Two Results on the Binding Numbers of Product Graphs

Wayne Goddard[1]
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
USA

and

Henda C. Swart
Department of Mathematics
University of Natal
4001 Durban
South Africa

Abstract

The binding number of a graph G is defined to be the minimum of $|N(S)|/|S|$ taken over all nonempty $S \subseteq V(G)$ such that $N(S) \neq V(G)$. In this paper, two general results for the binding numbers of product graphs are obtained. (1) For any G on m vertices, it is shown that $bind(G \times K_n) = (nm - 1)/(nm - \delta(G) - n + 1)$ for all n sufficiently large. (2) For arbitrary G and for H with $bind(H) \geq 1$, a (relatively) simple expression is derived for $bind(G[H])$.

[1]This research formed part of the first author's Ph.D. thesis and was supported by a grant from the Foundation for Research Development.

ARS COMBINATORIA 34(1992), pp. 119-128

1 Introduction

The concept of the binding number of a graph was introduced by Woodall [8] in 1973. It was an attempt to measure how "well-distributed" the edges of a graph are. Various results have been obtained including: basic results such as bounds; the binding numbers of specific graphs or families thereof, especially of various products of "common" graphs; and conditions on the binding number which (together with other simple conditions) guarantee the presence of a required subgraph.

We return to the calculation of the binding number of a product graph and prove two general results. The first results on product graphs were obtained by Kane, Mohanty and Hales [4]. Subsequently, Wang, Tian and Liu [6] considered more lexicographic products. They [7] also considered some cartesian products, as did Guichard [3] and Luo [5]. Almost all the results have dealt with the case where all the factors are "nice", i.e. complete, cycles, paths or complete bipartite. For example, most of [7] deals with the cartesian product graphs $G \times K_n$, where G is the cartesian product of cycles, or of paths.

In contrast, we show here that for any graph G on m vertices, $bind(G \times K_n) = (nm-1)/(nm-\delta(G)-n+1)$ for all sufficiently large n. We also derive a general expression for the binding number of the lexicographic product $G[H]$ holding for all G and for H with $bind(H) \geq 1$. We then exhibit some simplifications for classes of G and H. As a corollary follow most of the ad hoc results of [4] and [6]. A more general discussion of lexicographic products is to be found in [2].

•••

In this paper we consider only finite undirected graphs without loops or multiple edges. For definitions not given here, see [1].

Let G be a graph with vertex set $V(G)$. For a subset $S \subseteq V(G)$ we let $N(S)$ denote the neighbourhood of S, and $N[S]$ its closed neighbourhood

$S \cup N(S)$. Then $\mathcal{F}(G)$ is defined to be $\{ S : \phi \neq S \subseteq V(G) \,\&\, N(S) \neq V(G) \}$. Further, the *binding number* of G is given by

$$bind(G) := \min_{S \in \mathcal{F}(G)} \frac{|N(S)|}{|S|}.$$

A *binding set* of G is any set $S \in \mathcal{F}(G)$ such that $bind(G) = |N(S)|/|S|$. Also, for any $S \subseteq V(G)$, the *excess* of S, is given by $\mathrm{Exc}(S) = |N(S)| - |S|$. The following well-known results will prove useful:

Proposition 1 [8] *For any graph G on $p(G)$ vertices and with minimum degree $\delta(G)$, $bind(G) \leq (p(G)-1)/(p(G)-\delta(G))$.*

Proposition 2 [8]
a) For $n \geq 1$, $bind(K_n) = n-1$.
b) For $n \geq 3$, $bind(C_n) = 1$ if n is even, and $(n-1)/(n-2)$ if n is odd.
c) For $n \geq 1$, $bind(P_n) = 1$ if n is even, and $(n-1)/(n+1)$ if n is odd.
d) For $1 \leq a \leq b$, $bind(K(a,b)) = a/b$.

2 Cartesian Products

This section is devoted to evaluating $bind(G \times K_n)$ for n sufficiently large. Consider $P = G \times K_n$ with $n \geq 3$, and G a connected graph of order $m \geq 2$ and minimum degree $\delta(G)$.

Our strategy will be as follows. For an $S \in \mathcal{F}(P)$, we will define parameters f, d and u, and observe some constraints on these. We will find a lower bound $e(f,d,u)$ for $\mathrm{Exc}(S)$, and an upper bound $s(f,d,u)$ for $|S|$. Then, subject to the constraints, we evaluate

$$\min_{f,d,u} \frac{e(f,d,u)}{s(f,d,u)}. \tag{1}$$

This is a lower bound for $bind(P) - 1$.

Let $S \in \mathcal{F}(P)$. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$, and denote by B_i the vertex set of the copy of K_n corresponding to v_i ($i = 1, 2, \ldots, m$). Further let $S_i = S \cap B_i$ and $N_i = N(S) \cap B_i$.

We now define the following (weak) partition of $V(G)$:

$$\begin{aligned} F &= \{ v_i \in V(G) : |S_i| = n \} & |F| = f, \\ D &= \{ v_i \in V(G) : 1 < |S_i| < n \} & |D| = d, \\ U &= \{ v_i \in V(G) : |S_i| = 1 \} & |U| = u, \quad \text{and} \\ Z &= \{ v_i \in V(G) : |S_i| = 0 \} & |Z| = z. \end{aligned}$$

The constraints on f, d and u that we use are:

$$f + d + u \leq m, \quad f + d \leq m - 1, \quad \text{and} \quad f \leq m - \delta(G) - 1. \qquad (2)$$

The first is trivially true, and the second is clearly a consequence of $N(S) \neq V(P)$. But so is the third: for if $v_i \in F$ then $N_j = B_j$ for all $v_j \in N_G(v_i)$, and thus $N_G[F] \neq V(G)$. We will say that f, d and u are legal if they satisfy (2).

Lemma 1 *The following table gives lower bound $e(f, d, u)$ on* Exc(S)*:*

case	D	F	U	Exc(S) $\geq$
(a)	ϕ	ϕ	•	$(n-2)u$
(b)	ϕ	•	ϕ	n
(c)	ϕ	•	•	$(n-2)u + 1$
(d)	•	?	ϕ	$n + d - 1$
(e)	•	?	•	$(n-2)u + d$

where ϕ indicates that the set is empty, • that the set is non-empty and ? that it is immaterial whether the set is empty or not.

Proof

We note first that for all i, $|N_i| \geq |S_i|$. Indeed, for all $v_i \in D$, $|N_i| - |S_i| \geq 1$

and for all $v_i \in U$, $|N_i| - |S_i| \geq n - 2$. Then cases (a) and (e) follow immediately. The remainder of the proof is as follows.
(b) By the given conditions, there exist $v_i \in F$ and $v_j \in Z$ which are adjacent so that $|N_j| - |S_j| = n$.
(c) That $\sum_{v_i \in U}(|N_i| - |S_i|) \geq (n-2)u$ follows from the above discussion. Also there exist $v_i \in F$ and $v_j \in U \cup Z$ which are adjacent; if $v_j \in Z$ then $|N_j| - |S_j| = n$, while if $v_j \in U$ then $|N_i| - |S_i| = (n-2) + 1$.
(d) By the given conditions, there exist $v_i \in F \cup D$ and $v_j \in Z$ which are adjacent; if $v_i \in F$ then $|N_j| - |S_j| = n$, while if $v_i \in D$ then $|N_i| + |N_j| - |S_i| - |S_j| \geq |S_i| + n - |S_i| - 0 = n$. The remainder of Exc($S$) follows from (the rest of) D. □

Observe that $|S| \leq s(f,d,u) = nf + (n-1)d + u$. So what remains is to evaluate quantity (1).

Lemma 2 *In evaluating (1) subject to (2), we may assume that* $u, d \neq \phi$.

Proof
In the notation of the previous lemma, we show that case (e) is at least as good as each of the other cases. For case (a) the ratio $e(f,d,u)/s(f,d,u)$ equals $n-2$; this is beaten by $f = 0$ and $d = u = 1$, for example.

For the remainder of the proof, we show that given legal f, d and u in cases (b) through (d), there is always a legal value of d' and u' from case (e) such that $e(f,d,u) \geq e(f,d',u')$ and $s(f,d,u) \leq s(f,d',u')$:

case (b): $d' = u' = 1$;

case (c): $d' = 1$; if $u = 1$ then $u' = 1$ else $u' = u - 1$;

case (d): $u' = 1$ and $d' = d$.

It is easily checked that these values satisfy the requirements. □

Lemma 3 *In evaluating (1) subject to (2), we may assume that* $f = m - \delta(G) - 1$ *and* $u = 1$.

Proof

Suppose $f < m - \delta - 1$. Then f can be incremented, and, if necessary, one of u or d incremented, so that the resultant f', u' and d' are legal and u', $d' > 0$. But then $e(f,d,u) \geq e(f',d',u')$ and $s(f,d,u) \leq s(f',d',u')$.

Further, the ratio $e(f,d,u)/s(f,d,u)$ is increasing in u; hence we should choose $u = 1$. □

Thus we have reduced evaluating (1) to evaluating:

$$\min_{1\leq d\leq\delta} \frac{n+d-2}{n(m-\delta-1)+(n-1)d+1},$$

where $\delta = \delta(G)$.

Now, it can be verified that if $n \geq m + 2 - \delta$, then this expression is minimised at $d = \delta$. In this case

$$\begin{aligned} bind(P) &\geq 1 + \frac{n+\delta-2}{n(m-\delta-1)+(n-1)\delta+1} \\ &= \frac{nm-1}{nm-(n-1+\delta)} \\ &= \frac{p(P)-1}{p(P)-\delta(P)}. \end{aligned}$$

Using Proposition 1, this may be summarised in the following theorem.

Theorem 1 *Let G be a connected graph on m vertices.*
If $n \geq m + 2 - \delta(G)$ then

$$bind(G \times K_n) = \frac{mn-1}{mn-n-\delta(G)+1}.$$

Proof

This follows from the above discussion except in the case where G is trivial in which case $m = 1$ and $\delta(G) = 0$ and the formula follows. □

For example, the binding number of $K_m \times K_n$ follows immediately provided n or m is at least 3. Also, in [7] it is shown that $n \geq 3$ is sufficient if G is the cartesian product of paths or of cycles.

3 Lexicographic Product

We consider the lexicographic product $G[H]$ where $bind(H) \geq 1$.

The vertex sets of G and H are given by $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively. Let S be a subset of $V(G[H])$. Denote for $i = 1, 2, \ldots, m$, $S_i := \{ w_j \in V(H) : (v_i, w_j) \in S \}$, and let $Y := \{v_i \in V(G) : S_i \neq \phi\}$.

Certainly,

$$N(S) = N_G(Y) \times V(H) \cup \bigcup_{v_i \in Y - N_G(Y)} \{v_i\} \times N_H(S_i).$$

Now, let S be a binding set of $G[H]$ (so that $Y \neq \phi$). As $N(S)$ is independent of the choice of S_i for $v_i \in Y \cap N_G(Y)$, and S is obviously a maximum set with the given neighbourhood, it holds that $S_i = V(H)$ for all $v_i \in Y \cap N_G(Y)$. Thus

$$S = (Y \cap N_G(Y)) \times V(H) \cup \bigcup_{v_i \in Y - N_G(Y)} \{v_i\} \times S_i.$$

Observe that $|N_H(S_i)| \geq |S_i|$ for all possible S_i as $bind(H) \geq 1$. Thus for any fixed Y, the ratio $|N(S)|/|S|$ is minimised at $S_i = V(H)$ for all $v_i \in Y - N_G(Y)$. The only question is whether such an S is valid: for $N(S)$ is the whole graph iff $N_G[Y] = V(G)$ and $N_H(S_i) = V(H_i)$ for all $v_i \in Y - N_G(Y)$.

Hence we are guaranteed a binding set such that one of the following holds:

1. $N_G[Y] \neq V(G)$ and $S_i = V(H)$ for all $v_i \in Y$; or

2. $N_G[Y] = V(G)$, and there exists $v_i \in Y - N_G(Y)$ such that $S_i \in \mathcal{F}(H)$ while $S_j = V(H)$ for all $v_j \in Y - \{v_i\}$.

In the latter case

$$\frac{|N(S)|}{|S|} = \frac{(m-1)n + |N_H(S_i)|}{(|Y|-1)n + |S_i|}.$$

For any fixed $S_i \in \mathcal{F}(H)$, this attains a minimum value if Y is of maximum cardinality such that $N_G[Y] = V(G)$ and $N_G(Y) \neq V(G)$, i.e. if $Y = V(G) - N_G(v)$ where v is a vertex of G of minimum degree.

This yields the following theorem:

Theorem 2 *For any graphs G and H such that $bind(H) \geq 1$, $bind(G[H]) = \min\{E_1, E_2\}$ where*

$$\begin{aligned} E_1 &= \min_{\substack{Y \neq \phi \\ N_G[Y] \neq V(G)}} \frac{|N_G[Y]|}{|Y|}, \quad \textit{and} \\ E_2 &= \min_{T \in \mathcal{F}(H)} \frac{(m-1)n + |N_H(T)|}{(m-1-\delta(G))n + |T|}. \end{aligned}$$

We note in passing that the parameter E_1 may be thought of as a "closed binding number."

3.1 Simplifications

Most of the previous results on $G[H]$ have assumed that both G and H are complete graphs, paths, cycles or complete bipartite graphs. The following lemmas recover several of these results, inter alia.

Lemma 4 *If $bind(H) = (n-1)/(n - \delta(H))$ then*

$$E_2 = \frac{nm - 1}{n(m - \delta(G)) - \delta(H)} = \frac{p-1}{p - \delta(G[H])},$$

where p is the order of $G[H]$.

This follows as one obviously takes T to be a binding set of H. This lemma is applicable if, for instance, H is a non-trivial complete graph, path of even order or odd cycle. Another result is that:

Lemma 5 *For all G,*

$$E_1 \geq \frac{m-1}{m - \kappa(G) - 1},$$

with equality iff $\kappa(G) = \delta(G)$ or $\kappa(G) = 0$.

Proof

If G is complete then $E_1 = \infty$. So assume that G is noncomplete. Let $Y \neq \phi$ such that $N_G[Y] \neq V(G)$. Then $G - (N_G[Y] - Y)$ is disconnected so that

$$\frac{|N_G[Y]|}{|Y|} = 1 + \frac{|N_G[Y] - Y|}{|Y|} \geq 1 + \frac{\kappa(G)}{m - \delta(G) - 1},$$

which, as $\kappa(G) \leq \delta(G)$, proves the bound. For equality, take $Y = V(F)$ where F is a component of G if $\kappa(G) = 0$, and $Y = V(G) - N[v]$ where v is a vertex of minimum degree if $\kappa(G) = \delta(G)$. □

This lemma applies to all four classes mentioned above.

References

[1] G. Chartrand & L. Lesniak, "Graphs & Digraphs" (2nd Edition), Wadsworth, Monterey, (1986).

[2] W.D. Goddard, On the Vulnerability of Graphs, Doctoral Thesis, University of Natal, Durban, 1989.

[3] D.R. Guichard, Binding number of the cartesian products of two cycles, *Ars Combin.*, **19** (1985), 175–178.

[4] V.G. Kane, S.P. Mohanty & R.S. Hales, Product Graphs and Binding Number, *Ars Combin.*, **11** (1981), 201–224 .

[5] Y.X. Luo, The binding number of a graph—proof of a conjecture of V.C. Kane et al., *Fuzhou Daxue Xuebao* **2** (1984), 19–24.

[6] J. Wang, S. Tian & J. Liu, The Binding Number of Lexicographic Products of Graphs, *Graph Theory with Applications to Algorithms and Computer Science* (Y. Alavi et al, eds.), Wiley, New York, (1985), 761–776.

[7] J. Wang, S. Tian & J. Liu, The Binding Number of Product Graphs, *Lecture Notes in Maths* **V1073** (1984), 119–128.

[8] D.R. Woodall, The Binding Number of a Graph and its Anderson Number, *J. Combin. Theory (B)*, **15** (1973), 225–255 .