assert that, for $n \geq 4$,

$$g(n) = 3H(n-2) + 2h_B(n-2) + 3g(n-2). \tag{4.5}$$

Conclusion. Define

$$H(x) = \sum_{n=0}^{\infty} H(n)x^n, \quad h_B(x) = \sum_{n=0}^{\infty} h_B(n)x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} g(n)x^n.$$

Routine manipulations on (4.3)–(4.5) lead to

$$\begin{aligned} x^2 g(x) - h_B(x) + 2x^2 H(x) &= 0 \\ -6x^2 g(x) - 2x^2 h_B(x) + (1 - 8x^2)H(x) &= x^2 \\ (1 - 3x^2)g(x) - 2x^2 h_B(x) - 3x^2 H(x) &= x^2 \end{aligned}$$

Solving for $H(x)$, we obtain

$$H(x) = \sum_{n=0}^{\infty} H(n)x^n = \frac{x^2(1 + 3x^2)}{1 - 11x^2 - 2x^6}.$$

The recurrence relation stated in Theorem 2 now follows easily. ∎

Next, we derive an explicit formula for $H_5(n)$. Since $H_5(n) = 0$ for odd n, it suffices to consider

$$\sum_{n=0}^{\infty} H_5(2n) z^n = \frac{z(1+3z)}{1-11z-2z^3}.$$

Let β_i denote the zeros of $G(z) = z^3 - 11z^2 - 2$. Define $\beta_i = y_i + 11/3$ such that y_i are the zeros of $y^3 + py + q$, where $p = -121/2$ and $q = -2716/27$. If

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \quad \text{and} \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

then Cardan's formula leads to

$$\begin{aligned} \beta_1 &= \frac{11}{3} + u + v \approx 11.0165, \\ \beta_2 &= \frac{11}{3} - \frac{u+v}{2} + \frac{u-v}{2}\sqrt{3}i \approx -0.0082 + 0.4260i \\ \beta_3 &= \frac{11}{3} - \frac{u+v}{2} + \frac{u-v}{2}\sqrt{3}i \approx -0.0082 - 0.4260i \end{aligned} \tag{4.6}$$

Similar to Theorem 4 and Corollary 5, it can be shown that

Theorem 6. *For all* $n \geq 0$, $H_5(2n+1) = 0$, *and*

$$H_5(2n) = \sum_{i=1}^{3} \frac{\beta_i + 3}{G'(\beta_i)} \beta_i^n,$$

where β_i *are the zeros of* $G(z) = z^3 - 11z^2 - 2$ *as given in (4.6). Asymptotically,*

$$H_5(2n) \sim \frac{\beta_1 + 3}{G'(\beta_1)} \approx 0.1151(11.0165)^n.$$

4. Remarks

As closing remarks, we pose several questions for further investigation:

(1) Apply these techniques to find the value of $H_6(n)$. One may have to generalize the definition of twin cycle, because now two nonadjacent 0-cells in the same column can be "sealed" by a column of 1-cells on their left, so that outlet(s) must be found in order to satisfy (EC).

(2) Even for $m = 6$, the task is already immensely tedious. Is there any alternate approach to simplify the derivation of $H_m(n)$?

(3) Is it true that $H_m(n)$ always satisfies a certain homogeneous linear recurrence relation with constant coefficients? If this can be answered affirmatively, the recurrence relation can be derived by the method of undetermined coefficients.

(4) What are the reasonable bounds on $H_m(n)$?

(5) Are there any simple relationships between $H_m(n)$ and $H_i(j)$, where $i \leq m$ and $j \leq n$?

References

1. V. Chvátal, *Tough graphs and hamiltonian circuits*, Discrete Math. **5** (1973), 215–228.
2. R. Tošić, O. Bodroža, Y.H.H. Kwong and H. J. Straightt, *On the number of hamiltonian cycles of $P_4 \times P_n$*, Indian J. of Pure and Applied Math. (to appear).
3. J. V. Uspensky, "Theory of Equations", McGraw-Hill, New York, 1948.