On edge-graceful regular graphs and trees

Sharon Cabaniss
Richard Low
John Mitchem

Mathematics and Computer Science Department
San Jose State University
San Jose, CA 95192

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $p = |V(G)|$ and $q = |E(G)|$. In 1985, Lo [5] defined the edge-graceful labeling of G where $\ell: E(G) \to \{1, 2, \ldots, q\}$ is one-to-one and ℓ induces a label on the vertices defined by ℓ^*:

$$\ell^*(v) = \sum_{uv \in E(G)} \ell(uv)(\text{modulo} p).$$

The labeling is *edge-graceful* if all vertex labels are distinct modulo p in which case G is called an edge-graceful graph.

In his 1985 paper, Lo showed that if graph G is edge-graceful then p divides $(q^2 + q + p(p-1)/2)$.

In this paper for positive integers n and t, K_n denotes the complete graph on n vertices, C_n denotes the cycle graph with $|V(C_n)| = n = |E(C_n)|$, and C_n^k denotes the kth power of C_n. That is, $V(C_n^k) = V(C_n)$ and $uv \in (C_n^k)$ if and only if $d(u, v) \leq k$ where $d(u, v)$ denotes the distance from vertex u to vertex v in C_n.

2. Regular Graphs

In [4], Lee, Seah and Wang conjectured that the kth power of C_n is edge-graceful for odd n. At the 1989 Southeastern Conference on Combinatorics, Graph Theory and Computing, Sin-Min Lee and Eric Seah presented a "proof" of this conjecture. A flaw was discovered in the proof given in the paper available after the conference. Specifically, their "proof" relied upon the obviously false statement that if $gcd(n, k) \neq 1$, then $gcd(n, k+1) = 1$. We prove the conjecture by showing that C_n^k is edge-graceful for n odd and $1 \leq k < \frac{n}{2}$. From this, the known results that C_n, C_n^2 and K_n are edge-graceful for odd n are easy corollaries. The proofs make use of the simple observation that if $q = kp$, $k \geq 1$, then an edge-graceful labeling with edges labeled 1 to q is equivalent to using each edge label $1, 2, \ldots, p$ exactly k times so that the induced vertex labeling yields distinct labels modulo p. For example, for $C_5^2 = K_5$ we get the labeling shown in Figure 1.

We have recently received a preprint [1] of a paper by Ho, Lee, and Seah in which they give a very nice theorem which generalizes Theorems 1 and 4 given below.

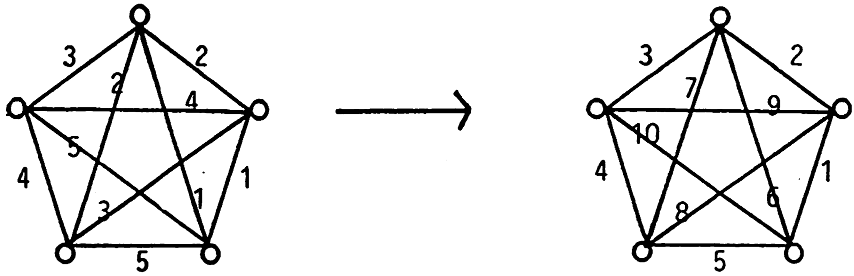

Figure 1.

Theorem 1. *Let n be odd and $1 \leq k < \frac{n}{2}$, then C_n^k is edge-graceful.*

Proof: Let r be the positive integer such that $2^{r-1} < k \leq 2^r$. Let $t = 2^r - k \geq 0$ and let $s = k - t = 2k - 2^r > 0$. Then $s + t = k$ and $s + 2t = 2^r$.

Label the edges of G as follows for all i, $1 \leq i \leq n$:

$$\ell(v_1, v_{i+1}) = \ell(v_i, v_{i+2}) = \cdots = \ell(v_i, v_{i+t}) = 2_i$$
$$\ell(v_i, v_{i+t+1}) = \ell(v_i, v_{i+t+2}) = \cdots = \ell(v_i, v_{i+t+s}) = i.$$

This is a legitimate edge labeling since each label i, $1 \leq i \leq n$, is used exactly k times. That is because the label i is used s times on edges incident to v_i, and the label $2i$ is used t times on edges incident to v_i. Since $gcd(n, 2) = 1$, $\{2 \cdot 1, 2 \cdot 2, \ldots, 2 \cdots n\}$ are all distinct in Z_n. Thus each i is used exactly $t + s = k$ times.

This edge labeling induces distinct labels on the vertices. Figure 2 shows the labeling of edges incident to v_i.

Note that

$$\begin{aligned}\ell(v_i) &= t(2i) + s(i) + t(2i) + s(i) - 2(1 + 2 + \cdots + t) \\ &\quad - ((t + 1) + (t + 2) + \cdots + (t + s)) \\ &= i(4t + 2s) - 2(1 + 2 + \cdots + t) - ts - (1 + \cdots + s)\end{aligned}$$

Similarly, $\ell(v_{i+1}) = (i + 1)(4t + 2s) - 2(1 + 2 + \cdots + t) - ts - (1 + \cdots + s)$. The difference between successive vertex labels is $\ell(v_{i+1}) - \ell(v_i) = 4t + 2s = 2(2t + s) = 2 \cdot 2^r = 2^{r+1}$. Since $gcd(n, 2^{r+1}) = 1$, 2^{r+1} generates Z_n and the n vertex labels are distinct.

Thus, C_n^k is edge-graceful for n odd. ∎

This theorem yields previously known results as simple corollaries:

Corollary 2. [3] K_n is edge-graceful for n odd.

Proof: Set $k = \frac{(n-1)}{2}$ in the theorem so that $C_n^k = K_n$. ∎

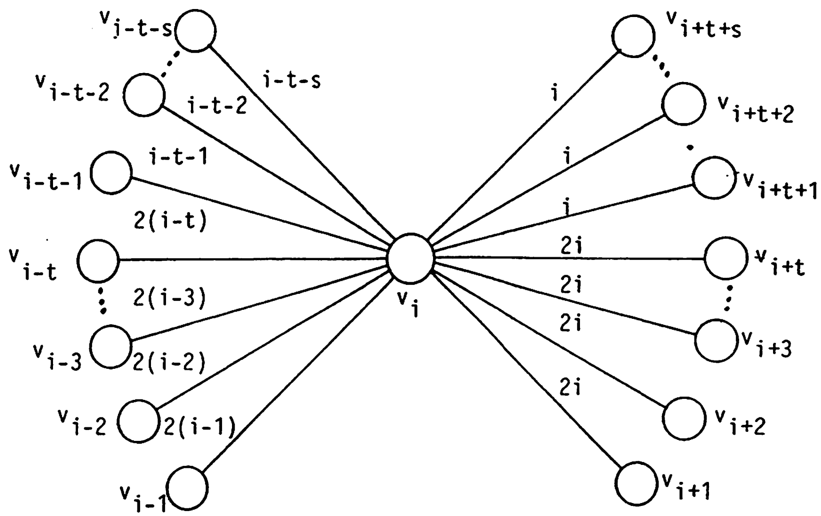

Figure 2

Corollary 3 [5] C_n edge-graceful for n odd.

Proof: Set $k = 1$ in the theorem. ∎

Let $C_{n,k}$ denote the graph G such that $V(G) = V(C_n)$ and $E(G) = E(C_n) \cup \{(v_i, v_j) \mid d(v_i, v_j) = k \text{ in } C_n\}$. For example, $C_n^2 = C_{n,2}$.

Theorem 4. *$C_{n,k}$ is edge-graceful for odd n and k positive.*

Proof: For all i, $1 \leq\leq n$, label the edges as follows:

$$\ell(v_i, v_{i+1}) = \ell(v_i, v_{i,k}) = i.$$

Thus for all i there are exactly two edges labeled i so we have a legitimate labeling of the edges.

This edge labeling induces distinct vertex labels because $\ell(v_i) = i+i+(i-1)+(i-k) = 4i-(k+1)$ and $\ell(v_{i+1}) = 4(i+1)-(k+1)$. The difference between successive vertices is $\ell(v_{i+1}) - \ell(v_i) = 4$. Since $gcd(n, 4) = 1$, 4 generates Z_n and G is edge-graceful. ∎

3. Trees

In the remainder of the paper we consider the problem of determining which trees are edge-graceful. Lo's condition is useful as it implies that any edge-graceful tree must have odd order. In [2] Lee conjectured that all trees of odd order are edge-graceful. Although this conjecture is far from settled, in the remainder of this paper we prove that two large classes of odd order trees are edge-graceful.

A *spider* is a tree with a unique core vertex c of degree greater than 2 and all other vertices of degree at most 2. A spider is called *regular* if the distance from the core vertex to all end vertices is the same.

In [6] Small showed that all regular spiders of odd order are edge-graceful. His proof that regular spiders of odd order with an even number of legs are edge-graceful is quite transparent. However, his proof that regular spiders of odd order with an odd number of legs are edge-graceful is considerably more complicated. We give an easier proof that Small's algorithm for regular spiders of odd order with an odd number of legs yields an edge-graceful labeling.

Theorem 5 [6]. *Any regular spider T of odd order is edge-graceful.*

Proof: As Small's proof is easy to follow when T has an even number of legs we consider only T with an odd number of legs.

Let T be a spider with $2k+1$ legs of length $2n$. Then $p = (2k+1)2n+1$ and $q = (2k+1)2n$. We divide the set of legs into one special leg and k pairs of legs. Furthermore, we divide the edge labels into inverse pairs modulo p. For each of the k leg pairs, we will describe the edge labeling of one leg and the other leg will be labeled by the corresponding inverses. Thus, if the tth edge from the core is labeled with a, in the corresponding leg the tth edge will be labeled with $-a$ (modulo p).

We first label consecutively the edges of the special leg starting with the edge incident with the core vertex: $n\cdot n, -1\cdot n, n(n-1), -2n, n(n-2), -3n, \ldots, n\cdot 1, -n\cdot n$. It is clear that the edges of the special leg are labeled with $n, 2n, \ldots, n^2$ and their inverses while its vertices are labeled $n, 2n, \ldots, n^2$ and their inverses and zero.

We use a matrix to describe the label of one leg from each leg pair. Each column read from the top describes the labeling of the consecutive edges of a leg starting with an edge incident with the core vertex. The matrix below describes the labeling when k is odd.

1	2	3	4	...	k
n^2+n	$5n^2$	$5n^2+n$	$9n^2$	...	$(2k-1)n^2+n$
$2n^2+n$	$4n^2$	$6n^2+n$	$8n^2$	...	$2kn^2+n$
n^2+2n	$5n^2-n$	$5n^2+2n$	$9n^2-n$	...	$(2k-1)n^2+2n$
$2n^2+2n$	$4n^2-n$	$6n^2+2n$	$8n^2-n$	...	$2kn^2+2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n^2+(n^2-n)$	$4n^2+2n$	$6n^2-n$	$8n^2+2n$	...	$2kn^2-n$
$2n^2+(n^2-n)$	$3n^2+2n$	$7n^2-n$	$7n^2+2n$	...	$(2k+1)n^2-n$
$n^2+n^2=2n^2$	$4n^2+n$	$6n^2$	$8n^2+n$	...	$2kn^2$
$2n^2+n^2=3n^2$	$3n^2+n$	$7n^2$	$7n^2+n$	...	$(2k+1)n^2$

If k is even, the last column is reversed.

We will next show that all edges have different labels modulo p. First consider the labels $1n, 2n, \ldots, n^2$ on the special leg and the labels given in the matrix. Recall that the inverse of each of the labels in the matrix is also used to label other edges. Each of the labels under consideration is a multiple of n, and the largest coefficient used is $(2k+1)n = \frac{1}{2}q < \frac{1}{2}p$. Since $\gcd(p, n) = 1$, it follows that all labels under consideration are distinct and thus their inverses are all distinct. We must also show that none of the labels is the inverse of another. In order to do this, we assume the opposite that $tn = -sn$ where $1 \leq t, s \leq \frac{q}{2}$. But then $(t+s)n = 0$ while $t + s \leq q < p$, which is impossible. Thus, all edge labels are distinct modulo p.

We must next show that each vertex label is distinct modulo p. First note that the core vertex is labeled n^2 and the middle vertex of the special leg is labeled 0. The matrix below gives the vertex labels induced by the edge labels given in the above matrix. Each column starts with the vertex adjacent to the core. Again, the matrix is given for k odd.

1	2	3	4	...	k
$3n^2+2n$	$9n^2$	$11n^2+2n$	$17n^2$	...	$(4k-1)n^2+2n$
$3n^2+3n$	$9n^2-n$	$11n^2+3n$	$17n^2-n$	...	$(4k-1)n^2+3n$
$3n^2+4n$	$9n^2-2n$	$11n^2+4n$	$17n^2-2n$	...	$(4k-1)n^2+4n$
$3n^2+5n$	.	.	.	...	.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$5n^2-2n$	$7n^2+4n$	$13n^2-2n$	$15n^2+4n$	...	$(4k+1)n^2-2n$
$5n^2-n$	$7n^2+3n$	$13n^2-n$	$15n^2+3n$	...	$(4k+1)n^2-n$
$5n^2$	$7n^2+2n$	$13n^2$	$15n^2+2n$	...	$(4k+1)n^2$
$3n^2$	$3n^2+n$	$7n^2$	$7n^2+n$	...	$(2k+1)n^2$

If k is even, the last element in the last column is $(2k-1)n^2 + n$ and the other elements in the last column are reversed.

The labels in the matrix and the labels $1n, 2n, \ldots, n^2$ on the vertices of the special leg are all multiples of n and their largest coefficient is $(4k+1)n < p$. Again, since $\gcd(n, p) = 1$, it follows that these vertex labels are all distinct. Furthermore, each label is of the form $tn^2 + rn$ where $0 \leq r \leq n$ and t is of the form $4x$ or $4x - 1$. Also note that if a and b are the labels on the two edges incident with some vertex w, then $\ell(w) = a + b$ (modulo p) and the corresponding edges are labeled $-a$ and $-b$ as stated earlier. Thus the vertex corresponding to w is labeled $(-a) + (-b) = -(a+b)$ (modulo p) so that the labels on the corresponding vertices of the corresponding leg pairs are inverses of each other. Hence the only way for two vertices to have the same label is if one label, L_2, is

L_1. Then $t_1 n^2 + r_1 n = L_1 = -L_2 = -(t_2 n^2 + r_2 n)$ where $0 \le r_1, r_2 \le n$ and both t_1 and t_2 are of the form $4x$ or $4x-1$. Thus, $(t_1+t_2)n^2 + (r_1+r_2)n = 0$ (modulo p) so that $((t_1+t_2)n^2 + (r_1+r_2)n$ is a multiple of $p = (4k+2)n+1$. Dividing by p, it follows that $(t_1+t_2) = (r_1+r_2)(4k+2)$. But $(t_1+t_2) \le (4k+1)+(4k+1) = 2(4k+2)-2$. Thus $(r_1+r_2) = 0$ or 1, and since (r_1+r_2) is not 0, it must be 1. Without loss of generality, let $r_1 = 0$. Then $L_1 = t_1 n^2$ and $L_2 = t_2 n^2 + n$. But then L_1 and L_2 are either both on the special leg or both in the last row of the vertex label matrix. Thus, $4k+2 = t_1 + t_2 < 2(2k+1)$, which is impossible. Hence all vertex labels are distinct modulo p and the graph T is edge-graceful. ∎

Theorem 6. *Let T be an odd tree with a root of even degree. If T has no adjacent degree 2 vertices, no two degree 2 vertices with the same parent, and an even number of non-root degree 2 vertices then T is edge-graceful.*

Before proving Theorem 6, we give several corollaries to it. The first which we discuss, Corollary 7, was stated by Lee in [2]. His "proof" given there, however, is not correct. The "proof" relies on choosing certain positive integer solutions to particular diophantine equations. However, it is easy to find trees whose corresponding equations have no appropriate solutions.

Corollary 7. *If T is an odd tree with at most one vertex of degree 2, then T is edge-graceful.*

Proof: Since T has odd order it has an odd number of even degree vertices. If one of them is of degree 2, let it be the root; otherwise, let any even vertex be the root. Now the tree satisfies the conditions of Theorem 6 and hence is edge-graceful. ∎

We define a rooted tree to be a *full n-ary tree*, $n \ge 2$, if every vertex has either n or no outgoing edges. The following then follows immediately from Corollary 7.

Corollary 8. *For $n \ge 2$, any odd full n-ary tree is edge-graceful.*

Corollary 9. *If T is an odd tree with a root of even degree at least 4 such that no two degree 2 vertices are adjacent and no two degree 2 vertices have a common parent, then T is edge-graceful.*

Proof: If T has an even number of degree 2 vertices, then it satisfies the conditions of Theorem 6 and is edge-graceful. If T has an odd number of degree 2 vertices, choose one such vertex v at minimum distance from the root. Let v be the new root of T. Now each degree 2 vertex except v still has its original parent. Thus, T is edge-graceful because the conditions of Theorem 6 hold. ∎

Proof (Theorem 6): We first describe our general labeling procedure and then show that the procedure can be accomplished. The edge labels $1, 2, 3, \dots, p-1$ are partitioned into inverse pairs modulo p, $\{1, p-1\}, \{2, p-2\}, \dots, \{\lfloor p/2 \rfloor, \lceil p/2 \rceil\}$, which will be used in our labeling. If a vertex u is the parent of vertex

v, then we will say that edge $e = uv$ is *outgoing* from u and *incoming* to v. The root r and the odd vertices of degree larger than 1 will have an even number of outgoing edges. These edges will be labeled with distinct inverse pairs. Non-root even vertices of degree at least 4 will have three outgoing edges labeled respectively $a_1, a_2, -a3$ where $a_1 + a_2 - a_3 \equiv 0$ (modulo p). The other outgoing edges from such vertices will be labeled with inverse pairs. Finally, non-root degree 2 vertices will be paired and the outgoing edges from each pair will be labeled with inverse pairs.

This labeling on the edges will induce a label of 0 on the root, and each degree 1 vertex will be labeled with the label of its incoming edge. In fact, each non-root vertex of degree different from 2 will be labeled with the label of its incoming edge. Finally, if v_1 and v_2 are paired degree 2 vertices, then the label on v_1 (respectively v_2) will be the label on the incoming edge to v_2 (respectively v_1). Thus, if distinct labels $1, 2, \ldots, p-1$ are used on the edges in the prescribed manner, then the induced labeling on the vertices will use $0, 1, \ldots, p-1$.

We now consider five different types of subtrees of T which partition the edges of T.

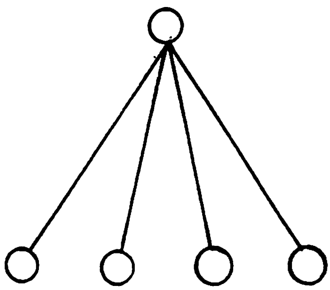

Figure 3

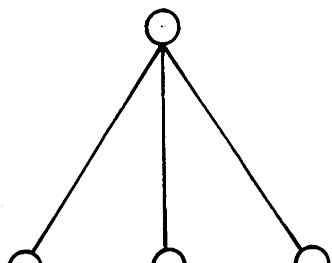

Figure 4.

Type 1. (Figure 3) A non-root vertex with an even number of outgoing edges and no degree 2 children.

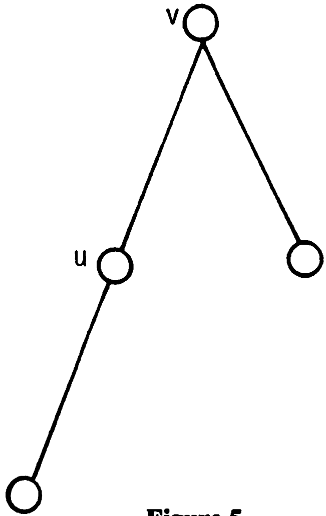

Figure 5.

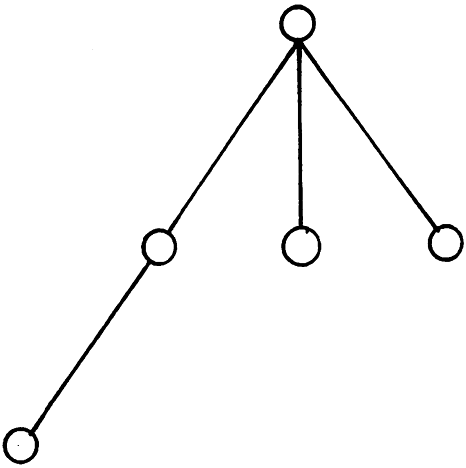

Figure 6.

Type 2. (Figure 4) A non-root vertex with an odd number at least 3 of outgoing edges and no degree 2 children.
Type 3. (Figure 5) A vertex v (root or non-root) with an even number of outgoing edges and exactly one degree 2 child u and the outgoing edge from u. Note that if v is the root it must have at least 4 outgoing edges.
Type 4. (Figure 6) A non-root vertex with an odd number at least 3 of outgoing edges and exactly one degree 2 child u and the outgoing edge from u.

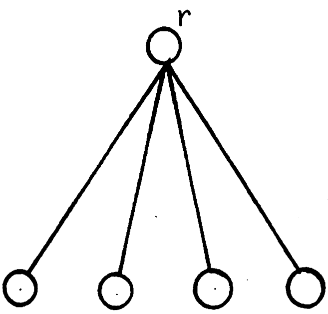

Figure 7.

Type 5. (Figure 7) The root r, if it has no degree 2 children.

We let the edges of T be partitioned into t_i subtrees of type i, $1 \leq i \leq 5$. It follows that $p-1 = q \geq 2t_1+3t_2+3t_3+4t-4+2$. If $t_5 = 1$, then the constant 2 comes from the root's outgoing edges which are not counted by t_3. If $t_5 = 0$, then the constant 2 comes from one of the type 3 trees having at least 2 extra outgoing edges. Now we partition the set of type 3 subtrees into pairs (as much as possible) and also partition the set of type 4 subtrees into pairs (as much as possible) in order to label the pairs in the following manner where $a_1 + a_2 - a_3 = 0 \pmod p$):

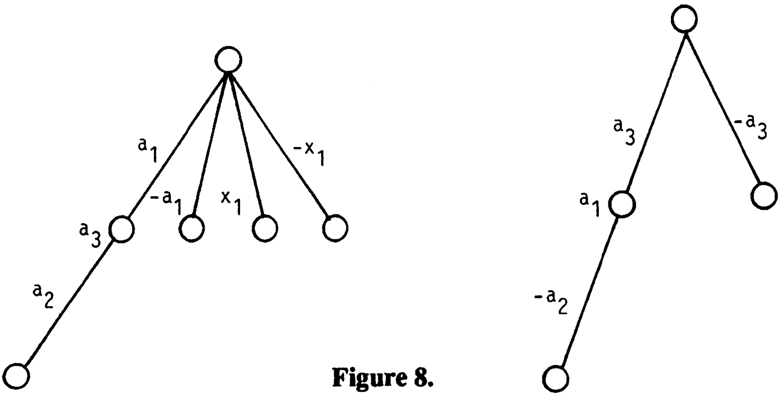

Figure 8.

Figure 8 shows the labeling on pairs of type 3.

Figure 9 shows the labeling on pairs of type 4.

Thus, in addition to using inverse pairs in our labeling, we make frequent use of triples $\{a_1, a_2, a_3\}$ where $a_1 + a_2 - a_3 \equiv 0$ (modulo p), and their inverses $\{-a_1, -a_2, -a_3\}$. We now consider two cases in order to show that we can partition the edge labels $1, 2, \ldots, p-1$ in such a way as to accomplish the labeling described above. Specifically, we will form disjoint triples from $\{1, 2, \ldots, \frac{(p-1)}{2}\}$. Of course, the inverse of such a triple will form another triple from $\{\frac{(p+1)}{2}, \ldots, p-1\}$.

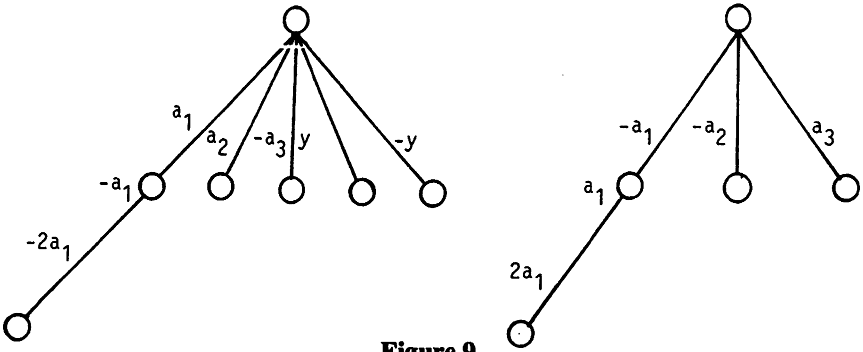

Figure 9.

The edges of a type 1 subtree will be labeled with inverse pairs. Three edges of a type 2 subtree will be labeled with a triple and the remaining edges of that subtree will be labeled with inverse pairs.

Case 1. t_3 and t_4 are both even.

Since T has odd order it must have an odd number of even vertices. Since r is an even vertex and t_4 is even, it follows that t_2 is even. Thus let $t_i = 2k_i, i = 2,3,4$. Then the set of subtrees of each type i, $i = 2,3,4$, can be partitioned into pairs of subtrees. For each pair we will use a triple $\{a_1, a_2, a_3\}$ $(a_1 + a_2 - a_3 \equiv 0$ (modulo p)) and its inverse. It remains to show that we can form the appropriate triples. We need to form $k_2 + k_3 + k_4$ triples from the set $\{1,2,3,\ldots,\frac{(p-1)}{2}\}$ where $p-1 \geq 2t_1 + 3t_2 + 3t_3 + 4t_4 + 2 = 2t_1 + 6k_2 + 6k_3 + 8k_4 + 2$. Thus we form our triples from $\{1,2,3,\ldots,t_1 + 3k_2 + 3k_3 + 4k_4 + 1\}$ and we consider two subcases:

Subcase 1. $k_2 + k_3 + k_4 = 2s$.

In this subcase we form the following triples:

$$\begin{array}{ccc}
3s+1 & 3s & 1 \\
3s+2 & 3s-1 & 3 \\
\vdots & \vdots & \vdots \\
4s-1 & 2s+2 & 2s-3 \\
4s & 2s+1 & 2s-1
\end{array}$$

$$\begin{array}{ccc}
5s+2 & 5s & 2 \\
5s+3 & 5s-1 & 4 \\
\vdots & \vdots & \vdots \\
6s & 4s+2 & 2s-2 \\
6s+1 & 4s+1 & 2s
\end{array}$$

All numbers from 1 to $6s + 1$ are used except $5s + 1$. Also $\frac{(p-1)}{2} \geq 6s + 1 +$

$t_1 + k_4$. Thus $6s+2, 6s+3, \ldots, 6s+1+t_1+k_4$ are also unused numbers between and $\frac{(p-1)}{2}$.

For each of the k_4 triples $\{a_1, a_2, a_3\}$ to be used with type 4 subtrees we must also find a corresponding $2a_1$ which is not part of any selected triple or its inverse triple. We use the integers $6s+2, 6s+4, \ldots, 6s+2k_4$ as shown in the following:

$$
\begin{array}{cccc}
3s+1 & 3s & 1 & 6s+2 \\
3s+2 & 3s-1 & 3 & 6s+4 \\
\vdots & \vdots & \vdots & \vdots \\
4s-1 & 2s+2 & 2s-3 & 8s-2 \\
4s & 2s+1 & 2s-1 & 8s \\
 & & & \\
6s+1 & 4s+1 & 2s & 8s+2 \\
6s & 4s+2 & 2s-2 & 8s+4 \\
\vdots & \vdots & \vdots & \vdots \\
. & . & . & 6s+2k_4 \\
5s+3 & 5s-1 & 4 & \\
5s+2 & 5s & 2 &
\end{array}
$$

The numbers $6s+2, 6s+4, 6s+6, \ldots, 6s+2k_4$ which are less than or equal to $\frac{(p-1)}{2}$ are clearly not part of any selected triple or its inverse triple. However, suppose $6s+2k_4 > \frac{(p-1)}{2} \geq 6s+1+t_1+k_4$. Then $-(6s+2k_4) = p-6s-2k_4 \geq (12s+3+2t_1+2k_4) - (6s+2k_4) = 6s+3+2t_1 > 6s+1$. It follows that none of $6s+2, 6s+4, \ldots, 6s+2k_4$ is part of any selected triple or its inverse triple. Thus in this subcase we can choose the labeling of edges as described and T is edge-graceful.

Subcase 2. $k_2 + k_3 + k_4 = 2s+1$.

Here we form the following triples (the first three columns) and the corresponding fourth number.

$$
\begin{array}{cccc}
3s+3 & 3s+2 & 1 & 6s+6 \\
3s+4 & 3s+1 & 3 & 6s+8 \\
\vdots & \vdots & \vdots & \vdots \\
4s+2 & 2s+3 & 2s-1 & 8s+4 \\
4s+3 & 2s+2 & 2s+1 & 8s+6 \\
6s+4 & 4s+4 & 2s & 8s+8 \\
6s+3 & 4s+5 & 2s-2 & 8s+10 \\
\vdots & \vdots & \vdots & \vdots \\
. & . & . & 6s+4+2k_4 \\
5s+6 & 5s+2 & 4 & \\
5s+5 & 5s+3 & 2 &
\end{array}
$$

All numbers from 1 to $6s + 4$ are used except $5s + 4$. Also, $\frac{(p-1)}{2} \geq 6s + 4 + t_1 + k_4$. Thus $6s + 6, 6s + 8, \ldots, 6s + 4 + t_1 + k_4$ are also unused numbers between 1 and $\frac{(p-1)}{2}$. Also $-(6s + 4 + 2k_4) = p - (6s + 4 + 2k_4) = 12s + 2t_1 + 2k_4 + 9 - (6s + 4 + 2k_4) = 6s + 2t_1 + 5 > 6s + 4$. That is, the inverse of the largest number used in column 4 is larger than the largest number in a triple. It follows that none of $6s + 6, 6s + 8, \ldots, 6s + 4 + 2k_4$ is part of any selected triple or its inverse triple. Thus in this subcase T is edge-graceful.

Case 2. t_3 and t_4 are both odd.

T has an odd number of even vertices. Since r is an even vertex and t_4 is odd, it follows that t_2 is odd. Thus let $t_i = 2k_i + 1, i = 2, 3, 4$. Then the set of subtrees of each type $i, i = 2, 3, 4$, can be partitioned into pairs of subtrees with exactly one subtree of each type not paired. For each pair we will use a triple $\{a_1, a_2, a_3\}$ with $a_1 + a_2 - a_2 \equiv 0$ (modulo p) and its inverse triple as in Case 1. We will label the three unpaired trees as in Figure 10 where $b = a_1 + c$.

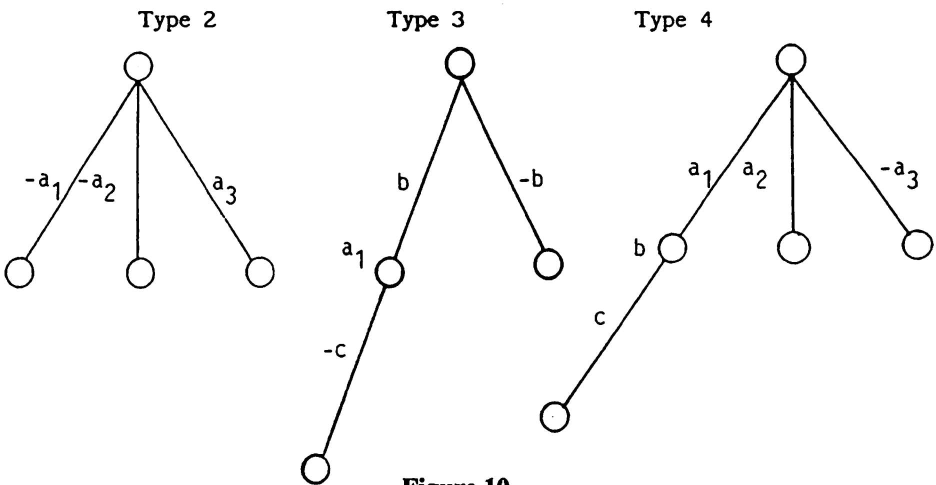

Figure 10.

As in Case 1, we need $k_2 + k_3 + k_4$ triples from the set $\{1, 2, \ldots, \frac{(p-1)}{2}\}$ such that k_4 of the triples $\{a_1, a_2, a_3\}$ have a corresponding $2a_1$ which is not part of any selected triple or its inverse triple. However, we also need one additional triple $\{a_1, a_2, a_3\}$ and its inverse triple as well as two special inverse pairs, $\{b, -b\}$ and $\{c, -c\}$, such that $b = a_1 + c$.

In this case $p - 1 \geq 2t_1 + 3(2k_2 + 1) + 3(2k_s + 1) + 3(2k_3 + 1) + 4(2k_2 + 1) + 2 = 2t_1 + 6k_2 + 6k_3 + 6k_4 + 12$. Thus $\frac{(p-1)}{2} \geq t_1 + 3k_2 + 3k_3 + 4k_4 + 6$.

Subcase 1. $k_2 + k_3 + k_4 + 1 = 2s$.

Of our $k_2 + k_3 + k_4 + 1$ triples we will also have $k_4 + 1$ additional corresponding numbers of the form $2a_1$. Then one of these triples will not be used with its corresponding $2a_1$. Instead, it will be used with special inverse pairs $\{b, -b\}$

and $\{c, -c\}$ for labeling the three non-paired subtrees of types 2,3 and 4 as in Figure 10.

We form the following triples and their corresponding $2a_1$'s.

$$
\begin{array}{cccc}
3s+1 & 3s & 1 & 6s+2 \\
3s+2 & 3s-1 & 3 & 6s+4 \\
\vdots & \vdots & \vdots & \vdots \\
4s-1 & 2s+2 & 2s-3 & 8s-2 \\
4s & 2s+1 & 2s-1 & 8s \\
6s+1 & 4s+1 & 2s & 8s+2 \\
6s & 4s+2 & 2s-2 & 8s+4 \\
\vdots & \vdots & \vdots & \vdots \\
. & . & . & 6s+2+2k_4 \\
 & & & \\
. & . & . & \\
5s+3 & 5s-1 & 4 & \\
5s+2 & 5s & 2 &
\end{array}
$$

All numbers from 1 to $6s+1$, except $5s+1$, are used. Since $\frac{(p-1)}{2} \geq t_1 + 6s + k_4 + 6$, the numbers $6s+2, 6s+3, \ldots, 6s+t_1+k_4+6$ are also unused numbers between 1 and $\frac{(p-1)}{2}$. We take our special inverse pairs from $c = 5s+1$ and $b = 6s+3$ which are unused. Thus the triple $\{a_1, a_2, a_3\}$ containing $a_1 = s+2$ is used to get $b = a_1 + c$ as required. It remains to show that b is not used as a $2a_1$ for some triple or its inverse triple. The largest value of any $2a_1$ is $6s+2+2k_4$ and $p - (6s+2+2k_4) \geq (2t_1 + 6(2s-1) + 2k_4 + 13) - (6s+2+2k_4) = 2t_1 + 6s + 5 > 6s+3$. It follows that $6s+3$ is not used as a value of $2a_1$ and that no value of $2a_1$ is part of any selected triple on its inverse triple. Thus in this subcase we can choose the labeling on edges as described and T is edge-graceful.

Subcase 2. $k_2 + k_3 + k_4 + 1 = 2s+1$.

Again $k_4 + 1$ of our triples will have corresponding $2a_1$'s as in the previous subcase, and one of these triples will not be used with its corresponding $2a_1$

$$
\begin{array}{cccc}
3s+3 & 3s+2 & 1 & 6s+6 \\
3s+4 & 3s+1 & 3 & 6s+8 \\
\vdots & \vdots & \vdots & \vdots \\
4s+2 & 2s+3 & 2s-1 & 8s+4 \\
4s+3 & 2s+2 & 2s+1 & 8s+6 \\
6s+4 & 4s+4 & 2s & 8s+8 \\
6s+3 & 4s+5 & 2s-2 & 8s+10 \\
\vdots & \vdots & \vdots & \vdots \\
. & . & . & 6s+6+2k_4 \\
5s+6 & 5s+2 & 4 & \\
5s+5 & 5s+3 & 2 &
\end{array}
$$

All numbers from 1 to $6s+4$ are used except $5s+4$. Also, $\frac{(p-1)}{2} \geq t_1 + 6s + k_4 + 6$ so $6s+6, 6s+8, \ldots, 6s+6+t_1+k_4$ are unused numbers from the set $\{1, 2, \ldots, \frac{(p-1)}{2}\}$. We take our special inverse pairs from $c = 5s+4$ and $b = 6s+5$. Thus, the triple $\{a_1, a_2, a_3\}$ with $a_1 = s+1$ will be used to get $b = a_1 + c$ as required. Since the largest value of $2a_1$ is $6s+6+2k_4$, we must show that its inverse is greater than b. But, $p - (6s+6+2k_4) \geq 2t_1 + 12s + 2k_4 + 13 - 6s - 6 - 2k_4 = 2t_1 + 6s + 7 > 6s + 5 = b$. Thus, $6s+5$ is not used as a value of $2a_1$ and no value of $2a_1$ is also part of a triple or its inverse. In this subcase also we can choose the labeling on edges as described and T is edge-graceful.

A proof of the conjecture that all odd trees are edge-graceful appears distant. However, it is interesting to compare Theorems 5 and 6. Theorem 5 essentially shows that odd trees with lots of degree 2 vertices and a certain symmetry are edge-graceful. Theorem 6 shows that odd trees with only scattered degree 2 vertices are edge-graceful.

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