Intersections of $S(2,4,v)$ designs

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Abstract. We give a complete solution to the intersection problem for a pair of Steiner system $S(2,4,v)$, leaving a handful of exceptions when $v = 25, 28$ and 37.

1. Preliminaries

A Steiner system $S(t,k,v)$ is a set V of *elements* and a set B of k-element subsets of V called *blocks*, with the property that every t-subset of V appears in precisely one block. Two Steiner Systems (V, B) and (V, B') intersect in s blocks if $|B \cap B'| = s$.

Kramer and Mesner [6] asked the following: for what values s does there exist two Steiner systems $S(t,k,v)$ intersecting in s blocks? Lindner and Rosa [7] solved this problem for all v for $S(2,3,v)$ systems. Gionfriddo and Lindner [3] examined intersections of $S(3,4,v)$ systems; that problem is now essentially solved [4]. No other general result on intersection sizes of $S(t,k,v)$ systems are known.

In this paper, we solve the intersection problem for $S(2,4,v)$ systems, also known as $(v,4,1)$ designs, leaving some possible exceptions for $v = 25$, 28 and 37. Let $b = v(v-1)/12$, and let $I_4[v] = \{0,1,\ldots,b_v\}\backslash\{b_v-7, b_v-5, b_v-4, b_v-3, b_v-2, b_v-1\}$. Denote by $J_4[v]$ the set of intersection sizes of $S(2,4,v)$ systems. We prove the following:

Main Theorem. *For* $v \equiv 1,4 \pmod{12}, v \geq 40, J_4[v] = I_4[v]$.

We employ two recursive constructions, along with a detailed examination of $S(2,4,13)$ and $S(2,4,16)$. In Section 2, we establish that $J_4[v] \subseteq I_4[v]$ for all v. In Section 3, we introduce the recursive constructions used. In Section 4, we analyze the cases $v = 13$ and $v = 16$ in detail, to produce the needed ingredients in the constructions. Then in Section 5, we apply the recursive constructions to prove the Main Theorem for $v \geq 49$. Finally, In Section 6, we examine the four remaining small cases: $v = 25$, 28, 37, and 40. We establish $J_4[40] = I_4[40]$ using a special tripling construction, and then apply various computational methods to the remaining cases.

2. Necessary Conditions

In this section, we establish necessary conditions for $J_4[v]$. We recall that $I_4[v]$ is the set

$$\{0,1,2,\ldots,b_v-8\}\cup\{b_v-6,b_v\}.$$

The main necessary condition follows.

Lemma 2.1. *For* $v\equiv 1,4 \pmod{12}$, $J_4[v]\subseteq I_4[v]$.

Proof: Two $S(2,4,v)$ designs (V,B_1) and (V,B_2) intersect in at least 0 and at most b_v blocks. If they intersect in b_v-s blocks, consider $D_1=B_1\backslash B_2$ and $D_2=B_2\backslash B_1$. D_1 and D_2 each contain s blocks. Now let G be the graph on vertex set V whose edges are the pairs appearing in blocks of D_1 (or D_2). G is a simple graph on $6s$ edges, with every vertex having degree $\equiv 0 \pmod 3$. G has two partitions into K_4's (D_1 and D_2), and the resulting sets of K_4's are disjoint. Hence G has no vertices of degree 3.

Now suppose that G has a vertex x of degree 6. In D_1, suppose x appears in $\{x,a,b,c\}$ and in $\{x,d,e,f\}$. Without loss of generality, D_2 then contains blocks $\{x,a,b,d\}$ and $\{x,c,e,f\}$. Now the two edges $\{a,c\}$ and $\{b,c\}$ appear in different blocks in D_2. Hence G contains at least two vertices of degree at least 9.

Suppose G has a vertex x of degree 9. In D_1 suppose that x appears in blocks $\{x,a,b,c\}$, $\{x,d,e,f\}$ and $\{x,g,h,i\}$. In D_2, there are (up to isomorphism) three possibilities for the blocks containing x.

I. $\{x,a,d,g\},\{x,b,e,h\},\{x,c,f,i\}$
II. $\{x,a,d,g\},\{x,b,c,e\},\{x,f,h,i\}$
III. $\{x,a,b,d\},\{x,c,g,h\},\{x,e,f,i\}$

In case III, edges $\{a,d\}$, $\{b,d\}$, $\{c,g\}$, $\{c,h\}$, $\{e,i\}$ and $\{f,i\}$ appear in distinct blocks of D_1 and hence $s\geq 9$. In Case II, edges $\{a,d\},\{b,e\},\{c,e\},\{f,i\}$ and $\{h,i\}$ appear in distinct blocks of D_1 and hence $s\geq 8$.

Finally, consider Case 1. Here, the edges of three triangles $\{a,d,g\}$, $\{b,e,h\}$ and $\{c,f,i\}$ must appear in blocks of D_1. This can be done with $s=6$ *or* $s\geq 8$. At this point, if $s\in\{1,2,3,4,5,7\}$, G contains no vertices of degree 9, and has at least one vertex of degree ≥ 12. If any vertex has degree ≥ 15, there are at least 16 vertices each of degree ≥ 6 forcing G to have at least 48 edges and hence $s\geq 8$.

It remains to consider the case where G has degrees 6 and 12. Here, G has at least two vertices of degree 12 and hence at least 13 vertices. But then G has at least 45 edges and hence $s\geq 8$.

Lemma 2.1 gives a general necessary condition. We develop next a necessary condition for $J_4[16]$.

Lemma 2.2. $J_4[16] \subseteq \{0,1,2,3,4,5,6,8,12,20\}$.

Proof: Suppose that B_1 and B_2 are two $(16,4,1)$'s with at least seven blocks in common. Two cases arise. Either B_1 and B_2 share a parallel class, or they contain two blocks in common from each of two parallel classes. To see this, observe that if B_1 and B_2 share three blocks of a parallel class, the (unique) structure of the $S(2,4,16)$ design forces them to share the whole parallel class.

For the first case, suppose that they contain a parallel class in common. The remaining sixteen blocks are equivalent to a pair of mutually orthogonal latin squares with rows and columns indexed by Z_4, and symbols from Z_4.

Two such pairs, L_1, L_2 and M_1, M_2 are said to intersect in $|S|$ positions, where $S = \{(i,j) \in Z_4^2 \mid L_1(i,j) = M_1(i,j) \text{ and } L_2(i,j) = M_2(i,j)\}$. We need only show $|S| \in \{0,1,2,4,8,16\}$. We define a collection B of 4-subsets of Z_4^2 as follows:

$$\text{For } i \in Z_4, \{(i,j) \mid j \in Z_4\} \in B,$$
$$\text{and } \{(j,i) \mid j \in Z_4\} \in B.$$

If π is a permutation on Z_4, then $\{(j,\pi(j)) \mid j \in Z_4\} \in B$. If $\{i_1,i_2\}$ and $\{j_1,j_2\}$ are 2-subsets of Z_4 with $i_1 < i_2$, then

$$\{i_1,i_2\} \times \{j_1,j_2\} \in B$$
$$\{(i_1,j_1),(i_1,j_2),(i_2,k_1),(i_2,k_2)\} \in B, \text{ and}$$
$$\{(j_1,i_1),(j_2,i_1),(k_1,i_2),(k_2,i_2)\} \in B,$$

where $\{k_1,k_2\} = Z_4 \ \{j_1,j_2\}$.

It is easy to see that every 3-subset of Z_4^2 is contained in exactly one block of B, that is, (Z_4^2, B) is a Steiner quadruple system of order 16. We claim that if L is a latin square of order 4 which has an orthogonal mate, and $b \in B$, then the four symbols of L occurring in the cells of b are of the form

$$xxxx, \ xxyy, \text{ or } wxyz,$$

where $Z_4 = \{w,x,y,z\}$.

This is easy to check. The only other possibillties are $xxxy$ and $xxyz$. The first cannot even yield a latin square, and none of the latin squares in the second possibility have an orthogonal mate.

Hence, glven the entries in any three cells T of L, the entry in the fourth cell of the unique block of B containing T is uniquely determined.

In particular, with L_1, L_2, M_1, M_2, and S as above, S must be a sub-Steiner quadruple system of (Z_4^2, B). It is well known that $|S| \in \{0,1,2,4,8,16\}$.

In the second case, B_1 and B_2 share two blocks from each of two parallel classes. The automorphism group of the $S(2,4,16)$ acts transitively on such sets of four blocks. There are 3192 permutations mapping this set of four blocks onto itself. For each permutation π of this type, we determined $B \cap \pi B$ for B a $S(2,4,16)$ design. This small exhaustive search by computer shows that $|B \cap \pi B| \in \{4,5,6,8,12,20\}$, completing the proof.

3. Recursive Constructions

We first describe simple tripling and quadrupling constructions, and then examine their use in the determination of intersection sizes.

Construction 3.1 (quadrupling): Let G, B be a group divisible design of order v with block size 4 and groups of sizes congruent to 0 (mod 3). Let G have t_i groups of size g_i. Then there exists

(1) a $(4v+1,4,1)$ design containing t_i subdesigns of order $4g_i+1$, all intersecting in a common element.

(2) a $(4v+4,4,1)$ design containing t_i subdesigns of order $4g_i+4$ all intersecting in a common block.

Proof: Let G, B be such a group divisible design on element set V. We form a $(4v+1,4,1)$ design on $(Z_4 \times V) \cup \{\infty_1\}$ or a $(4v+4,4,1)$ design on $(Z_4 \times V) \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$ as follows. For each block $b \in B$, $b = \{w,x,y,z\}$, form a $(16,4,1)$ on $Z_4 \times b$ omitting a parallel class on $(Z_4 \times \{w\}, Z_4 \times \{x\}, Z_4 \times \{y\}, Z_4 \times \{z\})$. Now for each group g of G, place a $(4|g|+1,4,1)$ design on $(Z_4 \times g) \cup \{\infty_1\}$, or a $(4|g|+4,4,1)$ design containing block $\{\infty_1,\infty_2,\infty_3,\infty_4\}$ on $(Z_4 \times g) \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$. In the latter case, the block on $\{\infty_i\}$ is taken once only.

There is much freedom in choosing the ingredients, which enables us to construct two systems intersecting in a specified number of blocks.

We apply Theorem 3.1 to GDDs with groups of size 3, and possibly one group of size 6. For both the $4v+1$ and $4v+4$ constructions, we can choose on $Z_4 \times b$, $b \in B$, any $(16,4,1)$ containing the required parallel class. Let $JP_4[v]$ for $v \equiv 4 \pmod{12}$ denote the number of blocks shared br two $(v,4,1)$ designs with a common parallel class, *in addition to* those shared in that parallel class.

In the $4v+1$ construction, we can choose any $(13,4,1)$ design on $(Z_4 \times g) \cup \{\infty\}$ when $|g| = 3$, and any $(25,4,1)$ design when $|g| = 6$. Hence we obtain the following:

Lemma 3.2. *Let G, B be a GDD or order v with block size 4 and group type $3^s 6^t$. Let $b = |B|$. For $1 \le i \le b$, let $a_i \in JP_4[16]$. For $1 \le i \le s$, let $c_i \in J_4[13]$; for $1 \le i \le t$, let $d_i \in J_4[25]$. Then there exist two $(4v+1,4,1)$ designs intersecting in precisely*

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{s} c_i + \sum_{i=1}^{t} d_i$$

blocks.

Proof: Using Construction 3.1, use the *same* GDD G, B to construct two $(4v + 1, 4, 1)$ designs. For blocks $B_1, \ldots, B_b$ place $(16, 4, 1)$'s sharing a parallel class and a_i other blocks on $Z_4 \times B_1$ in the two systems. For groups G_i of size 3, place $(13, 4, 1)$'s with c_i blocks in common, and for groups H_i of size 6, place $(25, 4, 1)$'s with d_i blocks in common.

The $4v + 4$ construction is slightly more complicated.

Lemma 3.3. *Let G, B be a GDD on $v = 3s + 6t$ elements with b blocks of size 4 and group type $3^s 6^t$, $s \geq 1$. For $1 \leq i \leq b$, let $a_i \in JP_4[16]$. For $1 \leq i \leq s - 1$, let $c_i + 1 \in J_4[16]$ and let $c_s \in J_4[16]$. For $1 \leq i \leq t$, let $d_i + 1 \in J_4[28]$. Then there exist two $(4v + 4, 4, 1)$ designs with precisely*

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{s} c_i + \sum_{i=1}^{t} d_i$$

blocks in common.

Proof: We proceed as in Lemma 3.2 on the blocks in B. For groups $g \in G$ except for one group S of size 3 we must choose $(16, 4, 1)$'s or $(28, 4, 1)$'s with at least the block $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ in common. This block is then omitted in both designs for all but one group, namely S. Finally on $(Z_4 \times S) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, we place two $(16, 4, 1)$'s intersecting in c_s blocks.

In applying Lemmas 3.2 and 3.3, we employ GDD's of group of type 3^s or $3^s 6^1$. Such GDD's exist for all $v \equiv 0 \pmod 3$ except $v = 9, 18$. For $v \equiv 0, 3 \pmod{12}$, these are obtained by omitting a point in a $(v + 1, 4, 1)$ design; for $v \equiv 6, 9$ (mod 12) they are obtained by omitting a point in the 7-block of a $(v + 1, \{4, 7^*\}, 1)$ design [1].

We employ a second general construction to handle values missed by quadrupling.

Construction 3.4 (tripling): Let G, B be a GDD of order v, blocksizes 4 and 5, and groups of size 4 and 5. Then there exists a $(3v + 1, 4, 1)$ design.

Proof: Let G, B be such a GDD on element set V. We form a $(3v + 1, 4, 1)$ design on $(Z_3 \times V) \cup \{\infty\}$. For each block $b \in B$, if $|b| = 4$ place a GDD with blocksize 4 and group type 3 on $Z_3 \times B$; similarlry if $|b| = 5$ place a GDD with group type 3^5. For each group $g \in G$, if $|g| = 4$ place a $(13, 4, 1)$ on $(Z_3 \times g) \cup \{\infty\}$; if $|g| = 5$ place a $(16, 4, 1)$ on $(Z_3 \times g) \cup \{\infty\}$.

The GDD's of type 3^4 and 3^5 are obtained by omitting an element from a $(13, 4, 1)$ or $(16, 4, 1)$, respectively.

Now we consider consequences for intersection sizes. The *flower* of an element is the set of blocks containing the element. Let $JF_4[v]$ denote the number of blocks shared by two $(v, 4, 1)$ designs in addition to those in a required common flower. Observe that $JF_4[v]$ is precisely the intersection sizes of $\{4\}$-GDD's of group type $3^{(v-1)/3}$ having all groups in common. Hence we obtain:

Lemma 3.5. *Let G, B be a GDD of order v with b_4 blocks of size 4, b_5 blocks of size 5 and group type $4^s 5^t$. Choose*

$$\begin{aligned} a_i &\in JF_4[13] \text{ for } 1 \le i \le b_4 \\ c_i &\in JF_4[16] \text{ for } 1 \le i \le b_5 \\ d_i &\in J_4[13] \text{ for } 1 \le i \le s \\ e_i &\in J_4[16] \text{ for } 1 \le i \le t \end{aligned}$$

Then there exist two $(3v + 1, 4, 1)$ designs intersecting in precisely

$$\sum_{i=1}^{b_4} a_i + \sum_{i=1}^{b_5} c_i + \sum_{i=1}^{s} d_i + \sum_{i=1}^{t} e_i$$

blocks.

Proof: Similar to Lemma 3.2.

4. Ingredients

First we consider $v = 13$. Mathon and Rosa [9] showed that $\{0, 1, 2, 3, 4, 5, 7, 13\} \subseteq J_4[13]$, and hence by Lemma 2.1 we obtain

Lemma 4.1. $J_4[13] = \{0, 1, 2, 3, 4, 5, 7, 13\}$.

Since $s \in JF_4[13]$ implies $s + 4 \in J_4[13]$, it is an easy exercise to establish

Lemma 4.2. $JF_4[13] = \{0, 1, 3, 9\}$.

Next we consider $v = 16$. Mathon and Rosa [9] established that $\{0, 1, 2, 3, 4, 5, 6, 8, 12, 20\} \subseteq J_4[16]$. By Lemmas 2.1 and 2.2, we obtain

Lemma 4.3. $J_4[16] = \{0, 1, 2, 3, 4, 5, 6, 8, 12, 20\}$.

Examples are easily produced which, together with the necessary conditions, establish the following.

Lemma 4.4. $JP_4[16] = \{0, 1, 2, 4, 8, 16\}$ *and* $JF_4[16] = \{0, 1, 3, 7, 15\}$.

At this point, the recursions of Section 3 require only a few values for $v = 25$ and $v = 28$. We establish these here, and return to small orders in Section 6.

Lemma 4.5. $\{0,29,50\} \subseteq J_4[25]$.

Proof: $50 \in J_4[25]$ by taking the same $(25,4,1)$ twice. $0 \in J_4[25]$ since every $(v,4,1)$, $v \geq 13$, has a disjoint isomorphic mate [8].

To obtain $29 \in J_4[25]$, take any $(25,4,1)$, (V,B). Let x, y, z be three elements on a common block. Let B' be the blocks obtained by applying the permutation (xyz) to (V,B). Then B and B' share 29 blocks.

Lemma 4.6. $\{1,63\} \subseteq J_4[28]$.

Proof: $63 \in J_4[28]$ by taking a $(28,4,1)$ design twice. To obtain $1 \in J_4[28]$, delete four points in a block to form a PBD on 24 elements with block sizes 3 and 4. Simple counting shows that this PBD has a disjoint isomorphic mate, which can then be completed to a $(28,4,1)$ sharing one block with the original.

5. Applying the Recursions

In this section, we prove the Main Theorem for all $v \geq 49$. First we treat the (easier) case $v \equiv 1 \pmod{12}$.

Lemma 5.1. *For* $v \equiv 1 \pmod{12}$, $v \geq 49$, $J_4[v] = I_4[v]$.

Proof: Necessity is established in Lemma 2.1. For sufficiency, let $w = (v-1)/4$, and $x = w/3$. If $w \equiv 0,3 \pmod{12}$, there is a GDD of order w with blocksize 4 and group type 3^x. Apply Lemma 3.2 using this GDD, $J_4[13]$ from Lemma 4.1 and $JP_4[16]$ from Lemma 4.4. When $w \equiv 6,9 \pmod{12}$, $w \neq 18$, choose instead a GDD with group type $3^{x-2}6^1$ and use Lemma 4.5 on the $(25,4,1)$ ingredient.

When $w = 18$, form instead a GDD on 24 elements with blocksize 5 and group type 4^6 (omit a point from a $(25,5,1)$ design) and apply Lemma 3.5 using $JF_4[16]$ and $J_4[13]$. Since six $(13,4,1)$'s are contained in the $(73,4,1)$ so constructed, Lemma 3.5 produces all of the required values.

Now we turn to $v \equiv 4 \pmod{12}$.

Lemma 5.2. *For* $v \equiv 4 \pmod{12}$, $v \geq 52$, $J_4[v] = I_4[v]$.

Proof: Proceed as in Lemma 5.1, applying Lemma 3.3 instead of 3.2. Provided $w \neq 18$, we use $J_4[16]$, $JP_4[16]$ (Lemma 4.4) and Lemma 4.6 for $J_4[28]$. This produces all values *except* b_v-6, b_v-9, b_v-10, b_v-11, b_v-13 and b_v-21.

When $w = 18$, apply Lemma 3.5 using the GDD of blocksize 5 and group type (i.e., the $(25,5,1)$ design itself); again all values except the six stated omissions are obtained, given the values for $JF_4[16]$ and $J_4[16]$.

Now we must handle the six remaining values. Rees and Stinson [10] proved that if $v \equiv 1,4 \pmod{12}$, $w \equiv 1,4 \pmod{12}$ and $v \geq 3w+1$, then there is a $(v,4,1)$ design containing a $(w,4,1)$ subdesign. By taking all blocks not in the subdesign identically, and two copies of the subdesign intersecting in all but s

blocks, we have that $b_v - s \in J_4[v]$ if $b_w - s \in J_4[w]$. Using the Rees-Stinson result with $w = 13$, we obtaln intersection numbers $b_v - 6$, $b_v - 9$, $b_v - 10$, $b_v - 11$ and $b_v - 13$ for $v \geq 40$. Similarly using $w = 25$, and the fact that $29 \in J_4[25]$, we obtain $b_v - 21$ for $v \geq 76$.

All that remains is $b_v - 21$ for $v = 52$ and 64. For $v = 52$, observe that there is a GDD on 52 points with block size 4 and group type 13 (essentially a pair of mutually orthogonal latin squares of side 13). In two $(52, 4, 1)$'s, take the blocks of the GDD identically. Replace the four groups by $(13, 4, 1)$'s intersecting in $0, 5, 13$ and 13 blocks respectively. This gives $b_{52} - 21$.

Finally consider $v = 64$. Let G, B be a GDD on 21 elements with blocksize 4 and 5, and group type $5^1 4^4$; this is obtained by omitting four points from a block of a $(25, 5, 1)$. Apply Lemma 3.5 to produce $(64, 4, 1)$ designs. These designs have four $(13, 4, 1)$ designs intersecting in a single point, and by choosing intersection sizes $0, 5, 13$ and 13 on these, we can obtain $b_{64} - 21 \in J_4[64]$.

6. Small Orders

Four small orders, $\{25, 28, 37, 40\}$, remain. The last of these can be handled by a recursion similar to Construction 3.4.

Lemma 6.1. $J_4[40] = I_4[40]$.

Proof: Let D be a Kirkman triple system of order 27 (briefly KTS(27)) containing three disjoint Kirkman triple systems of order 9. Let $P_1, \ldots, P_4, R_1, \ldots, R_9$ be the 13 parallel classes of the KTS(27) so that $P_1, \ldots, P_4$ each induce parallel classes in the three KTS(9)'s. We add 13 points $a_1, \ldots, a_4, b_1, \ldots, b_9$ to this KTS(27) and form blocks by adding a_i to each triple in P_i and b_i to each triple in R_i. Finally, place a $(13, 4, 1)$ on the 13 new points. Consider each ingredient in turn. On the $(13, 4, 1)$, we can get any intersection size from $J_4[13]$. On the (b_i, R_i) blocks, we can permute the R_i to obtain intersection numbers $\{0, 9, 18, 27, 36, 45, 54, 63, 81\}$. On the (a_i, P_i) blocks, we can permute the parallel classes of each of the KTS(9)'s to obtain intersection numbers $\{0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 36\}$. Combining these yields the desired result.

For $v = 25$, we find that $J_4[25] \neq I_4[25]$.

Lemma 6.2. $b_{25} - 6 \notin J_4[25]$.

Proof: There is a unique graph G on 36 edges having two partitions into K_4's (see the proof of Lemma 2.1). Hence if a $(v, 4, 1)$ design B intersects another in $b_v - 6$ blocks, B contains (without loss of generality) the blocks $\{x, a, b, c\}$, $\{x, d, e, f\}$, $\{x, g, h, i\}$, $\{y, a, d, g\}$, $\{y, b, e, h\}$ and $\{y, c, f, i\}$. Form a PBD B' from a $(25, 4, 1)$ design (V, B) by deleting the four elements $\{w, x, y, z\}$ in the block containing $\{x, y\}$. B' is a PBD with 28 blocks of size 3 and 21 blocks of size 4. The 28 triples resolve into four parallel classes. Now let $L =$

$\{a, b, c, d, e, f, g, h, i\}$ and let $R = V \backslash (L \cup \{w, x, y, z\})$. Call an edge connecting $\ell \in L$ and $r \in R$ a *cross edge* and call other edges *inside edges.* Now the triples arising from x and y consist entirely of inside edges. After eliminating such edges, there remain 18 edges inside L, 42 edges inside R, and 108 cross edges. We require 14 triples and 21 quadruples which partition the remaining edges, and the 14 triples are in two parallel classes. Call a triple or quadruple an (i, j)-block if it contains i elements of L and j elements of R. Let the excess of a block be the number of cross edges it uses less the number of inside edges which it uses.

We have the following:

blocktype	excess
(4,0), (0,4)	-6
(3,0), (0,3)	-3
(3,1), (1,3)	0
(2,1), (1,2)	1
(2,2)	2

We must account for a total excess of $48 = 108 - 60$. Since L has at most 18 edges, the number of $(2,2)$-blocks is at most 18. Also since the number of triples is 14, we obtain excess at most $36 + 14 = 50$. Since we require excess 48, block of types $(4,0), (0,4), (3,0)$ and $(0,3)$ cannot occur.

Consider a parallel class of triples. It contains necessarily 2 $(2,1)$-blocks and 5 $(1,2)$-blocks. But then the triples exhaust 4 edges inside L, and hence the number of $(2,2)$-blocks is at most 14. However, in this case the total excess cannot exceed 42. This establishes that no such $(25, 4, 1)$ design exists.

Lemma 6.3. $b_{25} - 8 \notin J_4[25]$.

Proof: There are two graphs on 48 edges having two partitions into K_4's. One has degree sequence $12^2 6^{12}$; let x and y be the vertices of degree 12. If this configuration appears, the other blocks appearing with x and y are forced to appear in the 36-edge configuration eliminated in Lemma 6.2, and hence is impossible.

The other 48-edge configuration is obtained by adding blocks $\{z, a, e, i\}$ and $\{z, b, f, g\}$ to the 36-edge configuration, and hence is also eliminated by Lemma 6.2.

Now we determine some values in $J_4[25]$. We employ the list of sixteen nonisomorphic $S(2,4,25)$'s of Kramer, Magliveras and Mathon [5]. Our initial strategy is to apply a permutaton π to a system (V, B), and determine $|B \cap \pi B|$. For system 1 of [5], employ the permutations specified in Table 6.1 to establish that $\{0, 1, 2, \ldots, 21\} \subseteq J_4[25]$.

For larger intersection sizes, we adapt the transformation method used by Kramer, Mathon, and Magliveras [5]. Let (V, B) be an $S(2,4,v)$ and b be a block in B. Partition $B \backslash \{b\}$ into two classes U_b and B_b, where U_b contains all blocks

Table 1.

$\lvert B \cap \pi B \rvert$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	8	7	13	5	6	21	17	4	9	14	3	11	15	24	20	19	12	2	25	22	18	1	10	16	23
1	3	7	13	5	6	21	17	4	9	10	3	11	15	24	20	19	12	2	25	22	18	1	14	16	23
2	17	18	3	4	15	6	16	8	7	10	11	12	13	14	20	19	22	2	9	23	21	1	5	24	25
3	13	17	3	23	15	18	5	4	9	10	8	11	6	16	20	19	22	2	25	12	21	1	14	24	7
4	17	2	3	4	15	6	16	8	7	10	11	12	13	14	20	19	22	18	9	23	21	1	5	24	25
5	3	7	13	23	6	21	17	4	9	10	8	11	15	24	20	19	22	2	25	12	18	1	14	6	5
6	17	2	3	4	15	6	19	8	7	10	11	12	13	14	20	16	22	18	9	23	21	1	5	24	25
7	17	2	3	4	15	6	19	8	7	10	11	12	13	14	23	16	22	18	9	20	21	1	5	24	25
8	6	17	3	11	15	18	14	4	9	10	8	23	13	16	20	19	22	2	7	12	21	1	5	24	25
9	17	9	13	5	6	21	3	4	7	14	8	11	23	24	20	19	12	22	25	2	18	1	10	16	15
10	6	17	3	8	15	18	14	4	9	10	11	23	13	16	20	19	22	2	7	12	21	1	5	24	25
11	17	2	3	4	15	6	8	7	19	10	11	12	13	14	23	16	22	18	9	20	21	1	5	24	25
12	13	19	3	4	21	22	6	5	18	17	2	8	14	9	7	25	20	10	23	24	1	12	11	15	16
13	3	9	13	5	6	21	17	4	7	14	8	11	23	24	20	19	12	22	25	2	18	1	10	16	15
14	25	9	22	23	6	5	7	12	17	13	4	3	18	2	16	19	20	21	14	24	1	15	10	11	8
15	17	2	3	4	15	6	7	8	19	10	11	12	13	14	23	16	1	18	9	20	21	22	5	24	25
16	3	17	13	23	6	18	7	4	9	10	8	11	15	16	20	19	22	2	25	12	21	1	14	24	5
17	25	9	17	23	6	5	7	12	22	13	4	3	18	2	16	19	20	21	14	24	1	15	10	11	8
18	3	7	13	5	6	21	17	4	9	14	8	11	23	24	20	19	12	22	25	2	18	1	10	16	15
19	6	17	3	4	15	18	16	8	9	10	11	23	13	14	20	29	22	2	7	12	21	1	5	24	25
20	17	19	15	4	18	13	6	25	16	5	2	8	14	9	7	24	20	10	23	3	1	12	11	22	21
21	17	2	3	4	15	6	7	8	9	10	11	12	13	14	23	16	1	18	19	20	21	22	5	24	25

which do not intersect b, and B_b contains all blocks intersecting b. Now form $T_b = \{b' - b : b' \in B_b\}$; that is, T_b is formed from B_b by removing the elements of the fixed block b. T_b is a partition into triangles of an 8-regular graph G_b on $v-4$ vertices, and thus $|T_b| = 4(v-4)/3$. Moreover, T_b consists of 4 parallel classes P_1, P_2, P_3, P_4 of triangles. Now suppose that G_b can be partitioned into four parallel classes Q_1, Q_2, Q_3, Q_4 of triangles. Then from b, U_b, and these parallel classes, one can construct an $S(2,4,v)$ (V, B'). Observe that B and B' intersect in

$$1 + |U_b| + |P_1 \cap Q_1| + |P_2 \cap Q_2| + |P_3 \cap Q_3| + |P_4 \cap Q_4|$$

blocks.

Hence if one can obtain different partitions of such a graph G_b into parallel classes of triangles, one can obtain different intersection sizes. Observe that for any system (V, B) and any block b, the graph G_b is partitionable into parallel classes in at least one way. By permuting the names of the four parallel classes, one obtains intersection numbers $b_v - 2(v-4)/3$, $b_v - 3(v-4)/3$ and $b_v - 4(v-4)/3$. To obtain further values, we require graphs G_b with more than one resolution.

We employ the numbering of the designs in [5], and their numbering of the blocks. Specifying a design number and a block number defines a graph G_b, for which we present multiple partitions into parallel classes. In Table 6.2, we give such partitions to establish that

$$\{22,23,24,25,26,27,28,29,30,32,35,36,38\} \subseteq J_4[25].$$

At present, this leaves the values $\{31,33,34,37,39,40,41\}$ in doubt for $v = 25$.

Next we consider $J_4[28]$. By applying random permutations π to an $S(2,4,28)$, we obtained intersection sizes $\{0,\ldots,29\} \cup \{32\}$; we do not include them here. Next we applied the transformation method to some $S(2,4,28)$ designs listed in [9]. From the so-called "classical unital", design (vii) of [9], using the transformation method one obtains intersection sizes $\{31,33,35,36,37,38,39,41,42,43,45,47,48,51,63\}$. Moreover, one can apply permutations to the blocks in U_b provided they form an automorphism of G_b. In this way, we obtain intersection sizes $\{30,40\}$. Design (i) of [9] under the transformation method provides intersection size 55. We found an $S(2,4,28)$ giving intersection size 34. Hence the values which remain in doubt for $v = 28$ are $\{44,46,49,50,52,53,54,57\}$.

For $v = 37$, we applied random permutations to obtain intersection sizes $\{0,\ldots,57\} \cup \{60\}$. For higher intersection sizes, we form a specific $S(2,4,37)$ which realizes many of the values. We exploit the fact that the partial $S(2,4,12)$ obtained by deleting a point from the $S(2,4,13)$ *can* be embedded in an $S(2,4,37)$. One particular embedding is on elements $\{\infty\} \cup (\{x,y,z\} \times Z_{12})$. Take starter

A. Block number 50 of Design number 3

Partition 1:

1 5 13	2 7 10	3 14 21	4 8 16	6 17 19	9 11 20	12 15 18
1 9 12	2 13 21	3 4 15	5 16 19	6 7 18	8 10 20	11 14 17
1 15 21	1 6 14	3 8 11	4 18 19	5 9 17	7 12 20	10 13 16
1 4 7	2 5 8	3 6 9	10 14 18	11 15 16	12 13 17	19 20 21

Partition 2:

1 5 9	2 6 7	3 4 8	10 13 16	11 14 17	12 15 18	19 20 21
1 12 13	2 5 8	3 14 21	4 15 16	6 17 19	7 10 18	9 11 20
1 15 21	2 10 14	3 6 9	4 18 19	5 13 17	7 12 20	8 11 16
1 4 7	2 13 2	3 11 15	5 16 19	6 14 18	8 10 20	9 12 17

Intersection Sizes: 23,24,26,27,30,32

B. Block number 1 of Design number 6

Partition 1:

3 12 20	4 14 17	5 7 21	8 11 24	9 16 18	10 19 22	13 15 23
3 15 18	4 10 24	5 11 19	7 12 17	8 16 23	9 13 22	14 20 21
3 19 21	4 12 15	5 18 24	7 8 9	10 13 17	11 14 23	16 20 22
3 7 11	4 18 20	5 10 15	8 12 13	9 17 19	14 16 24	21 22 23

Partition 2:

3 11 19	4 14 17	5 10 24	7 8 12	9 16 18	13 15 23	20 21 22
3 12 15	4 18 24	5 19 21	7 9 17	8 11 23	10 13 22	14 16 20
3 18 20	4 10 15	5 7 11	8 16 24	9 19 22	12 13 17	14 21 23
3 7 21	4 12 20	5 15 18	8 9 13	10 17 19	11 14 24	16 22 23

Intersections Sizes: 25

Table 6.2

C. Block number 23 of Design number 11

Partition 1:

1 11 23	2 14 15	3 19 21	7 8 22	9 10 20	12 16 18	13 17 24
1 13 14	2 20 21	3 10 22	7 9 24	8 12 19	11 17 18	15 16 23
1 19 20	2 12 24	3 13 15	7 11 21	8 9 23	10 16 17	14 18 22
1 2 3	7 10 12	8 11 14	9 12 15	16 19 22	17 20 23	18 21 24

Partition 2:

1 11 23	2 20 21	3 13 15	7 9 24	8 12 19	10 16 17	14 18 22
1 13 14	2 12 24	3 19 21	7 8 22	9 10 20	11 17 18	15 16 23
1 19 20	2 14 15	3 10 22	7 11 21	8 9 23	12 16 18	13 17 24
1 2 3	7 10 13	8 11 14	9 12 15	16 19 22	17 20 23	18 21 24

Intersection Sizes: 24,26,28,32,38

D. Block number 23 of Design number 13

Partition 1:

1 7 21	2 17 24	3 11 13	8 22 23	9 10 14	12 19 20	15 16 18
1 12 14	2 8 19	3 18 22	7 11 15	9 23 24	10 20 21	13 16 17
1 16 23	2 10 15	3 9 20	7 22 24	8 12 13	11 19 21	14 17 18
1 2 3	7 10 13	8 11 14	9 12 15	16 19 22	17 20 23	18 21 24

Partition 2:

1 7 21	2 10 15	3 18 22	8 11 14	9 23 24	12 19 20	13 16 17
1 12 14	2 17 24	3 9 20	7 10 13	8 22 23	11 19 21	15 16 18
1 16 23	2 8 19	3 11 13	7 22 24	9 12 15	10 20 21	14 17 18
1 2 3	7 11 15	8 12 13	9 10 14	16 19 22	17 20 23	18 21 24

Intersection Sizes: 26,27,28,30,32,35

Table 6.2 continued

blocks with subscripts mod 12 as follows:

∞	x_0	y_0	z_0
x_0	x_4	y_{11}	z_5
x_2	z_0	z_1	z_5
x_7	y_0	y_1	z_9
x_{10}	y_0	y_2	z_4
x_3	y_0	y_4	z_7
x_2	y_0	y_5	z_{10}
x_5	y_1	z_0	z_2

and the short orbits

y_0	y_3	y_6	y_9
z_0	z_3	z_6	z_9

Call the resulting set of 102 blocks B. The pairs left uncovered by B form a complete 4-partite graph $K_{3,3,3,3}$; hence there is a set C of nine blocks which cover the remaining pairs. We consider permutations of the resulting design which are automorphisms on the $K_{3,3,3,3}$ for C; this enables us to replace C by a different set C' of blocks covering the same pairs. Recall that $C \cap C'$ can be any of $\{0,1,3,9\}$—this is $JF_4[13]$.

Hence we consider permutations on the design which induce automorphisms on the $K_{3,3,3,3}$. In this way, we obtain intersections sizes $|B \cap \pi B| + \{0,1,3,9\}$. We employ the following permutations:

π	$\|B \cap \pi B\|$
identity	102
$(x_0 x_1)$	86
$(y_0 y_3)$	80
$(x_0 x_1 x_2)$	78
$(x_0 x_1)(x_4 x_5)$	74
$(x_0 x_1 x_2 x_3)$	70
$(y_0 y_3 y_6)$	69
$(x_0 x_1)(y_0 y_3)$	68
$(y_0 y_3)(z_0 z_3)$	62
$(y_0 y_3 y_6 y_9)$	58

Hence the values in doubt for $v = 37$ are

$$\{64, 66, 76, 82, 84, 85, 88, 90, 91, 92, 93, 94, 96, 97, 98, 99, 100, 101\}.$$

We expect that the values left in doubt for $v = 37$ are all intersection sizes; however, we also expect that a proof of this will be the result of tedious computations, or good luck.

7. Concluding Remarks

In this paper, we have obtained a complete solution of the intersection problem for $S(2,4,v)$ systems with $v = 13, 16$ and $v \geq 40$. The solution for the intermediate cases is partial, and is complicated by the lack of complete enumerations of $S(2,4,25)$ and $S(2,4,28)$ designs. The remaining problems could be solved to a large extent by determining minimal embeddings of partial $S(2,4,v)$ designs. This problem is open, and merits further study.

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References

1. A.E. Brouwer, *Optimal packings of K_4's into a K_n*, J. Comb. Theory **A26** (1979), 278–297.
2. A.E. Brouwer, *Some unitals on 28 points and their embeddings in projective planes of order 9*, Lecture Notes in Mathematics **893** (1981), 183–188.
3. M. Gionfriddo and C.C. Lindner, *Construction of Steiner quadruple systems having a prescribed number of blocks in common*, Discrete Math. **34** (1981), 31–42.
4. A. Hartman and Z. Yehudai, *Intersections of Steiner quadruple systems.* (preprint)
5. E.S. Kramer, S.S. Magliveras and R. Mathon, *The Steiner systems $S(2,4,25)$ with nontrivial automorphism group*, Discrete Math. **77** (1989), 137–157.
6. E.S. Kramer and D.M. Mesner, *Intersections among Steiner systems*, J. Comb. Theory **A16** (1974), 273–285.
7. C.C. Lindner and A. Rosa, *Steiner triple systems with a prescribed number of triples in common*, Canadian J. Math. **27** (1975), 1166–1175.
8. C.C. Lindner and A.P. Street, *Disjoint designs and irreducible designs without repeated blocks*, Ars Combinatoria **21A** (1986), 229–236.
9. R. Mathon and A. Rosa, *Some results on the existence and enumeration of BIBD's*, Mathematics Report 125, McMaster University (1985).
10. R. Rees and D.R. Stinson, *On the existence of incomplete designs of block size four having one hole.* (to appear).