On Powers of Strongly Chordal and Circular Arc Graphs

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Abstract. In this paper we study the powers of two important classes of graphs — strongly chordal graphs and circular arc graphs. We show that for any positive integer $k \geq 2$, G^{k-1} is a strongly chordal graph implies G^k is a strongly chordal graph. In case of circular arc graphs, we show that every integral power of a circular arc graph is a circular arc graph.

1. Introduction and definitions.

In this paper, we shall study the properties of the powers of two important classes of graphs, strongly chordal and circular arc graphs. Our interest in powers of graphs is motivated by earlier works by various authors as well as by applications in radio/television frequency assignment problem which gives rise to squares of unit interval graphs (see Raychaudhuri and Roberts [1985] for details).

Before proceeding any further, we shall give some definitions. The *length of a chain* from a vertex x to a vertex y is the number of edges on this chain. The *distance* $d_G(x, y)$ from a vertex x to a vertex y in a graph G is the length of a shortest chain between x and y. Let G be any graph. The kth *power of* G, G^k, where k is a positive integer, is a graph with vertex set $V(G)$, and $\{x, y\} \in E(G^k)$ iff $d_G(x, y) \leq k$.

Chordal graphs are graphs in which every cycle of length ≥ 4 has a chord (an edge joining two non-consecutive vertices on the cycle). Powers of chordal graphs have been studied earlier by various authors, including Laskar and Shier [1980, 1983], Balakrishnan and Paulraja [1981, 1983] and Chang and Nemhauser [1984].

An important subclass of chordal graphs is the class of strongly chordal graphs, which are characterized by Farber [1983] and more recently by Dahlhaus and Duchet [1987]. Strongly chordal graphs can be characterized in terms of an *incomplete trampoline* or an *n-sun* or a *sun-flower*, which is a chordal graph $G = (V, E)$, whose vertex set V can be partitioned into $Y = \{y_1, y_2, \dots, y_n\}$ and $Z = \{z_1, z_2, \dots, z_n\}$ satisfying the following three conditions:

(1) Y is a stable set in G;
(2) $\{z_1, z_2, \dots, z_n, z_1\}$ is a cycle in G;
(3) $\{y_i, z_j\} \in E(G)$ iff $i = j$ or $i = j + 1 \bmod n$.

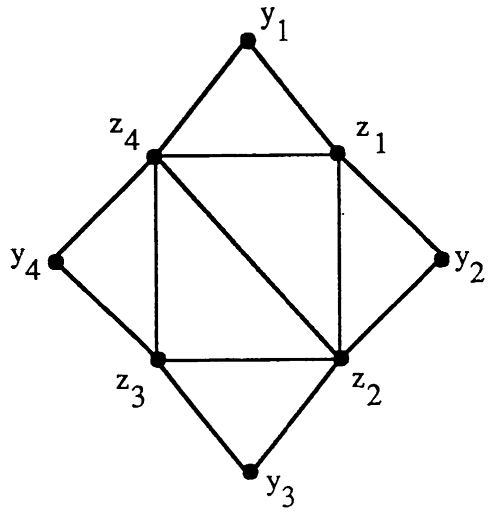

Figure 1: An n-sun, $n = 4$; a chordal graph whose square is not chordal

We show an n-sun $n = 4$ in Figure 1. Farber [1983] gave the following characterization of strongly chordal graphs: A *strongly chordal graph* is a chordal graph which contains no n-sun ($n \geq 3$) as a generated subgraph.

Dahlhaus and Duchet [1987] give another characterization of strongly chordal graphs. Let $C = [v_1, w_1, v_2, w_2, \ldots, v_p, w_p, v_1]$ be an even cycle of length $2p \geq 6$ in G. By a *skew chord* of G is meant an edge of G of type $\{v_i, w_j\}$ where $i \neq j, j-1$ (where arithmetic is modulo p). Then Dahlhaus and Duchet show that a *strongly chordal graph* is a chordal graph in which every even cycle of length ≥ 6 possesses a skew chord.

Powers of strongly chordal graphs have been studied by Lubiw [1982] and Dahlhaus and Duchet [1987]. They have independently showed that every power G^k of a strongly chordal graph G is again strongly chordal. In Section 2 of this paper we prove a more general result. We show that for any positive integer $k \geq 2$, G^{k-1} is a strongly chordal graph implies that G^k is a strongly chordal graph.

The other class of graphs which we discuss in this paper is the class of circular arc graphs, which belongs to the broad class of intersection graphs. An *intersection graph* of a family of sets is a graph G for which we can assign to each vertex x a set $S(x)$ in this family in such a way that $\{x, y\} \in E(G) \Leftrightarrow S(x) \cap S(y) \neq \phi$. A *circular arc graph* is an intersection graph of a family of arcs of a circle. They have been characterized by Tucker [1970] as follows: A graph G with v vertices is a circular arc graph if there is an ordering $x_1, x_2, \ldots, x_v$ of $V(G)$ such that $\{x_i, x_j\} \in E(G) \Rightarrow x_{i+1}, \ldots, x_j$ are adjacent to x_i, or $x_{j+1}, \ldots, x_i$ are adjacent to x_j where addition is modulo v.

In Section 3 of this paper we study the properties of powers of circular arc graphs. Using Tucker's conditions and other results, we have proved that every positive integral power G^k of a circular arc graph G is again a circular arc graph. However, the more general question of whether G^{k-1} is a circular arc graph implies G^k is a circular arc graph for $k > 2$, remains unanswered.

2. Strongly chordal graphs.

In this section, we shall show that if G^{k-1} is strongly chordal, then G^k is strongly chordal for all $k \geq 2$. We begin by giving a brief survey of some important related results.

Laskar and Shier [1980, 1983] noted that the square of a chordal graph is not necessarily chordal. The graph of Figure 1 is the smallest such example. In that same paper [1980] the authors establish that if G is chordal, then so are G^3 and G^5. It was also conjectured that any odd power of a chordal graph is again chordal. The conjecture has been shown true by Balakrishnan and Paulraja [1983]. Below we quote their results as Theorem 1.

Theorem 1 **(Balakrishnan and Paulraja [1983]).** *If G is chordal then so is G^{2k+1} for any integer $k \geq 1$.*

Duchet [1982] gives a more general result.

Theorem 2 **(Duchet [1982]).** *If G^k is chordal, then so is G^{k+2} for any integer $k \geq 1$.*

The example in Figure 1 gave the motivation to look at necessary and sufficient conditions for the square of a chordal graph to be chordal. Laskar and Shier [1983] and Chang and Nemhauser [1984] independently found the subclass of chordal graphs for which this is true. We quote their results as Theorem 3.

Theorem 3 **($1 \Leftrightarrow 2 \Leftrightarrow 3$: Chang and Namhauser [1984], $2 \Leftrightarrow 3$: Laskar and Shier [1983]).**

(1) *G^k is chordal for every positive integer k.*
(2) *G and G^2 are both chordal.*
(3) *G is chordal and if G has an n-sun with $n \geq 4$ as an induced subgraph, where the n-sun is defined on $\{y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n\}$, then $d_G(y_i, y_j) = 2$ for some i and j such that $j \notin \{i-1, i+1\}$.*

Lubiw [1982] studied powers of strongly chordal graphs and gives the following Theorem.

Theorem 4 **(Lubiw [1982]).** *If G is strongly chordal, then G^k is strongly chordal for any positive integer k.*

Motivated by (Duchet's) Theorem 2, which guarantees that G^{k+2} is chordal if G^k is chordal, we asked the following question: For any graph G, under what conditions do we have G^{k-1} chordal, and G^k not chordal, where k is any positive integer ≥ 2. We answer this question in the following Theorem.

Theorem 5. *Suppose G is any graph, and k is positive integer ≥ 2. Suppose G^{k-1} is chordal, and G^k is not chordal. Then G^{k-1} contains an n-sun, $n \geq 3$, as an induced subgraph.*

For the proof of Theorem 5, we shall need some lemmas.

Lemma 1 **(Laskar and Shier [1983]).** *Suppose C_n is any cycle of length n of a chordal graph. Then for every edge $\{u, v\}$ of this cycle, there is a vertex w of this cycle that is adjacent to both u and v.*

Proof: Follows easily by induction on n. ∎

Lemma 2. *Suppose G is any graph, k is an integer ≥ 2, G^{k-1} is chordal, but G^k is not chordal. Then if $x_0, x_1, \ldots, x_{r-1}$ is a chordless cycle C in G^k of length ≥ 4, then $d_G(x_i, x_{i+1}) = k$ (where $+$ is addition modulo r), for all $i = 0, 1, \ldots, r-1$.*

Proof: Suppose there are to vertices x_ℓ and $x_{\ell+1}$, for which $d_G(x_\ell, x_{\ell+1}) < k$. In G, let P_i be the shortest chain between x_i and x_{i+1}, as shown in Figure 2, $i = 0, 1, \ldots, r-1$. Let z_i be the vertex following x_i on P_i and y_i be the vertex preceding x_{i+1} on P_i. Note that some of z_i's y_i's may not be distinct from x_i or x_{i+1}, respectively. Then consider the following cycle Z in G^{k-1}: $x_\ell, x_{\ell+1}, y_{\ell+1}, x_{\ell+2}, y_{\ell+2}, x_{\ell+3}, \ldots, x_{\ell-2}, y_{\ell-2}, x_{\ell-1}, z_{\ell-1}, x_\ell$ ($-$ is subtraction modulo r). Then by Lemma 1, there is a vertex v in Z which is adjacent to both x_ℓ, and $x_{\ell+1}$, in G^{k-1}. Now, v cannot be x_i for any i since C is chordless in G^k. If $v = z_{\ell-1}$ then $\{x_{\ell+1}, x_{\ell-1},\}$ is a chord in G^k because $z_{\ell-1}$ is adjacent to $x_{\ell-1}$ in G and to $x_{\ell+1}$ in G^{k-1}. If $v = y_i$ for i in $\{\ell+1, \ell+2, \ldots, \ell-2\}$, there is a chord $\{x_i x_j\}$ in G^k such that $i \neq j-1, j+1$, which is a contradiction. Thus, $d_G(x_\ell, x_{\ell+1}) = k$. ∎

Proof of Theorem 5: ($\pm$ stands for arithmetic modulo r)

Suppose $k \geq 2, G^{k-1}$ chordal, and G^k has a chordless cycle $C = \{x_0, x_1, \ldots, x_{r-1}\}$ where $r \geq 4$. Let P_i be a shortest chain in G between x_i and x_{i+1}, $i \in \{0, 1, \ldots, r-1\}$. Then Lemma 2 $\rightarrow$ P_i is of length k in G. Let z_i be the vertex following x_i on P_i in G as shown in Figure 3. Note that the z_i's are distinct. We shall consider two cases:

Case 1: $\{x_i, z_{i+1}\} \notin E(G^{k-1})$ for all i in $\{0, 1, \ldots, r-1\}$. Then we have the following claim:

Claim 1: $\{z_{i-1}, z_i\} \in E(G^{k-1})$ for all i in $\{0, 1, \ldots, r-1\}$. Consider the cycle $Z \equiv x_i, z_i, x_{i+1}, z_{i+1}, \ldots, x_{i-1}, z_{i-1}, x_i$ in G^{k-1} and the edge $\{x_i, z_i\}$ on this cycle. By Lemma 1, some vertex v of Z must be adjacent to both x_i and z_i in G^{k-1}. Since by Lemma 2 $d_G(x_i, x_{i+1}) = k$, and C is chordless in G^k, v is not of the form x_j. If $v = z_j$, j in $\{i+2, i+3, \ldots, i-2\}$, then there is a chord

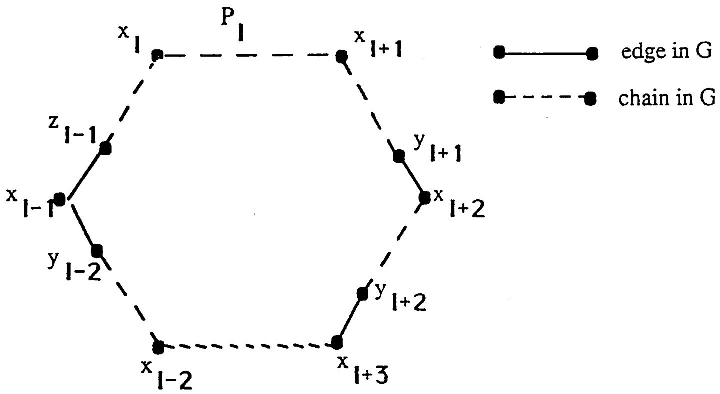

Figure 2:

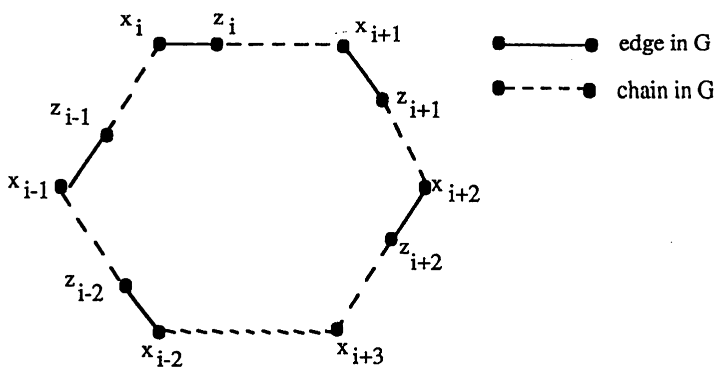

Figure 3:

$\{x_i, x_j\}$ in G^k, when $|i-j| \geq 2$. Thus, v must be z_{i-1} or z_{i+1}. But $v \neq z_{i+1}$ by assumption. So $v = z_{i-1}$. Therefore, $\{z_{i-1}, z_i\} \in E(G^{k-1})$.

Hence, $\{z_0, z_1, z_2, \ldots, z_{r-1}, z_0\}$ forms a cycle in G^{k-1}. Also, it follows easily that $\{x_i, z_j\} \in E(G^{k-1})$, when $i = 0, 1, \ldots, r-1$, and $j = i$ or $i-1$. By assumption $\{x_i, z_{i+1}\} \notin E(G^{k-1})$. Also, if $j \notin \{i+1, i, i-1\}$, then $\{x_i, z_j\} \notin E(G^{k-1})$ since that would imply that $\{x_i, x_j\} \in E(G^k)$, $|i-j| \geq 2$, which is a contradiction since C is chordless in G^k. So $\{x_i, z_j\} \in E(G^{k-1})$ only when $j = i$ or $i-1$. Finally, $\{x_0, x_1, \ldots, x_r\}$ is a stable set in G^{k-1} by Lemma 2 and the fact C is chordless in G^k. So $\{x_0, x_1, \ldots, x_r, z_0, z_1, \ldots, z_r\}$ forms a r-sun $G^{k-1}, r \geq 4$.

Case 2: $\{x_i, z_{i+1}\} \in E(G^{k-1})$ for some i in $\{0, 1, \ldots, r-1\}$. Then we can make the following claim:

Claim 2: $\{z_{i-1}, z_{i+1}\} \in E(G^{k-1})$. Considering the cycle $x_i, z_{i+1}, x_{i+2}, z_{i+2}, \ldots, x_{i-1}, z_{i-1}, x_i$ and the edge $\{x_i, z_{i+1}\} \in G^{k-1}$, it is easy to show by Lemma 1 that $\{z_{i-1}, z_{i+1}\} \in E(G^{k-1})$.

Consider the following vertices: Let $x'_1 = x_i$, $z'_1 = z_{i-1}$. Having chosen $x'_j = x_m$, and $z'_j = z_{m-1}$, let

$$\left.\begin{aligned} x'_{j+1} &= x_{m+1} \\ z'_{j+1} &= z_m \end{aligned}\right\} \quad \text{if } \{x_m, z_{m+1}\} \notin E(G^{k-1})$$

and

$$\left.\begin{aligned} x'_{j+1} &= x_{m+2} \\ z'_{j+1} &= z_{m+1} \end{aligned}\right\} \quad \text{if } \{x_m, z_{m+1}\} \in E(G^{k-1}).$$

We stop when x_{i-1} is the last chosen vertex, that is when $x'_j = x_{i-1}$, or when $x'_j = x_{i-2}$, and $\{x_{i-2}, z_{i-1}\} \in E(G^{k-1})$.

Let $\{x'_1, \ldots, x'_\ell\}$ and $\{z'_1, \ldots, z'_\ell\}$ be the vertices so generated. Note that either $x'_\ell = x_{i-1}$, or $x'_\ell = x_{i-2}$, and $\{x_{i-2}, z_{i-1}\} \in E(G^{k-1})$. We shall show that $\{x'_1, \ldots, x'_\ell, z'_1, \ldots, z'_\ell\}$ form a ℓ-sun, in G^{k-1}. By Lemma 2, and the fact that C is chordless in G^k, $\{x'_1, \ldots, x'_\ell\}$ forms a stable set in G^{k-1}. Next, we show that $\{z'_1, \ldots, z'_\ell\}$ forms a cycle in G^{k-1}. To see this, note that if $\{x_m, z_{m+1}\} \notin E(G^{k-1})$, then $\{z'_j, z'_{j+1}\} = \{z_{m-1}, z_m\} \in E(G^{k-1})$. If $\{x_m, z_{m+1}\} \in E(G^{k-1})$, then by Claim 2 $\{z'_j, z'_{j+1}\} = \{z_{m-1}, z_{m+1}\} \in E(G^{k-1})$. If $x'_\ell = x_{i-1}$, then $z'_\ell = z_{i-2}$, and $\{z'_\ell, z'_1\} = \{z_{i-2}, z_{i-1}\} \in E(G^{k-1})$ by Claim 1. If $x'_\ell = x_{i-2}$, then $\{x_{i-2}, z_{i-1}\} \in E(G^{k-1})$ and $z'_\ell = z_{i-3}$. So by Claim 2, $\{z'_\ell, z'_1\} = \{z_{i-3}, z_{i-1}\} \in E(G^{k-1})$. Also it is easy to verify that $\{x'_i, z'_j\} \in E(G^{k-1})$ (for all $i = 1, \ldots, \ell$) iff $i = j$ or $i = j-1$. So $\{x'_1, x'_2, \ldots, x'_\ell, z'_1, z'_2, \ldots, z'_\ell\}$ forms a ℓ-sun, $\ell \geq 3$, so long as $r \geq 5$. Therefore, let us assume that $r = 4$, that is, $\{x_0, x_1, x_2, x_3, x_0\}$ is a chordless cycle in G^k. If $\{x_i, z_{i+1}\} \notin E(G^{k-1})$ for any i in $\{0, 1, 2, 3\}$, then obviously $\{x_0, x_1, x_2, x_3, z_0, z_1, z_2, z_3\}$ forms a 4-sun in G^{k-1}. Thus, suppose that for some i, $\{x_i, z_{i+1}\} \in E(G^{k-1})$ as shown in Figure 4(a). It can be shown that $k > 2$ in this case. Let y_{i-1} be the vertex preceding x_i, in the chain of shortest length in G between x_{i-1} and x_i. We claim that $\{z_{i+2}, y_{i-1}, z_{i+1}, z_{i+2}\}$ forms a cycle in G^{k-1}. To see this, consider the cycle $z_{i+2}, x_{i-1}, y_{i-1}, x_i, z_{i+1}, x_{i+2}, z_{i+2}$ in G^{k-1} and apply Lemma 1 to edges $\{z_{i+2}, x_{i-1}\}$, $\{x_i, y_{i-1}\}$ and $\{x_{i+2}, z_{i+2}\}$ of

G^{k-1}. Finally, it can be shown that $\{x_{i+2}, x_{i-1}, x_i, z_{i+2}, y_{i-1}, z_{i+1}\}$ forms a 3-sun in G^{k-1} as shown in Figure 4(b). (Note that y_{i-1} is different from z_{i+1} or z_{i-1}, since $k > 2$).

So we have proved Theorem 5, showing that G^{k-1} chordal but G^k not chordal implies that G^{k-1} contains an n-sun ($n \geq 3$) as an induced subgraph. ∎

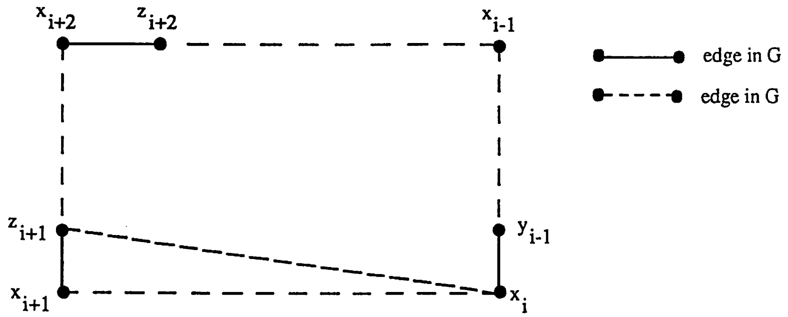

Figure 4(a):

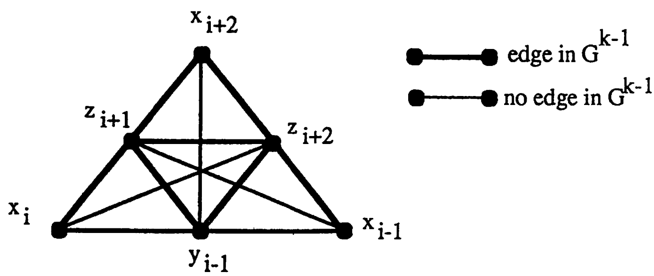

Figure 4(b): An n-sun ($n = 3$) in G^{k-1}

Corollary 5.1. *For all positive integers $k \geq 2$, if G^{k-1} is strongly chordal, then G^k is chordal.*

Proof: The above corollary is the contrapositive of Theorem 5. ∎

In Figure 5 we give an example of G, which illustrates Theorem 5. G^2 is chordal, but G^3 is not; $\{v_1, v_2, v_3, v_4\}$ forms a chordless cycle in G^3. Note that $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$ forms a 4-sun in G^2.

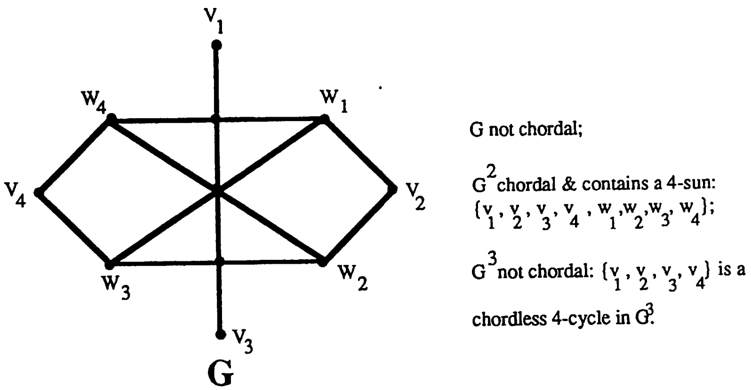

Figure 5:

Theorem 6. *Let k be any positive integer ≥ 2. Suppose G^{k-1} is strongly chordal. Then G^k cannot contain a n-sun, $n \geq 3$.*

In order to prove this Theorem, we shall use a lemma proved by Dahlhaus and Duchet [1987], which we quote below.

Lemma 3 (Dahlhaus and Duchet [1987]). *Suppose G is strongly chordal. Let v, a, b be three vertices mutually adjacent in G^k, where k is any positive integer ≥ 2. Then either v or a neighbor of v (in G) is adjacent to a and b in G^{k-1}.*

Proof of Theorem 6: We shall prove this theorem in two parts — first for $k = 2$, then for any $k \geq 3$.

Suppose that by contradiction $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ forms a n-sun, $n \geq 3$ in G^k, where $\{v_1, v_2, \ldots, v_n\}$ forms the stable set and $\{w_1, w_2, \ldots, w_n\}$ forms a cycle in G^k. Also, $\{v_i, w_j\} \in E(G^k)$ iff $j = i$ or $i - 1$ modulo n.
Case 1: $k = 2$. By Lemma 3, since v_i, w_{i-1}, w_i are pairwise adjacent in G^k, either v_i or a neighbor of v_i in G is adjacent to w_{i-1} and w_i in G^{k-1}. Therefore, $k_1, w_1, k_2, w_2, \ldots, k_n, w_n, k_1$, (where k_i is either v_i or a neighbor of v_i in G) forms an even cycle in G^{k-1} without a skew chord. It is easy to see that these k_i's are distinct, since the v_i's form a stable set in G^k, and skew chords are absent since $\{v_i, w_j\} \in E(G^k)$ iff $j = i$ or $i - 1$ modulo n. So G^{k-1} is not strongly chordal, which is a contradiction.
Case 2: $k \geq 3$. We shall consider two subcases:
Case 2a: $d_G(v_i, w_j) > 1$ for all i and $j = i$ or $i - 1$ mod n. In the shortest chain from v_i to w_i in G, let z_i be the vertex following v_i and in the shortest chain from w_{i-1} to v_i in G, let y_i be the vertex preceding v_i. Then z_i's and y_i's are distinct from

v_i's and w_i's. Assuming that they are also distinct from each other, (we shall consider the other possibility later), $v_1, z_1, w_1, y_2, v_2, z_2, w_2, \ldots, v_n, z_n, w_n, y_1, v_1$ form an even cycle C in G^{k-1}. Let us rename these vertices as $v'_1, w'_1, v'_2, w'_2, \ldots, v'_{2n}, w'_{2n}$ as circled in Figure 6a. In this cycle C of G^{k-1} all skew chords are of the form $\{v'_i, w'_j\}, j \neq i, i-1$. It can be shown that some of these skew chords are not present while some may or may not be present. For instance, if i is odd then no skew chord of the form $\{v'_i, w'_j\}$ could be present in G^{k-1} since $\{v_1, v_2, \ldots, v_n\}$ forms a stable set in G^k. If i is even, then the only skew chords which may be in G^{k-1} are $\{v'_i, w'_j\}, j = i+1$, or $i-2$ modulo $2n$. Any other skew chord of the form $\{v'_i, w'_j\}$, where i is even cannot be present because of the absence of any skew chord in the even cycle $v_1, w_1, \ldots, v_n, w_n, v_1$ of G^k. Suppose in the even cycle C of G^{k-1}, there is no skew chord. Then Theorem 6 follows easily. Therefore, let us assume that some skew chord $\{v'_i, w'_j\}$, (where i is even and $j = i+1$ or $i-2 \mod 2n$) is present in G^{k-1}. If $j = i+1 \mod 2n$, we remove the vertices w'_i and v'_{i+1} from C. If $j = i-2 \mod 2n$, we remove the vertices w'_{i-1} and v'_{i-1} $(\mathrm{mod}\ 2n)$. Thus, we get a new cycle C', each time including a skew chord $\{v'_i, w'_{i+1}\}$ or $\{w'_{i-2}, v'_i\}$ and omitting two vertices of C. However, once some vertices of C are omitted we do not include the skew chords with one end on these omitted vertices even if they are present in G^{k-1}. Thereby we form a new cycle C' in G^{k-1} which is of length $2p \geq 6$ since every time we either omit two vertices or none. Note C' has no skew chords in G^{k-1} since every skew chord of C' is a skew chord of C which is absent in G^{k-1}. We describe this case in Figure 6a.

Next, let us suppose that z_i's and y_i's are distinct from v_i's and w_i's but suppose that $z_i = y_j, j = i$ for some i. Then we identify z_i with y_j on C for all such pairs, thus, omitting v_i from C and obtaining a new even cycle C' of length ≥ 6 without any skew chords in G^{k-1} as shown in Figure 6b.

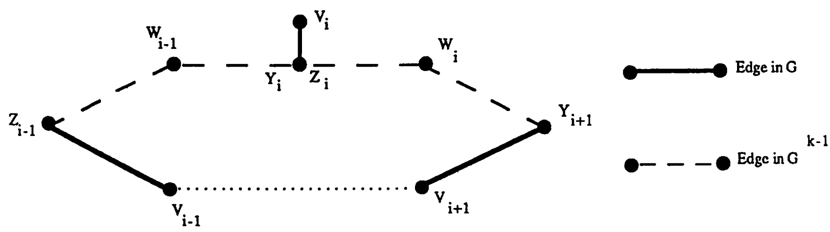

Figure 6(b):

Case 2b: Suppose $d_G(v_i, w_j) = 1$ for some i and $j = i$ or $i-1$. Let z_i, y_i and C be as described in Case 2a. Note that C may not be an even cycle. If $d_G(v_i, w_j) = 1$ for either $j = i$ or $i-1$ but not both, then z_i and y_i are not distinct from w_i and w_{i-1}, respectively. Suppose $d_G(v_i, w_i) = 1$, then $d_G(y_i, w_i) = 2$.

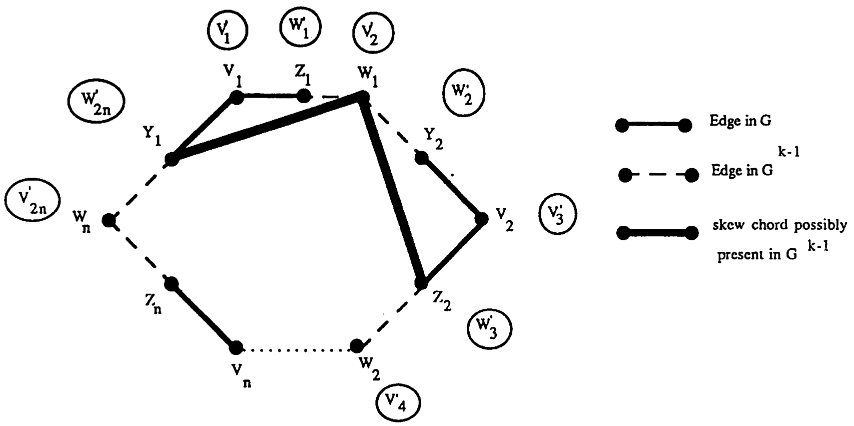

Figure 6(a):

So $\{y_i, w_i\} \in E(G^{k-1})$ since $k \geq 3$. So omit v_i and include edge $\{y_i, w_i\}$ on C and if $d_G(w_{i-1}, v_i) = 1$, then $d_G(w_{i-1}, z_i) = 2$ and so $\{w_{i-1}, z_i\} \in E(G^{k-1})$. So in that case we omit v_i and include edge $\{w_{i-1}, z_i\}$ on C. So the new cycle C' of G^{k-1} that we will generate in this way is even and is of length ≥ 6 and there is no skew chord in C', if we include the skew chords present and omit some vertices as described in Case 2a. If $d_G(v_i, w_j) = 1$ for $j = i$ and $i - 1$, then $z_i = w_i$ and $y_i = w_{i-1}$ and including any skew chord present and omitting some vertices as described in Case 2a, we generate an even cycle C' (from C) in G^{k-1} without a skew chord. So in all cases we have proved that if G^k contains a n-sun $n \geq 3$, then G^{k-1} contains an even cycle without a skew chord. ∎

Theorem 7. *Let k be any positive integer ≥ 2. Suppose G^{k-1} is strongly chordal. Then G^k is strongly chordal.*

Proof: The contrapositive of Theorem 5 says that G^k is chordal and Theorem 6 says that G^k cannot contain a n-sun, $n \geq 3$. So G^k must be strongly chordal. ∎

3. Circular arc graphs.

The main result of this subsection is the fact that every positive integral power G^k of a circular arc graph G is a circular arc graph.

In order to prove this result we shall need a few Theorems.

Theorem 8. *If G is a circular arc graph, then so is G^2. In addition, if $x_1, x_2, \ldots, x_v$, is an ordering of $V(G)$ which satisfies Tucker's [1970] condition, then the same ordering of $V(G^2)$ also satisfies Tucker's condition.*

Proof: Suppose G is a circular arc graph on v vertices, and let $x_1, x_2, \ldots, x_v$ be a circular ordering of $V(G)$ which satisfies Tucker's condition. Suppose $\{x_i, x_j\} \in$

$E(G^2)$. Then there is a vertex x_s which is adjacent to both x_i and x_j in G. There are essentially three cases to consider:
Case 1: $s = i$ or $s = j$.
Case 2: i, s, and j are 3 distinct vertices, and $i < s < j$ in the clockwise order.
Case 3: i, s, and j are 3 distinct vertices and $i < j < s$ in the clockwise order.
Case 1: $s = i$ or $s = j$. In this case $\{x_i, x_j\} \in E(G)$. So by Tucker's condition either $x_{i+1}, \dots, x_j$ are adjacent to x_i in G and so in G^2, or $x_{j+1}, \dots, x_i$ are adjacent to x_j in G and so in G^2.

For Case 2 and Case 3, since $\{x_i, x_s\} \in E(G)$, by Tucker's condition either

(a) $x_{i+1}, \dots, x_s$ are adjacent to x_i in G; or
(b) $x_{s+1}, \dots, x_i$ are adjacent to x_s in G.

Similarly, since $\{x_s, x_j\} \in E(G)$

(c) $x_{s+1}, \dots, x_j$ are adjacent to x_s in G; or
(d) $x_{j+1}, \dots, x_s$ are adjacent to x_j in G.

We shall now consider Case 2 and Case 3 separately.
Case 2: $i < s < j$. It is easy to see that if case (b) or (d) holds, then $x_1, x_2, \dots, x_v$ satisfy Tucker's condition in G^2. So it will suffice to assume that cases (a) and (c) hold. Case (a) implies that $x_{i+1}, \dots, x_s$ are adjacent to x_i in G and so in G^2, and case (c) implies that $x_{s+1}, \dots, x_j$ are adjacent to x_s in G, and since $\{x_s, x_i\} \in E(G)$, it follows that $x_{s+1}, \dots, x_j$ are adjacent to x_i in G^2.
Case 3: $i < j < s$. Proof is similar to Case 2. ∎

Let us digress at this point and recall that circular arc graphs belong to the larger class of intersection graphs of families of sets. In this context we next give a result which is related to odd powers of some intersection graphs. (This proof is a generalization of a similar proof valid for chordal graphs, given by Chang and Nemhauser.)

Theorem 9. *Suppose G is the intersection graph of sets belonging to a family D. Let $S(x)$ be the set associated with vertex x. Suppose m is any positive integer ≥ 1. Let $S'(x) = \{US(y)\colon d_G(x,y) \leq m\}$. If for all x, $S'(x) \in D$, then G^{2m+1} is an intersection graph of sets belonging to D.*

Proof: We shall show that $S'(x)$ is an intersection representation for G^{2m+1}.

Suppose $S'(x) \cap S'(y) \neq \phi$. Then there are vertices u and v such that $d_G(x,u) \leq m$, $d_G(y,v) \leq m$ and $S(u) \cap S(v) \neq \phi$. It follows that $\{u, v\} \in E(G)$. Thus, $d_G(x,y) \leq d_G(x,u) + d_G(u,v) + d_G(v,y) \leq m + 1 + m = 2m + 1$. Thus, $\{x, y\} \in E(G^{2m+1})$.

Next, suppose that $\{x, y\} \in E(G^{2m+1})$. Then in the shortest chain from x to y in G, there are two vertices u and v, which are adjacent in G, such that $d_G(x,u) \leq m$, $d_G(y,v) \leq m$. Now $S(u) \cap S(v) \neq \phi$, so $S'(x) \cap S'(y) \neq \phi$. ∎

Corollary 9.1. *For all positive integers $m \geq 1$, G is a circular arc graph $\Rightarrow$ G^{2m+1} is a circular arc graph.*

Proof: Let $S(x)$ be a circular arc representation of G. Then $\{US(y): d_G(x,y) \leq m\}$ is a union of some arcs of a circle, in which the first arc is $S(x)$ and every consecutive pair of arcs intersect. Thus, this union is itself an arc of the circle (may be the complete circle). ∎

Lemma 4. *Let G be any graph and m and n be positive integers. Then $(G^m)^n = G^{mn}$.*

Proof:

(a) Let $\{x, y\}$ be an edge of $(G^m)^n$. Then, $d_{G^m}(x,y) \leq n$. So in G^m there is a chain $P_{x,y}$ between x and y of length $\leq n$. Consequently, between any two vertices joined by an edge in $P_{x,y}$ in G^m, there is a chain of length $\leq m$ in G. So in G there is a chain of length $\leq mn$ between x and y. So $d_G(x,y) \leq mn$. So $\{x, y\}$ is an edge of G^{mn}.

(b) Let $\{x, y\}$ be an edge of G^{mn}. Now it is easy to see that $d_G(x,y) \leq k$ (where k is a positive integer) $\Rightarrow d_{G^m}(x,y) \leq \lfloor \frac{k}{m} \rfloor + (k - \lfloor \frac{k}{m} \rfloor \cdot m)$, where m is a positive integer $\leq k$. So if $k = mn$, it easily follows that $d_{G^m}(x,y) \leq n$. So $\{x, y\}$ is an edge of $(G^m)^n$.

∎

Theorem 10. *Suppose G is a circular arc graph and k is any positive integer. Then G^k is a circular arc graph.*

Proof: We first note that any positive even integer is of the form $x \cdot 2^m$ where $m = 0, 1, 2, \ldots$ and x is a positive odd integer. We repeatedly divide the even integer by 2 until the quotient x is an odd integer.

Suppose G is a circular arc graph. If k is any odd positive integer then by Corollary 9.1 G^k is a circular arc graph. If k is an even positive integer, then $k = x \cdot 2^m$. Thus, $G^k = G^{x \cdot 2^m} = (G^{2^m})^x$ (by Lemma 4). Now by repeated application of Theorem 8, and by Lemma 4, (G^{2^m}) is a circular arc graph. So again by Corollary 9.1 $(G^{2^m})^x$ is a circular arc graph. ∎

Apart from applications in communication problems, there are certain important graph theoretic parameters which are closely related to powers of graphs. We shall introduce one of them here.

A *k-stable set* is a vertex set $S \subseteq V$ such that $d_G(x,y) > k$ for every distinct pair of vertices x and y in S. The *k-stability problem* is to find the *k-stability number*, $\alpha_k(G)$, which is the maximum cardinality of a k-stable set. The k-stability problem is NP-complete for general graphs. However, Theorem 9 enables us to solve the k-stability problem for a circular arc graph in polynomial time.

Theorem 11. *The k-stability problem on a circular arc graph G can be solved in $O(v^4)$ steps.*

Proof: By the discussions of this section, the k-stability problem for a circular arc graph G is equivalent to the problem of finding a maximum cardinality stable set in another circular arc graph G^k.

Given G, construction of G^k takes $O(v^4)$ steps. If k is 2^m, then by Theorem 8, the ordering $x_1, \ldots, x_v$ of $V(G)$ which arises in Tucker's charcterization of circular arc graphs is also the required ordering of $V(G^k)$ which guarantees that G^k is a circular arc graph. Then using Tucker's [1970] construction, we can get a circular arc representation of G^k in $O(v)$ steps. If $k(= 2m + 1)$ is odd, then from Theorem 9 we can build a circular arc representation of G^k from that of G in $O(v^4)$ steps (used in the construction of G^m).

Given a circular arc representation of a circular arc graph, Gupta *et al* [1982] gives an $O(v^2)$ algorithm to calculate its maximum size stable set. Thus, the k-stability problem for a circular arc graph can be solved in $O(v^4)$ steps. ∎

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