A New Sufficient Condition for Panconnected Graphs①

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ABSTRACT: Let G be a simple graph of order n with independence number α. We prove in this paper that if, for any pair of nonadjacent vertices u and v, $d(u)+d(v) \geqslant n+1$ or $|N(u) \cap N(v)| \geqslant \alpha$, then G is (4,n−1)−connected unless G is some special graphs. As corollary we investigate edge−pancyclity of graphs.

We only consider undirected, simple graphs in this paper. let Gbe a simple graph of order n. G being (r,m)−connected means that for any two vertices u,v of G, there exists a u−v path of each length from r−1 to m−1 in G, where $2 \leqslant r \leqslant m \leqslant n$. G is edge−pancyclic if edge of G lies on a cycle of each length from 3 to n. Let α denote the independence number of G, that is, the size of a maximal independent set in G. Let $K \subset N = \{3,4,\cdots,n\}$. G is edge K^-−pancyclic if every edge of G lies on a cycle of each length r, $r \in N \setminus K$. Particularly, if $K=\{k\}$, we say G is edge k^-−pancyclic. Similarly we can define an edge is K^-−pancyclic.

We use the notation G(r,t) to denote the following special class of graphs. For any $G \in G(r,t)$, $V(G)=V_1 \cup V_2$ where $|V_1|=r$ and $G[V_1]$ is any simple graph, $V_2=V_{21} \cup V_{22} \cup \cdots \cup V_{2t}$ and $G[V_{2j}]$ is complete for any j, $1 \leqslant j \leqslant t$. Moreover, every vertex in V_1 is adjacent with every vertex in V_2. Obviously, $K_{r,t}$ is a special element of G(r,t). Terms not found here see [1].

Theorem 1 *Let G be a simple graph of order $n(\geqslant 5)$ with independence number α. If for any pair of nonadjacent vertices u and v, $|N(u) \cap N(v)| \geqslant \alpha$ or $d(u)+d(v) \geqslant n+1$, then G is (5,n)−connected, unless G is belong to $G(\alpha,\alpha)$.*

Proof: If $\alpha=1$, i.e. G is complete, the theorem holds. So, suppose $\alpha \geqslant 2$.

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① The project supported by NSFC

Let x, y be any two vertices of G.

Firstly, we prove that if there is no x–y path of length 3 in G, then there must exists an x–y path of length 4.

In fact, if G contains no x–y path of length 3, then when x,y are nonadjacent, for any $u,v \in N(x) \cap N(y)$, $uv \notin E$ and $N(w) \cap N(z) = \phi$, where $w \in \{u,v\}$, $z \in \{x,y\}$. If $d(u)+d(v) \geqslant n+1$, or $\alpha \geqslant 3$, then $|N(u) \cap N(v)| \geqslant 3$. We can easily get an x–y path of length 4. So we assume $\alpha = 2$. This implies $N(x) \cap N(y) = \{u,v\}$ and there exists a vertex w in $G-\{x,y\}$, such that $wu \in E$ or $wv \in E$. Without loss of generality, say $wu \in E$. Clearly $xw \notin E$. But now $|N(x) \cap N(w)| \geqslant 2$ implies the existence of an x–y path of length 4. When $xy \in E$, for any $u \in N(x)$, $u \neq y$, we have $uy \in E$ (Otherwise, since $|N(u) \cap N(y)| \geqslant 2$, let $w \in N(u) \cap N(y)$, the path xuwy is an x–y path of length 3). Hence $N(x)-\{y\} = N(y)-\{x\}$ and $N(x)-\{y\}$ is independent. As above, we can get an x–y path of length 4.

Suppose now G contains an x–y path of length $r (3 \leqslant r \leqslant n-2)$, but no x–y path of length r+1. We prove that G belongs to $G(\alpha,\alpha)$.

Let $P = v_0 v_1 \cdots v_r$ be an x–y path of length r, where $v_0 = x$, $v_r = y$. By the connectivity, there exists $u \in V(G) \setminus V(P)$ with $d_p(u) > 0$. Obviously the following two Claims hold:

Claim 1 *If* $uv_j \in E, 0 \leqslant j \leqslant r$, *when* $j<r, uv_{j+1} \notin E$; *when* $j>0, uv_{j-1} \notin E$.

Claim 2 *For any* $v_i, v_j \in N_p(u)$, $v_i \neq v_j$, $v_{i+1}v_{j+1}, v_{i-1}v_{j-1} \notin E$.

Let $N_p(u) = \{v_{i_1}, v_{i_2}, \cdots, v_{i_m}\}$, $0 < i_1 < i_2 < \cdots < i_m < r$ and $A_j = \{v_{i_1+1}, v_{i_2+1}, \cdots, v_{i_{j-1}+1}\}$, $B_m = \{v_{i_{j+1}-1}, v_{i_{j+2}-1}, \cdots, v_{i_m-1}\}$, $C_j = \{v_{i_j+1}, v_{i_j+2}, \cdots v_{i_{j+1}-1}\}$. By Claims 1 and 2, $A_m \cup \{u\}$ and $B_1 \cup \{u\}$ are independent. Therefore, by the definition of α, the following Claim holds.

Claim 3 $m \leqslant \alpha$ *and if* $m = \alpha$, *then* $i_j = 0, i_m = r$.

Claim 4 *Let* $v \in C_j$. *If* $N(u) \cap N(v) \subseteq V(P)$ *and* $d(u)+d(v) \geqslant n+1$, *then thereexist integers s, t such that* $vv_{i_s}, vv_{i_s+1}, vv_{i_t}, vv_{i_t-1}$ *appear in G where* $s \neq j$, $t \neq j+1$ *and* $1 \leqslant s \leqslant m-1$, $2 \leqslant t \leqslant m$.

Proof: By symmetry, we only prove the existence of s. In fact, if there exisitsno such an integer, then for any $g \neq j$, $1 \leqslant g \leqslant m-1$, there holds $|[\{v\},\{v_{i_g}, v_{i_g+1}\}]| \leqslant 1$. Since $N(u) \cap N(v) \subseteq V(P)$, $d(u)+d(v) \leqslant n$. This is a contradiction.

The following is divided into two cases.

Case 1 There exists k ($1\leqslant k\leqslant$ m) such that $i_{k-1}\leqslant i_k-2$.
Now we have:

Claim 5 *$N(u)\cap N(v_{i_k-2})\subseteq V(P)$.*

Subcase 1.1 $d(u)+d(v_{i_k-2})\geqslant n+1$

By Claims 4, 5, there exists $j\neq k-1$, $1\leqslant j\leqslant m-1$ such that $v_{i_k-2}v_{i_j}\in E(G)$, $v_{i_k-2}v_{i_j+1}\in E(G)$ and $i_{k-1}<i_k-3$. Without loss of generality, suppose $j\geqslant k$.
If there exists $w\in N(v_{i_j+1})\cap N(u)\setminus V(P)$, then the path $v_0v_1\cdots v_{i_k-2}v_{i_j}v_{i_j-1}\cdots v_{i_k}uwv_{i_j+1}v_{i_j+2}\cdots v_r$ is an x−y path of length r+1. This is a contradiction.Hence $N(v_{i_j+1})\cap N(u)\subseteq V(P)$. By Claims 2 and 4, $d(u)+d(v_{i_j+1})<n$. This implies $\left|N(v_{i_j+1})\cap N(u)\right|\geqslant\alpha$. Therefore there holds the following Claim.

Claim 6 *$m=\alpha$ and $N(v_{i_j+1})\cap N(u)=N_p(u)$.*

Claim 6 implies $v_{i_j+1}v_{i_k}\in E$. Replacing the segment on P from v_{i_k-2} to v_{i_j+1} by $v_{i_k-2}v_{i_j}v_{i_j-1}\cdots v_{i_k}v_{i_j+1}$. One gets a path P′ of length r−1. Considering the path P′, one can see that $N(u)\cap N(v)\subseteq V(P)$ for any $v\in A_m\cup B_1\setminus\{v_{i_k-1}\}$. Hence, by Claims 2 and 4, we get

Claim 7 *For any $v\in A_m\cup B_1\setminus\{v_{i_k-1}\}$, $N(u)\cap N(v)=N_P(u)$.*

Claim 8 *$A_m\cup\{v_{i_k-1},u\}$ is independent.*

For otherwise, there exists $v_{i_t+1}\in A_m$ with $v_{i_t+1}v_{i_k-1}\in E$. If $t=k-1$, an x−y path $v_0v_1\cdots v_{i_{k-1}}uv_{i_j}v_{i_j-1}\cdots v_{i_k-1}v_{i_{k-1}+1}v_{i_{k-1}+2}\cdots v_{i_k-2}v_{i_j+1}v_{i_j+2}\cdots v_r$ is of length r+1. For $t\neq k-1$, without loss of generality, suppose $t<k-1$, an x−y path $v_0v_1\cdots v_{i_t}uv_{i_{k-1}}v_{i_{k-1}-1}\cdots v_{i_t+1}v_{i_k-1}v_{i_k-2}\cdots v_{i_{k-1}+1}$

$v_{i_k} v_{i_k+1} \cdots v_r$ is of length r+1.A contradictions.

Claim 8 implies the existence of an independent set of cardinality $\alpha+1$. A contradiction.

Subcase 1.2 $|N(v_{i_k-2}) \cap N(u)| \geqslant \alpha$.

By the proof of case 1.1, we can suppose that for any j ($2 \leqslant j \leqslant m$), if $i_k > i_{k-1}+2$, then $|N(v_{i_k-2}) \cap N(u)| \geqslant \alpha$. By Claims 3 and 5 and the symmetry, we get

Claim 9 *$m=\alpha$ and for any j, $2 \leqslant j \leqslant m$, if $i_j > i_{j-1}+2$, then $N(u) \cap N(v_{i_j-2}) = N_P(u)$, $N(v_{i_{j-1}+2}) \cap N(u) = N_P(u)$.*

By Claim 9, it is easy to prove that for any $v \in A_m \cup B_1$, $N(u) \cap N(v) \subseteq V(P)$. Subsequently, by Claims 2, 4 and the intersection condition, we get

Claim 10 *For any $v \in A_m \cup B_1$, $N(u) \cap N(v) = N_P(u)$.*

Consider C_{k-1}. If $v_{i_{k-1}+1} v_{i_k-1} \notin E$, with a similar proof of Claim 8, Claim 10 implies that $A_m \cup \{v_{i_k-1}, u\}$ is independent. A contradiction. Hence, $v_{i_{k-1}+1} v_{i_k-1} \in E$. By Claim 9, we can replace the segment on P from $v_{i_{k-1}}$ to v_{i_k} by $v_{i_{k-1}} v_{i_{k-1}+2} v_{i_{k-1}+3} \cdots v_{i_k-1} v_{i_{k-1}+1} v_{i_k}$ and hence, we can suppose that for any $v \in C_{k-1}$, $N(u) \cap N(v) = N_P(u)$ and $G[C_{k-1}]$ is complete. And further, we have

Claim 11 *For any j ($1 \leqslant j \leqslant m-1$), $G[C_j]$ is complete, and for any $v \in C_j$, $N(u) \cap N(v) = N_P(u)$.*

By the assumption and Claim 11 we have

Claim 12 *For any j, s, $1 \leqslant j < s \leqslant m-1$, $[C_j, C_s] = \phi$.*

Set $V_1 = V(G) \setminus (V(P) \cup \{u\} \cup N(u))$. If $V_1 \neq \phi$, let $w \in V_1$, then $wu \notin E$. Therefore w is adjacent with some vertex, say v_{i_j+1}, of A_m. Suppose $N(w) \cap N(u) \subseteq N_P(u)$ (Otherwise we can easily get an x–y path of length r+1). By claim 11, there exists an x–y path of length r+1. A contradiction. Hence $V_1 = \phi$. By Claim 11 and the definition of independence number, $G[N(u) \setminus V(P)]$ is complete. That is, $G \in G(\alpha,\alpha)$.

Case 2 All cases but not case 1.

Let $V(P)=V_1\bigcup V_2$, where $V_1=\{v_i\in V(P)|i$ is odd$\}$, $V_2=\{v_i\in V(P)|i$ is even$\}$. For any $u\in V(G)\setminus V(P)$, if u is adjacent with some vertex of V_i, then u is adjacent with all vertices of $V_i(i=1$ or $2)$. Set $V_3=\{u\in V(G)\setminus V(P)| d_{V_1}(u)>0\}$, $V_4=\{u\in V(G)\setminus V(P)|, d_{V_2}(u)>0\}$. Obviously, $V_3\bigcap V_4=\phi$, and V_3 and V_4 are independent. Let $V_5=V(G)\setminus(\bigcup_{i=1}^{4}V_i)$. If $V_5\neq\phi$, let $w\in V_5$. Without loss of generality, suppose $d_{V_4}(w)>0$. If also $d_{V_3}(w)>0$, we can easily get an x−y path of length r+1. Hence $d_{V_3}(w)=0$. Let $u\in V_1$. Then $wu\notin E$, $d(u)+d(w)<n$ and $N(w)\bigcap N(u)=\phi$. This is a contradiction. Hence $V_5=\phi$.

If $V_4=\phi$ or $V_3=\phi$, without loss of generality, suppose $V_4=\phi$, then $V_3\neq\phi$. For any $u\in V_3$, $v\in V_2$, $uv\notin E$. Hence $N(u)\bigcap N(v)\subseteq V(P)$. Since $V_2\setminus\{v_1\}$ is independent, by Claim 4, $|V_1|\geqslant\alpha$. But $v_0u\notin E$, which contradicts Claim 3. If $V_3=\phi$, but $V_4\neq\phi$, let $u\in V_4$. Clearly, V_1 is independent. Since $N(u)\bigcap N(v_1)\subseteq V(P)$, by Claim 4, $|V_2|\geqslant\alpha$. Hence $|V_2|=\alpha$ and $i_1=0$, $i_m=1$. This implies $|V_1|=\alpha-1$. So $|V_0|=1$. That is, $G\in G(\alpha,\alpha)$.

If both V_3 and V_4 are nonempty, then, when r is odd or, r is even but $v_0v_1\notin E$, both V_1 and V_2 are independent. And further, $V_1\bigcup V_4$, $V_2\bigcup V_3$ are independent. So G is isomorphic to $K_{\alpha,\alpha}$. When r is even and $v_0v_1\in E$, V_1 is independent, $V_2-\{v_0\}$ or $V_2-\{v_1\}$ is independent too. It is easy to see that G is belong to $G(\alpha,\alpha)$.

Theorem 1 is proved.

Corollary 2[2,3] *Let G be a simple graph of order ($n\geqslant 3$). If for any pair of nonadjacent vertices u,v, $d(u)+d(v)\geqslant n+1$, then G is (5,n)− connected.*

Corollary 3[4] *Let G be a simple graph of order ($n\geqslant 3$) with independence number α. If for any pair of nonadjacent vertices u,v, $|N(u)\bigcap N(v)|\geqslant\alpha$, then G is (5,n)−connected, unless G is belong to $G(\alpha,\alpha)$.*

Corollary 4 *Let G be a simple graph of order $n(\geqslant 4)$ with independence number α. If for any pair of nonadjacent vertices u,v, $d(u)+d(v)\geqslant n+1$ or $|N(u)\bigcap N(v)|\geqslant\alpha$, then each edge of G is either 3^-−pancyclic or 4^-−pancyclic, unless G is isomorphic to $K_{\alpha,\alpha}$.*

References

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