THE MINIMUM SIZE OF A MAXIMAL PARTIAL PLANE

ZOLTÁN FÜREDI* AND L. SPISSICH†

*Mathematical Institute of the Hungarian Academy of Sciencies, 1364 Budapest, P. O. B. 127, Hungary

and

†8500 Pápa, Koltói A. u. 21., Hungary

Abstract. A partial plane of order n is a family $\mathcal{L}$ of $n+1$-element subsets of an n^2+n+1-element set, such that no two sets meet more than 1 element. Here it is proved, that if $\mathcal{L}$ is maximal, then $|\mathcal{L}| \geq \lfloor 3n/2 \rfloor + 2$, and this inequality is sharp.

1. EXAMPLES FOR MAXIMAL PARTIAL PLANES

Let n be a positive integer, P a set of n^2+n+1 elements. It will be convenient to set $P = \{1, 2, \ldots, n^2+n+1\}$. A family $\mathcal{L}$ of $(n+1)$-element subsets of P is called a *partial plane of order n* if

$$|L \cap L'| \leq 1$$

holds for every pair $L, L' \in \mathcal{L}$. (By another terminology, $(P, \mathcal{L})$ is a $(n^2+n+1, n+1, 2)$-packing, and $\mathcal{L}$ is a *nearly-disjoint* family.) $\mathcal{L}$ is *maximal* if there is no other partial plane containing it. Let $f(n)$ denote the minimum number of sets in a maximal partial plane.

Let the lines $A_0, A_1, \ldots, A_n$ form a spread with center $\{n^2+n+1\}$ (e.g., $A_i := \{in+1, in+2, \ldots, in+n\} \cup \{n^2+n+1\}$ for $0 \leq i \leq n$), and $B_1, \ldots, B_n$ an orthogonal equipartition of $P \setminus \{n^2+n+1\}$, (e.g., $B_i = \{i, i+n, \ldots, i+n^2\}$). Then $\{A_0, \ldots, A_n, B_1, \ldots, B_n\}$ is a maximal partial plane. Considering this example Mullin [M] conjectured that $f(n) = 2n+1$. It is easy to check that $f(1) = 3$ and $f(2) = 5$. Mullin had several more maximal partial planes of size $2n+1$ as well. However, the conjecture fails to be true for $n \geq 3$, we have

THEOREM 1.1. $f(n) = \lfloor 3n/2 \rfloor + 2$.

*Supported in part by the Hungarian National Science Foundation grant No. 1812
This paper was written while the first author visited Dept. Math., Massachusetts Institute of Technology, Cambridge, MA 02139, USA
new address: Dept. Math., University of Illinois at U. C., Urbana, IL 61801

Example for n odd. Let $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-1)/2}$ be a partition, where $|P_0| = \frac{1}{2}(n+1)(n+2)$ and $|P_1| = \cdots = |P_{(n-1)/2}| = n$. Let $L_1, \ldots, L_{n+2}$ be a system of $(n+1)$-element sets over P_0 such that every pairwise intersection is nonempty, and every element of P_0 is contained in exactly two of these sets. Moreover, let $L_i' = P_i \cup \{p_i\}$, where $p_i \in P_0$ is choosen arbitrarily, $1 \le i \le (n-1)/2$. Then, $\mathcal{L} := \{L_1, \ldots, L_{n+2}\} \cup \{L_1', \ldots, L_{(n-1)/2}'\}$ is a maximal partial plane. Indeed, if $|C \cap L| \le 1$ for all $L \in \mathcal{L}$ for some $(n+1)$-set C, then

$$(1.1) \qquad n+2 \ge \sum_{i=1}^{n+2} |C \cap L_i| = 2|C \cap P_0|$$

implies that $|C \cap P_0| \le \lfloor (n+2)/2 \rfloor = (n+1)/2$. Hence $|C \cap P| = \sum_{i=0,\ldots,(n-1)/2} |C \cap P_i| \le n$.

Example for n even. Let again $P = P_0 \cup P_1 \cup \cdots \cup P_{(n-2)/2}$, where $|P_0| = \frac{1}{2}(n+1)(n+3) - \frac{1}{2}$, $|P_1| = \cdots = |P_{(n-2)/2}| = n$. There exists a nearly-disjoint system of $(n+1)$-element sets $L_1, \ldots, L_{n+3} \subset P_0$, such that every element of P_0 is covered twice or 3 times. To see this, label the elements of P_0 by sets of size 2 and 3 as follows: $P_0 = \{p(B) : B \in \mathcal{B}\}$, where $\mathcal{B} = \{\{1,2,3\}\} \cup \{\{i,j\} : 1 \le i < j \le n+3,\ \{i,j\} \ne \{4,5\}, \{6,7\}, \ldots, \{n+2, n+3\}\}$. We get $L_i = \{p(B) : i \in B\}$ for $1 \le i \le n+3$.

Moreover, let $L_i' = P_i \cup \{p_i\}$, where $p_i \in P_0$, $1 \le i \le (n-2)/2$. Then $\{L_1, \ldots, L_{n+3}, L_1', \ldots, L_{(n-2)/2}'\}$ is a maximal partial plane. To prove the maximality we use (1.1) but the left hand side is replaced by $n+3$, and the equality sign $=$ by a greater-or-equal sign $\ge$.

2. The lower bound is sharp

In the proof of Theorem 1.1 we will use the following result of Seymour [S]: If $\mathcal{D}$ is a nearly-disjoint family over the underlying set Y, then it contains at least $|\mathcal{D}|/|Y|$ pairwise disjoint members. (This theorem is a special case of the Erdős-Faber-Lovász conjecture [E].) As the *dual* of a nearly-disjoint family is again nearly-disjoint, Seymour's theorem gives that there is a set $I \subset Y$ such that $|I \cap D| \le 1$ for all $D \in \mathcal{D}$ and

$$(2.1) \qquad |I| \ge |Y|/|\mathcal{D}|.$$

Proof of 1.1. The upper bound on $f(n)$ was given in the previous section. Now suppose that $\mathcal{L}$ is a maximal family over P with $|\mathcal{L}| = f(n)$. First we show, that one can suppose that

$$(2.2) \qquad \cup\mathcal{L} = P.$$

If the point $p \in P$ is uncovered, and $q \in P$ is contained in at least two lines $L, L' \in \mathcal{L}$, $q \in L \cap L'$, then $\mathcal{L}' := \mathcal{L} \setminus \{L\} \cup \{L \setminus \{q\} \cup \{p\}\}$ is also a maximal partial plane. Indeed, if $\mathcal{L}' \cup \{A\}$ is partial plane for some $A \subset P$, $|A| = n+1$, then $\mathcal{L}$ also can be extended by either A or by $A \setminus \{q\} \cup \{p\}$. Repeating this operation, we either obtain an $\mathcal{L}^*$ consisting of pairwise disjoint sets, a contradiction to its maximality, or an $\mathcal{L}^*$ covering the whole P, proving (2.2).

Denote by $L_1, \dots, L_b \in \mathcal{L}$ the lines having a point of degree one, i.e. for $1 \le i \le b$ one has $p_i \in L_i$ such that $p_i \notin L$ for all $L \in \mathcal{L} \setminus \{L_i\}$. The set $\{p_1, \dots, p_b\}$ intersects every $L \in \mathcal{L}$ in at most one element, hence $b \le n$. Let $C := P \setminus \cup\{L_i : 1 \le i \le b\}$. We have that $|C| \ge |P| - (n+1)b > 0$.

Considering the valencies of the points of P we obtain that

$$(n+1)|\mathcal{L}| \ge |P| + |C| \ge 2(n^2+n+1) - (n+1)b.$$

This implies that

$$|\mathcal{L}| \ge 2n + 1 - b. \tag{2.3}$$

Apply (2.1) to the restriction of $\mathcal{L}$ into C. We get the points $q_1, \dots, q_c \in C$ such that no pair $q_i q_j$ is contained in any $L \in \mathcal{L}$, and $c \ge |C|/(|\mathcal{L}| - b)$. Then $\{p_1, \dots, p_b, q_1, \dots, q_c\}$ is nearly-disjoint to $\mathcal{L}$, so

$$n \ge b + c \ge b + (n^2 + n + 1 - (n+1)b)/(|\mathcal{L}| - b).$$

Rearranging we have $(n-b)(|\mathcal{L}| - n - 1 - b) \ge 1$, implying

$$|\mathcal{L}| \ge n + 2 + b. \tag{2.4}$$

Finally, the sum of (2.3) and (2.4) gives $2|\mathcal{L}| \ge 3n + 3$, finishing the proof.

3. A remark on the lotto problem

The above discussed question is related to the following, so-called lotto problem (see, e.g., [BV]). For $v \ge k \ge t$, let $l(v,k,t)$ denote the smallest cardinality of a family $\mathcal{F}$ of k-subsets of the v-element underlying set V such that $K \subset V$, $|K| = k$ implies that $|F \cap K| \ge t$ for some $F \in \mathcal{F}$. It is easy to see, that $l(n^2+n+1, n+1, 2) = n+2$, in contrast with Theorem 1.1.

References

[BV] A. E. Brouwer and M. Voorhoeve, Turán theory and the lotto problem, *in* Packing and Covering in Combinatorics, (A. Schrijver, ed.), pp. 99–105. Math. Centre Tracts, **106**, Math. Centrum, Amsterdam, 1979.

[E] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.

[M] R. C. Mullin, personal communication, 1984.

[S] P. Seymour, Packing nearly-disjoint sets, *Combinatorica* **2** (1982), 91–97.