On the Chromatic Coefficients of a Bipartite Graph

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ABSTRACT

We give explicit expressions for the sixth and seventh chromatic coefficients of a bipartite graph. As a consequence we obtain a necessary condition for two bipartite graphs to be chromatically equivalent.

1980 *Mathematical Subject Classification. Primary* 05C15

§1. Introduction

Farrell [1] has given explicit expressions for the fourth and fifth coefficients of the chromatic polynomial of a graph. The results were obtained by using the following theorem for the chromatic polynomial $P(G;\lambda)$ of a graph G, due to Whitney [4].

Theorem A. *Let G be a graph of order p and size q. Then*

$$P(G;\lambda) = \sum_{k=1}^{p} \Big(\sum_{r=0}^{q} (-1)^r N(k,r) \Big) \lambda^k, \tag{1}$$

where $N(k,r)$ denotes the number of spanning subgraphs of G having exactly k components and r edges.

In this paper we give explicit expressions for the sixth and seventh chromatic coefficients of a bipartite graph. As a consequence we establish a necessary condition for two bipartite graphs to be chromatically equivalent. Two graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. Our results were obtained by using the technique introduced by Farrell in [1].

§2. Definitions and Known Results

All graphs considered in this paper are finite, undirected, simple and loopless. Let $V(G)$ be the vertex set of the graph G and $E(G)$ be the edge set of G. An *edge-gluing* $G \vee H$ of two graphs G and H is a graph obtained from $G \cup H$ by identifying an edge in G with an edge in H. $K^-_{m,n}$ denotes the graph obtained by deleting one edge from the complete bipartite graph $K_{m,n}$. C_n denotes the cycle with n vertices. A graph is said to be *of type* G^{+r} if it is obtained from G by adding r extra edges in such a way that no new cycles are created (so that the number of vertices is increased by r more than the increase in the number of components, which may be anything from 0 to r). $N_G(H)$ will denote the number of subgraphs of G isomorphic to H, and $I_G(H)$ will denote the number of induced subgraphs of G isomorphic to H (so that $N_G(H) = I_G(H)$ if H is complete, or if G is bipartite and H is complete bipartite). If S is a set of graphs, we denote by $N_G(S)$ the number of subgraphs of G isomorphic to graphs in S.

Let G be a bipartite graph with p vertices and q edges. The first three coefficients of $P(G;\lambda) = \sum_{i=0}^{p} a_i \lambda^{p-i}$ can easily be deduced to be 1, $-q$, and $\binom{q}{2}$. By using Theorems 1 and 2 of Farrell [1], the expressions for a_3 and a_4 are $N_G(C_4) - \binom{q}{3}$ and $\binom{q}{4} - (q-3)N_G(C_4) + N_G(K_{2,3})$, respectively.

The following lemma will be used to prove our main results.

Lemma B (Farrell [1]). *Let G be a graph with p vertices and $p-n$ components. Then G consists of $p-c$ isolated vertices together with $c-n$ non-trivial components, where $n \le c \le 2n$. If $1 \le n \le p-2$, then $n+1 \le c \le 2n$.*

§3. The Sixth Chromatic Coefficient

We will now prove the following result.

Theorem 1. *Let G be a bipartite graph of order p and size q. Then the coefficient of λ^{p-5} in the chromatic polynomial of G is*

$$-\binom{q}{5} + \binom{q-3}{2} N_G(C_4) + I_G(C_6) - (q-3)N_G(K_{2,3}) + N_G(K_{2,4}) - I_G(K^-_{3,3}) - 4N_G(K_{3,3}). \qquad (2)$$

Proof. By Theorem A, the coefficient of λ^{p-5} is given by

$$a_5 = \sum_{r=0}^{q}(-1)^r N(k,r) \quad \text{with } k = p-5,$$

where $N(k,r)$ is the number of spanning subgraphs of G with $p-5$ components and r edges. From Lemma B, we know that the only spanning subgraphs of G with $p-5$ components are those having $p-c$ isolated vertices and $c-5$ non-trivial components, where $6 \leq c \leq 10$.

<u>$c = 6$</u>. In this case, the bipartite graphs have $p-6$ isolated vertices and 1 component with 6 vertices. The connected bipartite graphs with 6 vertices are in Figure 1.

<u>$c = 7$</u>. The bipartite graphs with 7 vertices and 2 non-trivial components are in Figure 2.

<u>$c = 8$</u>. The bipartite graphs with 8 vertices and 3 non-trivial components are in Figure 3.

<u>$c = 9$</u>. There is only one bipartite graph with 9 vertices and 4 non-trivial components, which is $3K_2 \cup K_{1,2}$. We will call it graph (31).

<u>$c = 10$</u>. In this case, there is also one bipartite graph with 10 vertices and 5 non-trivial components — the graph consisting of 5 independent edges. We will call it graph (32).

In order to obtain the contributions of these graphs to a_5, it is convenient to put them into categories as follows.

$S_1 = \{1,2,3,4,5,6,18,,19,20,24,25,26,28,29,30,31,32\}$,
$S_2 = \{8,9,10,11,21,23,27\}$, $S_3 = \{13,14,22\}$,
$S_4 = \{15\}$, and $S_5 = \{7,12,16,17\}$.

All the graphs in S_1 contain 5 edges. The only other bipartite graphs with 5 edges are the two graphs of type C_4^{+1}. The number of subgraphs of this type is $(q-4)N_G(C_4)$. Hence

$$N_G(S_1) = \binom{q}{5} - (q-4)N_G(C_4).$$

The contribution of the graphs in S_1 is therefore

$$\theta_1 = (-1)^5 \left[\binom{q}{5} - (q-4)N_G(C_4)\right].$$

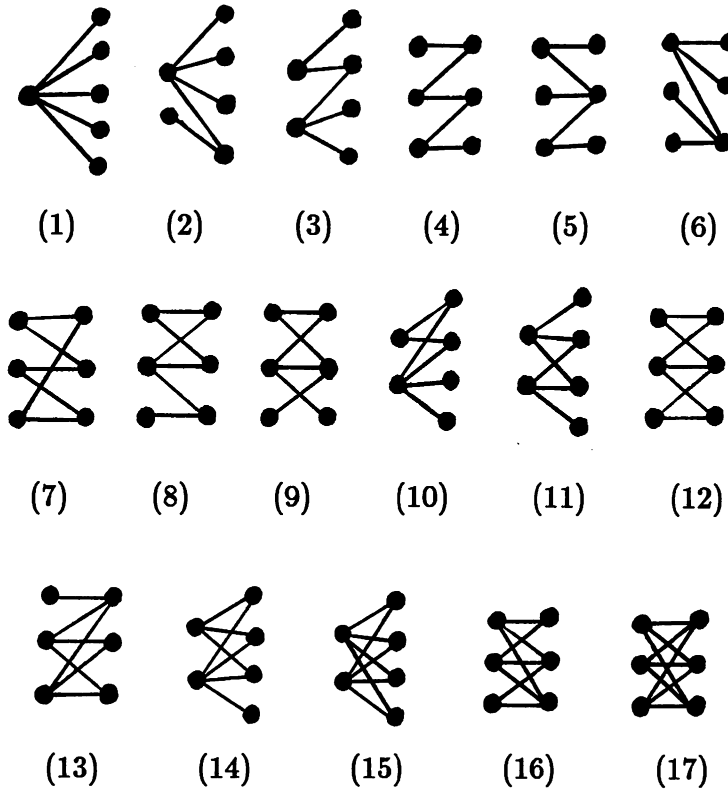


Figure 1.

All the graphs in S_2 are of type C_4^{+2}. The only other bipartite graph which consists of C_4 plus two edges is $K_{2,3}$. Thus

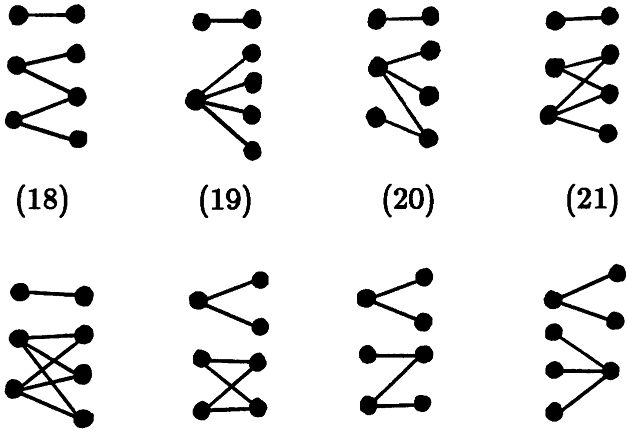
$$N_G(S_2) = \binom{q-4}{2} N_G(C_4) - 3N_G(K_{2,3}),$$

since each $K_{2,3}$ is counted three times in $\binom{q-4}{2} N_G(C_4)$. The contribution of these graphs to a_5 is therefore

$$\theta_2 = (-1)^6 \left[\binom{q-4}{2} N_G(C_4) - 3N_G(K_{2,3}) \right].$$

All the graphs in S_3 are of type $K_{2,3}^{+1}$. There is no other such graph. So

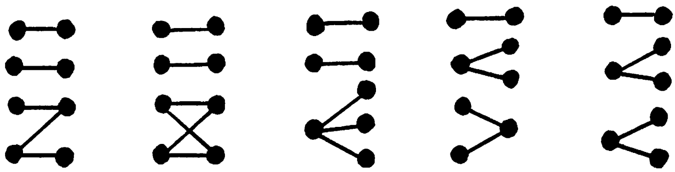
$$N_G(S_3) = (q-6) N_G(K_{2,3})$$



(18) (19) (20) (21)

(22) (23) (24) (25)

Figure 2.



(26) (27) (28) (29) (30)

Figure 3.

and the contribution of the graphs in S_3 to a_5 is therefore

$$\theta_3 = (-1)^7(q-6)N_G(K_{2,3}).$$

Graph (15) in S_4 is the complete bipartite graph $K_{2,4}$. The net contributionof all graph (15)'s to a_5 is

$$\theta_4 = (-1)^8 N_G(K_{2,4}).$$

The graphs in S_5 are C_6, $C_4 \vee C_4$, $K_{3,3}^-$ and $K_{3,3}$. We shall see that the contribution of all these graphs to a_5 is

$$\theta_5 = I_G(C_6) - I_G(K_{3,3}^-) - 4N_G(K_{3,3}).$$

Each occurrence of one of these graphs as a subgraph of G corresponds to an occorrence of a (possibly different) graph in this set as an *induced* subgraph of G. If G has induced subgraphs isomorphic to C_6, then their contribution to a_5 will be $(-1)^6 I_G(C_6)$. An induced subgraph isomorphic to $C_4 \vee C_4$ contains one C_6 and so contributes $(-1)^6 + (-1)^7 = 0$ to a_5. An induced subgraph isomorphic to $K_{3,3}^-$ contains two C_6's and four $C_4 \vee C_4$'s and so contributes $2(-1)^6 + 4(-1)^7 + (-1)^8 = -1$ to a_5; thus the total contribution of all such graphs is $-I_G(K_{3,3}^-)$. Finally, an induced subgraph isomorphic to $K_{3,3}$ contains 6 C_6's, 18 $C_4 \vee C_4$'s and 9 $K_{3,3}^-$'s and so contributes $6(-1)^6 + 18(-1)^7 + 9(-1)^8 + (-1)^9 = -4$ to a_5; thus the total contribution of all such graphs is $-4N_G(K_{3,3})$.

By adding the contributions of all graphs with $p-5$ components and r edges, we get (2) as required. □

The following necessary conditions for two bipartite graphs to be chromatically equivalent can be deduced and are well-known (see, for example [2] and [1]). In the remaining of this section, we give another necessary condition for two bipartite graphs to be chromatically equivalent.

Theorem C. *Let G_1 and G_2 be two chromatically equivalent bipartite graphs. Then*

(i) $|V(G_1)| = |V(G_2)|$;
(ii) $|E(G_1)| = |E(G_2)|$;
(iii) $N_{G_1}(C_4) = N_{G_2}(C_4)$;
(iv) $N_{G_1}(K_{2,3}) = N_{G_2}(K_{2,3})$;
(v) *G_1 is connected if and only if G_2 is connected;*
(vi) *G_1 is 2-connected if and only if G_2 is 2-connected.*

The following necessary condition for two bipartite graphs to be chromatically equivalent follows from Theorems 1 and C.

Theorem 2. *Let G_1 and G_2 be two chromatically equivalent bipartite graphs. Then*

$$I_{G_1}(C_6) + N_{G_1}(K_{2,4}) - I_{G_1}(K_{3,3}^-) - 4N_{G_1}(K_{3,3}) = I_{G_2}(C_6) + N_{G_2}(K_{2,4}) - I_{G_2}(K_{3,3}^-) - 4N_{G_2}(K_{3,3}).$$

Corollary. *Let G be a bipartite graph which has no $K_{2,3}$. If a graph H is chromatically equivalent with G, then $I_H(C_6) = I_G(C_6)$.*

Remark. We are able to show that there are infinitely many pairs of bipartite graphs which satisfy all the conditions in Theorem C but do not satisfy the necessary condition in Theorem 2.

§4. The Seventh Chromatic Coefficient

Let G be a bipartite graph of order p and size q. In order to find the seventh chromatic coefficient of G, we will need all the bipartite graphs with p vertices and $p-6$ components. According to Lemma B, these graphs consist of $p-c$ isolated vertices and $c-6$ non-trivial components, where $7 \le c \le 12$. It is not difficult to confirm that all these graphs (the non-trivial components) can be partitioned into the following 8 categories.

$$S_1 = \Big\{ \text{Forests with 6 edges} \Big\};$$

$$S_2 = \Big\{ \text{Graphs of type } C_4^{+3} \Big\};$$

$$S_3 = \Big\{ \text{Graphs of type } K_{2,3}^{+2} \Big\};$$

$$S_4 = \Big\{ \text{Graphs of type } K_{2,4}^{+1} \Big\};$$

$$S_5 = \Big\{ K_{2,5} \Big\}; \qquad S_6 = \Big\{ K_{3,4}^{-} \text{ and } K_{3,4} \Big\};$$

$$S_7 = \Big\{ \text{Graphs of types } C_6^{+1},\ (C_4 \vee C_4)^{+1},\ (K_{3,3}^{-})^{+1}, \text{ and } K_{3,3}^{+1} \Big\};$$

$$S_8 = \Big\{ \text{The Graphs } H_i\,(i = 1, 2, \ldots, 7) \text{ in Figure 4} \Big\}.$$

We shall now calculate the contributions of all the graphs in S_i $(1 \le i \le 8)$ to a_6.

All the graphs in S_1 contain 6 edges. The only other bipartite graphs with 6 edges are C_4 plus two edges and C_6. Also, the number of subgraphs of G that consist of C_4 plus two edges is $\binom{q-4}{2} N_G(C_4) - 2N_G(K_{2,3})$, since $K_{2,3}$ is counted

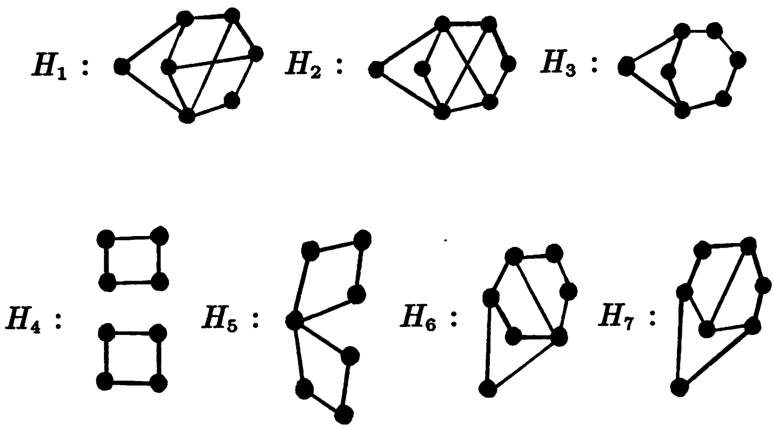


Figure 4.

three times in $\binom{q-4}{2}N_G(C_4)$. Hence

$$N_G(S_1) = \binom{q}{6} - N_G(C_6) - \binom{q-4}{2}N_G(C_4) + 2N_G(K_{2,3}).$$

The contribution of all graphs in S_1 to a_6 is therefore

$$\xi_1 = (-1)^6\left[\binom{q}{6} - N_G(C_6) - \binom{q-4}{2}N_G(C_4) + 2N_G(K_{2,3})\right].$$

All the graphs in S_2 had one C_4 plus three edges. Other such bipartite graphs which are not in S_2 are $C_4 \vee C_4$ and $K_{2,3}$ plus one edge. Hence

$$N_G(S_2) = \binom{q-4}{3}N_G(C_4) - 3(q-6)N_G(K_{2,3}) - 2N_G(C_4 \vee C_4),$$

since each $K_{2,3}$ is counted 3 times and each $C_4 \vee C_4$ is counted twice in $\binom{q-4}{3}N_G(C_4)$. The contribution of all the graphs in

S_2 to a_6 is therefore

$$\xi_2 = (-1)^7 \left[\binom{q-4}{3} N_G(C_4) - 3(q-6)N_G(K_{2,3}) - 2N_G(C_4 \vee C_4)\right]$$

$$= -\binom{q-4}{3} N_G(C_4) + 3(q-6)N_G(K_{2,3}) + 2N_G(C_4 \vee C_4).$$

All the graphs in S_3 contain $K_{2,3}$ plus two edges. Other such bipartite graphs which are not in S_3 are $K_{3,3}^-$ and $K_{2,4}$. Thus

$$N_G(S_3) = \binom{q-6}{2} N_G(K_{2,3}) - 2N_G(K_{3,3}^-) - 4N_G(K_{2,4}),$$

since each $K_{3,3}^-$ and $K_{2,4}$ is counted two and four times, respectively, in $\binom{q-6}{2} N_G(K_{2,3})$. The graphs in S_3 contribute to a_6 by the following amount.

$$\xi_3 = (-1)^8 \left[\binom{q-6}{2} N_G(K_{2,3}) - 2N_G(K_{3,3}^-) - 4N_G(K_{2,4})\right]$$

$$= \binom{q-6}{2} N_G(K_{2,3}) - 2N_G(K_{3,3}^-) - 4N_G(K_{2,4}).$$

The contribution of all the graphs in S_4 to a_6 is

$$\xi_4 = (-1)^9 (q-8)N_G(K_{2,4}) = -(q-8)N_G(K_{2,4}).$$

The contribution of the graph $K_{2,5}$ in S_5 to a_6 is

$$\xi_5 = (-1)^{10} N_G(K_{2,5}) = N_G(K_{2,5}).$$

The graphs in S_6 are $K_{3,4}^-$ and $K_{3,4}$. Each occurrence of one of these two graphs as a subgraph of G corresponds to an occurrence of a graph in this set as an *induced* subgraph of G. If G has an induced subgraphs isomorphic to $K^-_{3,3}$, then their contribution to a_6 will be $(-1)^{11} I_G(K^-_{3,4})$. An induced subgraph isomorphic to $K_{3,4}$ contains 12 $K^-_{3,4}$'s and so contributes $(-1)^{12} + 12(-1)^{11} = -11$ to a_6; thus the total contribution of all such graphs is $-11N_G(K_{3,4})$. Thus the contribution of all graphs in S_6 to a_6 is

$$\xi_6 = -I_G(K^-_{3,4}) - 11N_G(K_{3,4}).$$

Now we consider graphs in S_7. We shall see that the contribution of the four types of graphs in this category to a_6 is

$$\xi_7 = -(q-6)I_G(C_6) + (q-8)I_G(K^-_{3,3}) + 4(q-9)N_G(K_{3,3}).$$

Each occurrence of one of these graphs as a subgraph of G corresponds to an occurrence of a (possibly different) graph in this category as an *induced* subgraph of G. If G has induced subgraph isomorphic to a graph of type C_6^{+1}, then the contribution to a_6 will be $(-1)^7(q-6)I_G(C_6)$. An induced subgraph isomorphic to a graph of type $(C_4 \vee C_4)^{+1}$ contains one graph of type C_6^{+1} and so contributes $(-1)^8 + (-1)^7 = 0$ to a_6. An induced subgraph isomorphic to a graph of type $(K^-_{3,3})^{+1}$ contains 4 subgraphs of type $(C_4 \vee C_4)^{+1}$ and two subgraphs of type C_6^{+1}, and so contributes $4(-1)^8 + 2(-1)^7 + (-1)^9 = 1$ to a_6; thus the total contribution of all graphs of type $(K^-_{3,3})^{+1}$ is $(q-8)I_G(K^-_{3,3})$. Finally, an induced subgraph isomorphic to a graph of type $K^{+1}_{3,3}$ contains 6 subgraphs of type C_6^{+1}, 18 subgraphs of type $(C_4 \vee C_4)^{+1}$ and 9 subgraphs of type $(K^-_{3,3})^{+1}$, and so contributes $6(-1)^7 + 18(-1)^8 + 9(-1)^9 + (-1)^{10} = 4$ to a_6; thus the total contribution of all graphs of type $K^{+1}_{3,3}$ is $4(q-9)N_G(K_{3,3})$.

The net contribution to a_6 of all the graphs in S_8 is

$$\xi_8 = \sum_{i=1}^{5} N_G(H_i) - \sum_{i=6}^{7} N_G(H_i).$$

By adding all the contributions ξ_i $(1 \le i \le 8)$ to a_6, we get

Theorem 3. *Let G be a bipartite graph of order p and size q. Then the coefficient of λ^{p-6} in the chromatic polynomial of G is*

$$\begin{aligned}&\binom{q}{6} - (q-6)I_G(C_6) - N_G(C_6) - \binom{q-3}{3}N_G(C_4)\\&\quad + 2N_G(C_4 \vee C_4) + \left[2 + \frac{1}{2}(q-1)(q-6)\right]N_G(K_{2,3})\\&\quad - (q-4)N_G(K_{2,4}) + N_G(K_{2,5}) + (q-10)I_G(K_{3,3}^-)\\&\quad + (4q-54)N_G(K_{3,3}) - I_G(K_{3,4}^-) - 11N_G(K_{3,4})\\&\quad + \sum_{i=1}^{5} N_G(H_i) - \sum_{i=6}^{7} N_G(H_i).\end{aligned}$$

The next result follows immediately from Theorems 1 and 3.

Theorem 4. *Let G be a bipartite graph with p vertices and q edges. If G has no cycles of length 4, but n of length 6, then*

$$a_5 = -\binom{q}{5} + n, \quad \text{and} \quad a_6 = \binom{q}{6} - qn + 5n.$$

Remark. The coefficient a_5 can also be deduced from Theorem 2 in [3] and the coefficient a_6 (when $n = 1$) can be obtained from Theorem 3 in the same paper.

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